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SIMULATION OF A FRACTIONAL BROWNIAN MOTION IN THE SPACE $L_p([0,T])$

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Dedicated to the memory of our teacher Mykhailo Yosypovych Yadrenko

ABSTRACT. A model that approximates the fractional Brownian motion with parameter $\alpha \in (0,2)$ with a given reliability $1-\delta$, $0<\delta<1$, and accuracy $\varepsilon>0$ in the space $L_p([0,T])$ is constructed. An example of a simulation in the space $L_2([0,1])$ is given.

1. Introduction

Stochastic simulation of stochastic processes and fields is used in various fields of natural and social science such as economics, mathematics, physics, engineering, meteorology, biology, and sociology where it provides the base for thorough analysis and decision making. Stochastic simulation has been an actively developing area since the second half of the 20th century. A special place in stochastic simulation is occupied by methods and procedures for simulation of Wiener and generalized Wiener processes (like a fractional Brownian motion). Many studies (see, for example, [35]) exhibit the properties of the self-similarity and long-range dependence of the data observed in queuing theory and telecommunication networks. One of the processes possessing those properties is the fractional Brownian motion. Kolmogorov [8] was the first to consider this process when studying some problems in the theory of turbulence [9]. Kolmogorov [8] investigated the fractional Brownian motion in a Hilbert space, in particular he found the covariance function for this process by using a condition known today as *self-similarity*.

Among earlier papers, one should mention Yaglom [37] where stochastic processes with stationary nth order increments are studied in order to extend the spectral theory of a stationary process to a wider class of processes. The fractional Brownian motion is considered in Yaglom [37] as an example of a stochastic process with stationary increments of the first order. Yaglom [37] defines the fractional Brownian motion in terms of its spectral density.

Mandelbrot and van Ness [23] represent the fractional Brownian motion as an integral with respect to a Wiener process over the whole real line. Since then the fractional Brownian motion has been studied extensively. In particular, some classical representations for the fractional Brownian motion are obtained in the papers [29, 32]. The paper [34] is a brief survey of properties of fractional Brownian motions, while the books [1, 24] contain a systematic analysis as well as generalizations and applications of results concerning this process.

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A detailed survey of methods for simulation of fractional Brownian motions is given in [2] and [4]. A discrete sequence of independent Gaussian random variables is used in earlier papers as a model of generalized Wiener processes. Models of this kind are used, for example, for numerical evaluation of integrals [6] and solving stochastic boundary value problems [33]. The spectral representation of a generalized Wiener process is used in [3, 25, 30, 31] to construct a spectral model for this process. Expansions in the form of random series (see, for example, [5, 10, 19]) are used in [7, 11, 12, 15, 17, 21, 26, 27, 36] to construct the models in the form of finite sums of these series and to study the reliability and accuracy of these procedures. The papers [13, 14, 16] are devoted to simulation of stochastic processes and fields. In particular, the case of Gaussian processes and fields is considered in [22, 26], while a more general case of φ-sub-Gaussian processes is studied in [18, 36].

The approaches mentioned above have their own advantages and drawbacks. The main drawback of the procedure based on the representation of the fractional Brownian motion in the form of a random series is an enormous amount of preliminary calculations needed to model this stochastic process. For example, the papers [21, 26] determine the parameters of a model with a given accuracy in terms of zeros of two Bessel functions of the first kind. A numerical evaluation of these zeros with a given accuracy is a time-consuming procedure even for modern computers. In contrast, this drawback is not present if one uses a spectral representation to model the fractional Brownian motion.

The papers cited above propose spectral models for simulation of the fractional Brownian motion and study the convergence of covariance functions and finite-dimensional distributions of models to those of the fractional Brownian motion. On the other hand, the reliability and accuracy of the models are not investigated in those papers.

In the current paper, we construct a spectral model that approximates a fractional Brownian motion with a given reliability $1-\delta$, $0<\delta<1$, and accuracy $\varepsilon>0$ in the space $L_p([0,T])$.

The paper is organized as follows. We provide some auxiliary results in Section 2 and define a model for a fractional Brownian motion with parameter $\alpha \in (0,2)$. Conditions for a model to approximate a fractional Brownian motion with a given reliability and accuracy in the space $L_p([0,T])$ are obtained in Section 3. A particular case of the space $L_2([0,T])$ is also discussed in Section 3. Section 4 contains an example of simulation of a fractional Brownian motion with a given reliability and accuracy in the space $L_2([0,1])$ for certain values of the parameter α.

2. A model for the fractional Brownian motion

Let $(\Omega, \Sigma, \mathsf{P})$ be a standard probability space and let T be a parameteric set ($T=[0,T]$ or $T=[0,\infty]$).

Definition 2.1. A stochastic process $\{W_\alpha(t), t \in T\}$ is called a fractional Brownian motion with a parameter $\alpha \in (0,2)$ if it is a zero mean Gaussian process, $\mathsf{E}W_\alpha(t)=0$, whose covariance function is given by

$$R(t,s)=\frac{1}{2}\left(|t|^\alpha+|s|^\alpha-|t-s|^\alpha\right)$$

and such that $W_\alpha(0)=0$.

It is known that a fractional Brownian motion with a parameter $\alpha \in (0,2)$ can be represented as follows:

$$W_\alpha(t)=\frac{A}{\sqrt{\pi}}\left(\int_0^\infty \frac{\cos(\lambda t)-1}{\lambda^{\frac{\alpha+1}{2}}}\,d\xi(\lambda)-\int_0^\infty \frac{\sin(\lambda t)}{\lambda^{\frac{\alpha+1}{2}}}\,d\eta(\lambda)\right) \qquad t\in[0,T],$$

(see, for example, [29–31]), where $\xi(\lambda)$ and $\eta(\lambda)$ are independent real-valued standard Wiener processes such that

$$\mathsf{E}\xi(\lambda) = \mathsf{E}\eta(\lambda) = 0,$$

$$\mathsf{E}\big(d\xi(\lambda)\big)^2 = \mathsf{E}\big(d\eta(\lambda)\big)^2 = d\lambda,$$

$$A^2 = \left\{\frac{2}{\pi}\int_0^\infty \frac{1-\cos(\lambda t)}{\lambda^{\alpha+1}}\,d\lambda\right\}^{-1} = \left\{-\frac{2}{\pi}\Gamma(-\alpha)\cos\left(\frac{\alpha\pi}{2}\right)\right\}^{-1}.$$

Fix an interval $[0,\Lambda]$, $\Lambda>0$, and represent the process $W_\alpha = \{W_\alpha(t), t\in[0,T]\}$ as

$$W_\alpha(t) = W_\alpha(t,[0,\epsilon]) + W_\alpha(t,[\epsilon,\Lambda]) + W_\alpha(t,[\Lambda,\infty]),$$

where $0<\epsilon<\Lambda$ and

$$W_\alpha(t,[a,b]) = \frac{A}{\sqrt{\pi}}\left(\int_a^b \frac{\cos(\lambda t)-1}{\lambda^{\frac{\alpha+1}{2}}}\,d\xi(\lambda) - \int_a^b \frac{\sin(\lambda t)}{\lambda^{\frac{\alpha+1}{2}}}\,d\eta(\lambda)\right).$$

Let $0=\lambda_0<\lambda_1<\cdots<\lambda_M=\Lambda$ be a partition of the interval $[0,\Lambda]$ with $\lambda_1=\epsilon$. Then a *model* of the process W_α is constructed as a sum

$$\begin{aligned} S_M(t,\Lambda) &= \frac{A}{\sqrt{\pi}}\left(\sum_{i=1}^{M-1}\frac{\cos(\lambda_i t)-1}{\lambda_i^{\frac{\alpha+1}{2}}}\big(\xi(\lambda_{i+1})-\xi(\lambda_i)\big)\right.\\ &\qquad\left.-\sum_{i=1}^{M-1}\frac{\sin(\lambda_i t)}{\lambda_i^{\frac{\alpha+1}{2}}}\big(\eta(\lambda_{i+1})-\eta(\lambda_i)\big)\right)\\ &= \frac{A}{\sqrt{\pi}}\left(\sum_{i=1}^{M-1}\frac{\cos(\lambda_i t)-1}{\lambda_i^{\frac{\alpha+1}{2}}}X_i - \sum_{i=1}^{M-1}\frac{\sin(\lambda_i t)}{\lambda_i^{\frac{\alpha+1}{2}}}Y_i\right), \qquad t\in[0,T],\ M\in\mathbb{N}, \end{aligned}$$

where $\{X_i, Y_i\}$, $i=1,2,\dots,M-1$, are independent Gaussian random variables such that

$$\mathsf{E}X_i = \mathsf{E}Y_i = 0, \qquad \mathsf{E}X_i^2 = \mathsf{E}Y_i^2 = \lambda_{i+1}-\lambda_i.$$

The following result is used in the rest of this paper (Proposition 2.1 below is a particular case of Theorem 2.1 and Corollary 2.1 of [7]).

Proposition 2.1 ([7])**.** *Let $X=\{X(t), t\in[0,T]\}$ be a centered Gaussian stochastic process such that*

$$\int_0^T c := \big(\mathsf{E}(X(t))^2\big)^{p/2}\,dt < \infty.$$

Then the integral $\int_0^T |X(t)|^p\,dt$ is well defined with probability one and

$$\mathsf{P}\left\{\int_0^T |X(t)|^p\,dt > \varepsilon\right\} \le 2\exp\left\{-\frac{\varepsilon^{2/p}}{2c^{2/p}}\right\}$$

for all $\varepsilon > cp^{\frac{p}{2}}$.

3. Simulation of a fractional Brownian motion with a given reliability and accuracy in the space $L_p([0,T])$

Definition 3.1. We say that a model $S_M=\{S_M(t,\Lambda), t\in[0,T]\}$ approximates the stochastic process $W_\alpha=\{W_\alpha(t), t\in[0,T]\}$ with a given reliability $1-\delta$, $0<\delta<1$, and accuracy $\varepsilon>0$ in the space $L_p([0,T])$, $p\ge1$, if

$$\mathsf{P}\left\{\left(\int_0^T |W_\alpha(t)-S_M(t,\Lambda)|^p\,dt\right)^{1/p} > \varepsilon\right\} \le \delta.$$

Since the deviation $X_M(t,\Lambda) := W_\alpha(t) - S_M(t,\Lambda)$, $t\in[0,T]$, is a centered Gaussian stochastic process, one can apply Proposition 2.1. As a result, we obtain sufficient conditions under which the model $S_M(t,\Lambda)$, $t\in[0,T]$, approximates the process W_α with a reliability $1-\delta$, $0<\delta<1$, and accuracy $\varepsilon>0$ in the space $L_p([0,T])$.

Theorem 3.1. *A model S_M approximates the process W_α with a given reliability $1-\delta$, $0<\delta<1$, and accuracy $\varepsilon>0$ in the space $L_p([0,T])$, $p\ge 1$, if*

$$\int_0^T \left(\frac{2A^2}{\pi}\left(\frac{t^2\lambda_1^{2-\alpha}}{2(2-\alpha)} + \frac{1}{\alpha\Lambda^\alpha} + 2t^2\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^3}{3\lambda_i^{\alpha+1}} + \left(1+2^{2-\frac{3}{2}\alpha}\right)t^{\frac{3}{2}\alpha}\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^{\alpha+2}}{\lambda_i^{\frac{\alpha}{2}+2}}\right)\right)^{p/2} dt < \varepsilon^p\cdot\min\left\{\frac{1}{p^{\frac{p}{2}}}, \frac{1}{\left(-2\ln\frac{\delta}{2}\right)^{p/2}}\right\}$$

in the case of $\alpha\in(0,1]$, or if

$$\int_0^T \left(\frac{2A^2}{\pi}\left(\frac{t^2\lambda_1^{2-\alpha}}{2(2-\alpha)} + \frac{1}{\alpha\Lambda^\alpha} + 2t^2\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^3}{3\lambda_i^{\alpha+1}} + 4t^2\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^{\frac{\alpha}{2}+\frac{3}{2}}}{\lambda_i^{\frac{3\alpha}{2}-\frac{1}{2}}}\right)\right)^{p/2} dt < \varepsilon^p\cdot\min\left\{\frac{1}{p^{\frac{p}{2}}}, \frac{1}{\left(-2\ln\frac{\delta}{2}\right)^{p/2}}\right\}$$

in the case of $\alpha\in(1,2)$, where $M\in\mathbb{N}$ and $0=\lambda_0<\lambda_1<\cdots<\lambda_M=\Lambda$ is a partition of the interval $[0,\Lambda]$.

Proof. As mentioned above, one can apply Proposition 2.1 to the deviation process $X_M(t,\Lambda)$, $t\in[0,T]$. This means that the process $X_M(t,\Lambda)$ admits the inequality

$$\mathsf{P}\left\{\int_0^T |X_M(t,\Lambda)|^p\,dt > u\right\} \le 2\exp\left\{-\frac{u^{\frac{2}{p}}}{2c^{\frac{2}{p}}}\right\}$$

for all $u > cp^{\frac{p}{2}}$, where

$$c = \int_0^T \left(\mathsf{E}(X_M(t,\Lambda))^2\right)^{p/2}\,dt.$$

If we choose $\varepsilon = u^{1/p}$ and

$$\delta \ge 2\exp\left\{-\frac{u^{2/p}}{2c^{2/p}}\right\} = 2\exp\left\{-\frac{\varepsilon^2}{2c^{2/p}}\right\}, \qquad 0<\delta<1,$$

then the above bound is rewritten as

$$\mathsf{P}\left\{\left(\int_0^T |X_M(t,\Lambda)|^p\,dt\right)^{1/p} > \varepsilon\right\} \le \delta.$$

Note that the latter inequality coincides with that of Definition 3.1. Therefore, in order to obtain sufficient conditions under which the model S_M approximates the process W_α with a given reliability $1-\delta$ and accuracy $\varepsilon>0$ in the space $L_p([0,T])$ one only needs to choose appropriate parameters to satisfy the assumptions of Proposition 2.1.

First, the condition $u > cp^{p/2}$ implies $\varepsilon = u^{1/p} > \left(cp^{p/2}\right)^{1/p} = c^{1/p}p^{1/2}$, whence

$$c < \frac{\varepsilon^p}{p^{p/2}}.$$

Second, if $\delta \in (0,1)$ is such that

$$\delta \geq 2\exp\left\{-\frac{\varepsilon^2}{2c^{2/p}}\right\},$$

then

$$\exp\left\{-\frac{\varepsilon^2}{2c^{\frac{2}{p}}}\right\} \leq \frac{\delta}{2}; \qquad -\frac{\varepsilon^2}{2c^{\frac{2}{p}}} \leq \ln\frac{\delta}{2}; \qquad 2c^{\frac{2}{p}} \leq \frac{\varepsilon^2}{-\ln\frac{\delta}{2}};$$

$$c \leq \left(\frac{\varepsilon^2}{-2\ln\frac{\delta}{2}}\right)^{p/2} = \frac{\varepsilon^p}{\left(-2\ln\frac{\delta}{2}\right)^{p/2}}.$$

Hence

$$c = \int_0^T \left(\mathsf{E}(X_M(t,\Lambda))^2\right)^{p/2} dt < \varepsilon^p \cdot \min\left\{\frac{1}{p^{\frac{p}{2}}}, \frac{1}{(-2\ln\frac{\delta}{2})^{p/2}}\right\}. \tag{1}$$

Further we consider the second moment $\mathsf{E}(X_M(t,\Lambda))^2$:

$$\begin{aligned}\mathsf{E}\big(X_M(t,\Lambda)\big)^2 &= \mathsf{E}\big(W_\alpha(t) - S_M(t,\Lambda)\big)^2\\ &= \mathsf{E}\big(W_\alpha(t,[0,\lambda_1]) + W_\alpha(t,[\lambda_1,\Lambda]) + W_\alpha(t,[\Lambda,\infty]) - S_M(t,\Lambda)\big)^2\\ &= \mathsf{E}\big(W_\alpha(t,[0,\lambda_1])\big)^2 + \mathsf{E}\big(W_\alpha(t,[\Lambda,\infty])\big)^2 + \mathsf{E}\big(W_\alpha(t,[\lambda_1,\Lambda]) - S_M(t,\Lambda)\big)^2.\end{aligned}$$

Now we estimate each term in the above expression separately:

$$\begin{aligned}\mathsf{E}\big(W_\alpha(t,[0,\lambda_1])\big)^2 &= \frac{A^2}{\pi}\mathsf{E}\left(\int_0^{\lambda_1} \frac{\cos(\lambda t)-1}{\lambda^{\frac{\alpha+1}{2}}}\,d\xi(\lambda) - \int_0^{\lambda_1} \frac{\sin(\lambda t)}{\lambda^{\frac{\alpha+1}{2}}}\,d\eta(\lambda)\right)^2\\ &= \frac{A^2}{\pi}\left\{\mathsf{E}\left(\int_0^{\lambda_1} \frac{\cos(\lambda t)-1}{\lambda^{\frac{\alpha+1}{2}}}\,d\xi(\lambda)\right)^2 + \mathsf{E}\left(\int_0^{\lambda_1} \frac{\sin(\lambda t)}{\lambda^{\frac{\alpha+1}{2}}}\,d\eta(\lambda)\right)^2\right\}\\ &= \frac{A^2}{\pi}\left\{\int_0^{\lambda_1} \frac{(\cos(\lambda t)-1)^2}{\lambda^{\alpha+1}}\,d\lambda + \int_0^{\lambda_1} \frac{\sin^2(\lambda t)}{\lambda^{\alpha+1}}\,d\lambda\right\}\\ &= \frac{A^2}{\pi}\left\{\int_0^{\lambda_1} \frac{(\cos(\lambda t)-1)^2 + \sin^2(\lambda t)}{\lambda^{\alpha+1}}\,d\lambda\right\}\\ &= \frac{A^2}{\pi}\left\{\int_0^{\lambda_1} \frac{2-2\cos(\lambda t)}{\lambda^{\alpha+1}}\,d\lambda\right\} = \frac{A^2}{\pi}\left\{\int_0^{\lambda_1} \frac{4\sin^2\left(\frac{\lambda t}{2}\right)}{\lambda^{\alpha+1}}\,d\lambda\right\}\\ &\leq \frac{A^2}{\pi}\left\{t^2\int_0^{\lambda_1} \lambda^{2-1-\alpha}\,d\lambda\right\} = \frac{A^2}{\pi}\left\{\frac{t^2\lambda_1^{2-\alpha}}{2-\alpha}\right\}.\end{aligned}$$

The second term in that expression admits the bound

$$\begin{aligned}\mathsf{E}\big(W_\alpha(t,[\Lambda,\infty])\big)^2 &= \mathsf{E}\left(\frac{A}{\sqrt{\pi}}\int_\Lambda^\infty \frac{\cos(\lambda t)-1}{\lambda^{\frac{\alpha+1}{2}}}\,d\xi(\lambda) - \int_\Lambda^\infty \frac{\sin(\lambda t)}{\lambda^{\frac{\alpha+1}{2}}}\,d\eta(\lambda)\right)^2\\ &= \frac{A^2}{\pi}\left\{\int_\Lambda^\infty \frac{2-2\cos(\lambda t)}{\lambda^{\alpha+1}}\,d\lambda\right\} \leq 2\,\frac{A^2}{\pi}\left\{\int_\Lambda^\infty \frac{d\lambda}{\lambda^{\alpha+1}}\right\}\\ &\leq \frac{2A^2}{\alpha\pi\Lambda^\alpha}.\end{aligned}$$

The third term $\mathsf{E}\big(W_\alpha(t,[\lambda_1,\Lambda]) - S_M(t,\Lambda)\big)^2$ is such that

$$\begin{aligned}
&\mathsf{E}\big(W_\alpha(t,[\lambda_1,\Lambda]) - S_M(t,\Lambda)\big)^2 \\
&= \frac{A^2}{\pi}\mathsf{E}\Bigg(\int_{\lambda_1}^{\Lambda} \frac{\cos(\lambda t)-1}{\lambda^{\frac{\alpha+1}{2}}}\,d\xi(\lambda) - \int_{\lambda_1}^{\Lambda} \frac{\sin(\lambda t)}{\lambda^{\frac{\alpha+1}{2}}}\,d\eta(\lambda) \\
&\qquad - \sum_{i=1}^{M-1} \frac{\cos(\lambda_i t)-1}{\lambda_i^{\frac{\alpha+1}{2}}}\big(\xi(\lambda_{i+1})-\xi(\lambda_i)\big) \\
&\qquad + \sum_{i=1}^{M-1} \frac{\sin(\lambda_i t)}{\lambda_i^{\frac{\alpha+1}{2}}}\big(\eta(\lambda_{i+1})-\eta(\lambda_i)\big)\Bigg)^2 \\
&= \frac{A^2}{\pi}\left\{\mathsf{E}\left(\sum_{i=1}^{M-1}\int_{\lambda_i}^{\lambda_{i+1}}\left(\frac{\cos(\lambda t)-1}{\lambda^{\frac{\alpha+1}{2}}} - \frac{\cos(\lambda_i t)-1}{\lambda_i^{\frac{\alpha+1}{2}}}\right)d\xi(\lambda)\right.\right. \\
&\qquad \left.\left. - \sum_{i=1}^{M-1}\int_{\lambda_i}^{\lambda_{i+1}}\left(\frac{\sin(\lambda t)}{\lambda^{\frac{\alpha+1}{2}}} - \frac{\sin(\lambda_i t)}{\lambda_i^{\frac{\alpha+1}{2}}}\right)d\eta(\lambda)\right)^2\right\} \\
&= \frac{A^2}{\pi}\left\{\sum_{i=1}^{M-1}\int_{\lambda_i}^{\lambda_{i+1}}\left(\frac{\cos(\lambda t)-1}{\lambda^{\frac{\alpha+1}{2}}} - \frac{\cos(\lambda_i t)-1}{\lambda_i^{\frac{\alpha+1}{2}}}\right)^2 d\lambda\right. \\
&\qquad \left. + \sum_{i=1}^{M-1}\int_{\lambda_i}^{\lambda_{i+1}}\left(\frac{\sin(\lambda t)}{\lambda^{\frac{\alpha+1}{2}}} - \frac{\sin(\lambda_i t)}{\lambda_i^{\frac{\alpha+1}{2}}}\right)^2 d\lambda\right\} \\
&= \frac{A^2}{\pi}\left(\Sigma_{\cos} + \Sigma_{\sin}\right).
\end{aligned}$$

Now we estimate the integral

$$\begin{aligned}
&\int_{\lambda_i}^{\lambda_{i+1}}\left(\frac{\sin(\lambda t)}{\lambda^{\frac{\alpha+1}{2}}} - \frac{\sin(\lambda_i t)}{\lambda_i^{\frac{\alpha+1}{2}}}\right)^2 d\lambda \\
&\quad = \int_{\lambda_i}^{\lambda_{i+1}}\left(\frac{\sin(\lambda t)}{\lambda^{\frac{\alpha+1}{2}}} - \frac{\sin(\lambda_i t)}{\lambda^{\frac{\alpha+1}{2}}} + \frac{\sin(\lambda_i t)}{\lambda^{\frac{\alpha+1}{2}}} - \frac{\sin(\lambda_i t)}{\lambda_i^{\frac{\alpha+1}{2}}}\right)^2 d\lambda \\
&\quad \le 2\left(\int_{\lambda_i}^{\lambda_{i+1}}\left(\frac{\sin(\lambda t)-\sin(\lambda_i t)}{\lambda^{\frac{\alpha+1}{2}}}\right)^2 d\lambda + \int_{\lambda_i}^{\lambda_{i+1}}\big(\sin(\lambda_i t)\big)^2\left(\frac{1}{\lambda^{\frac{\alpha+1}{2}}} - \frac{1}{\lambda_i^{\frac{\alpha+1}{2}}}\right)^2 d\lambda\right) \\
&\quad = 2(W_{i1} + W_{i2})
\end{aligned}$$

and W_{i1} and W_{i2} on the right-hand side of the latter inequality:

$$W_{i1} \le \int_{\lambda_i}^{\lambda_{i+1}} \frac{4\left(\sin\left(\frac{\lambda t-\lambda_i t}{2}\right)\right)^2}{\lambda_i^{\alpha+1}}\,d\lambda \le \int_{\lambda_i}^{\lambda_{i+1}} \frac{(\lambda-\lambda_i)^2 t^2}{\lambda_i^{\alpha+1}}\,d\lambda = \frac{t^2(\lambda_{i+1}-\lambda_i)^3}{3\lambda_i^{\alpha+1}}. \tag{2}$$

When estimating W_{i2}, we consider separately the cases $\alpha \le 1$ and $1 < \alpha < 2$. In the case of $\alpha \le 1$, we apply the inequality $|\sin x| \le |x|^\beta$, $0 < \beta \le 1$. Then

$$W_{i2} \le \int_{\lambda_i}^{\lambda_{i+1}} \lambda_i^{2\beta} t^{2\beta} \left(\frac{1}{\lambda_i^{\frac{\alpha+1}{2}}} - \frac{1}{\lambda_{i+1}^{\frac{\alpha+1}{2}}} \right)^2 d\lambda = \int_{\lambda_i}^{\lambda_{i+1}} \lambda_i^{2\beta} t^{2\beta} \frac{\left(\lambda_{i+1}^{\frac{\alpha+1}{2}} - \lambda_i^{\frac{\alpha+1}{2}}\right)^2}{\left(\lambda_i^{\frac{\alpha+1}{2}} \lambda_{i+1}^{\frac{\alpha+1}{2}}\right)^2} d\lambda \tag{3}$$

for $\beta < \alpha$.

Recalling the inequality $a^\nu - b^\nu \le (a-b)^\nu$ if $\nu \le 1$ and $a > b$ we deduce from (3) that

$$W_{i2} \le \lambda_i^{2\beta} t^{2\beta} \frac{(\lambda_{i+1} - \lambda_i)^{\alpha+1}}{\lambda_i^{2(\alpha+1)}} \int_{\lambda_i}^{\lambda_{i+1}} d\lambda = t^{2\beta} \frac{(\lambda_{i+1} - \lambda_i)^{\alpha+2}}{\lambda_i^{2(\alpha-\beta)+2}},$$

since $0 < \lambda_i < \lambda_{i+1}$ and $(\alpha+1)/2 \le 1$ for $\alpha \le 1$. Choose $\beta = \frac{3}{4}\alpha$. Then

$$W_{i2} \le t^{\frac{3}{2}\alpha} \frac{(\lambda_{i+1} - \lambda_i)^{\alpha+2}}{\lambda_i^{\frac{\alpha}{2}+2}}. \tag{4}$$

In the case of $1 < \alpha < 2$, we have

$$\begin{aligned} W_{i2} &\le \int_{\lambda_i}^{\lambda_{i+1}} \lambda_i^2 t^2 \frac{\left(\lambda_{i+1}^{\frac{\alpha+1}{2}} - \lambda_i^{\frac{\alpha+1}{2}}\right)^2}{(\lambda_i \lambda_{i+1})^{\alpha+1}} d\lambda \\ &= \int_{\lambda_i}^{\lambda_{i+1}} \lambda_i^2 t^2 \frac{\left(\lambda_{i+1}^{\frac{\alpha+1}{4}} - \lambda_i^{\frac{\alpha+1}{4}}\right)^2 \left(\lambda_{i+1}^{\frac{\alpha+1}{4}} + \lambda_i^{\frac{\alpha+1}{4}}\right)^2}{(\lambda_i \lambda_{i+1})^{\alpha+1}} d\lambda \\ &\le \int_{\lambda_i}^{\lambda_{i+1}} \lambda_i^2 t^2 \frac{(\lambda_{i+1} - \lambda_i)^{\frac{\alpha+1}{2}} 4\lambda_{i+1}^{\frac{\alpha+1}{2}}}{(\lambda_i \lambda_{i+1})^{\alpha+1}} d\lambda = \frac{4t^2 (\lambda_{i+1} - \lambda_i)^{\frac{\alpha}{2}+\frac{3}{2}}}{\lambda_i^{\alpha-1} \lambda_{i+1}^{\frac{\alpha+1}{2}}} \\ &\le \frac{4t^2 (\lambda_{i+1} - \lambda_i)^{\frac{\alpha}{2}+\frac{3}{2}}}{\lambda_i^{\frac{3\alpha}{2}-\frac{1}{2}}}. \end{aligned} \tag{5}$$

Therefore inequalities (2)–(5) imply that

$$\Sigma_{\sin} \le 2 \left(t^2 \sum_{i=1}^{M-1} \frac{(\lambda_{i+1} - \lambda_i)^3}{3\lambda_i^{\alpha+1}} + t^{\frac{3}{2}\alpha} \sum_{i=1}^{M-1} \frac{(\lambda_{i+1} - \lambda_i)^{\alpha+2}}{\lambda_i^{\frac{\alpha}{2}+2}} \right) \tag{6}$$

for $\alpha \le 1$, or that

$$\Sigma_{\sin} \le 2t^2 \left(\sum_{i=1}^{M-1} \frac{(\lambda_{i+1} - \lambda_i)^3}{3\lambda_i^{\alpha+1}} + \sum_{i=1}^{M-1} \frac{4(\lambda_{i+1} - \lambda_i)^{\frac{\alpha}{2}+\frac{3}{2}}}{\lambda_i^{\frac{3\alpha}{2}-\frac{1}{2}}} \right) \tag{7}$$

for $1 < \alpha < 2$.

Now we estimate the integral $\int_{\lambda_i}^{\lambda_{i+1}}\left(\frac{\cos(\lambda t)-1}{\lambda^{\frac{\alpha+1}{2}}}-\frac{\cos(\lambda_i t)-1}{\lambda_i^{\frac{\alpha+1}{2}}}\right)^2 d\lambda$:

$$\begin{aligned}
&\int_{\lambda_i}^{\lambda_{i+1}}\left(\frac{\cos(\lambda t)-1}{\lambda^{\frac{\alpha+1}{2}}}-\frac{\cos(\lambda_i t)-1}{\lambda_i^{\frac{\alpha+1}{2}}}\right)^2 d\lambda\\
&\quad=\int_{\lambda_i}^{\lambda_{i+1}}\left(\frac{\cos(\lambda t)-1}{\lambda^{\frac{\alpha+1}{2}}}-\frac{\cos(\lambda_i t)-1}{\lambda^{\frac{\alpha+1}{2}}}+\frac{\cos(\lambda_i t)-1}{\lambda^{\frac{\alpha+1}{2}}}-\frac{\cos(\lambda_i t)-1}{\lambda_i^{\frac{\alpha+1}{2}}}\right)^2 d\lambda\\
&\quad\le 2\left(\int_{\lambda_i}^{\lambda_{i+1}}\left(\frac{\cos(\lambda t)-\cos(\lambda_i t)}{\lambda^{\frac{\alpha+1}{2}}}\right)^2 d\lambda\right.\\
&\qquad\qquad\left.+\int_{\lambda_i}^{\lambda_{i+1}}(\cos(\lambda_i t)-1)^2\left(\frac{1}{\lambda^{\frac{\alpha+1}{2}}}-\frac{1}{\lambda_i^{\frac{\alpha+1}{2}}}\right)^2 d\lambda\right)\\
&\quad=2(W_{i3}+W_{i4}).
\end{aligned}$$

For the term W_{i3}, we get

$$\begin{aligned}
W_{i3}&=\int_{\lambda_i}^{\lambda_{i+1}}\frac{\left(\cos(\lambda t)-\cos(\lambda_i t)\right)^2}{\lambda^{\alpha+1}}d\lambda=\int_{\lambda_i}^{\lambda_{i+1}}\frac{\left(2\sin\left(\frac{\lambda t+\lambda_i t}{2}\right)\sin\left(\frac{\lambda_i t-\lambda t}{2}\right)\right)^2}{\lambda^{\alpha+1}}d\lambda\\
&\le\int_{\lambda_i}^{\lambda_{i+1}}\frac{4\left|\frac{\lambda_i t-\lambda t}{2}\right|^2}{\lambda_i^{\alpha+1}}d\lambda\le\int_{\lambda_i}^{\lambda_{i+1}}\frac{t^2\left(\lambda_{i+1}-\lambda_i\right)^2}{\lambda_i^{\alpha+1}}d\lambda=\frac{t^2\left(\lambda_{i+1}-\lambda_i\right)^3}{3\lambda_i^{\alpha+1}}.
\end{aligned}\tag{8}$$

For $\alpha\le 1$, the term W_{i4} is estimated as follows:

$$\begin{aligned}
W_{i4}&=\int_{\lambda_i}^{\lambda_{i+1}}\left(\cos(\lambda_i t)-1\right)^2\left(\frac{1}{\lambda^{\frac{\alpha+1}{2}}}-\frac{1}{\lambda_i^{\frac{\alpha+1}{2}}}\right)^2 d\lambda\\
&=\int_{\lambda_i}^{\lambda_{i+1}}\left(2\sin^2\left(\frac{\lambda_i t}{2}\right)\right)^2\left(\frac{1}{\lambda^{\frac{\alpha+1}{2}}}-\frac{1}{\lambda_i^{\frac{\alpha+1}{2}}}\right)^2 d\lambda\\
&=\int_{\lambda_i}^{\lambda_{i+1}}4\sin^4\left(\frac{\lambda_i t}{2}\right)\left(\frac{1}{\lambda^{\frac{\alpha+1}{2}}}-\frac{1}{\lambda_i^{\frac{\alpha+1}{2}}}\right)^2 d\lambda\\
&\le\int_{\lambda_i}^{\lambda_{i+1}}4\left(\left|\frac{\lambda_i t}{2}\right|^{\frac{\beta}{2}}\right)^4\frac{\left(\lambda_{i+1}^{\frac{\alpha+1}{2}}-\lambda_i^{\frac{\alpha+1}{2}}\right)^2}{(\lambda_i\lambda_{i+1})^{\alpha+1}}d\lambda\\
&\le 2^{2-2\beta}\lambda_i^{2\beta}t^{2\beta}\frac{(\lambda_{i+1}-\lambda_i)^{\alpha+1}}{\lambda_i^{2(\alpha+1)}}\int_{\lambda_i}^{\lambda_{i+1}}d\lambda=\frac{2^{2-2\beta}t^{2\beta}(\lambda_{i+1}-\lambda_i)^{\alpha+2}}{\lambda_i^{2(\alpha-\beta)+2}}.
\end{aligned}$$

We choose $\beta=\frac{3}{4}\alpha$ again. Then

$$W_{i4}\le\frac{2^{2-\frac{3}{2}\alpha}t^{\frac{3}{2}\alpha}(\lambda_{i+1}-\lambda_i)^{\alpha+2}}{\lambda_i^{\frac{\alpha}{2}+2}}.\tag{9}$$

In the case of $1 < \alpha < 2$, we get

$$
\begin{aligned}
W_{i4} &= \int_{\lambda_i}^{\lambda_{i+1}} 4\sin^4\left(\frac{\lambda_i t}{2}\right) \frac{\left(\lambda_{i+1}^{\frac{\alpha+1}{2}} - \lambda_i^{\frac{\alpha+1}{2}}\right)^2}{(\lambda_i\lambda_{i+1})^{\alpha+1}}\, d\lambda \\
&\le \int_{\lambda_i}^{\lambda_{i+1}} 4\left(\left|\frac{\lambda_i t}{2}\right|^{\frac{1}{2}}\right)^4 \frac{\left(\lambda_{i+1}^{\frac{\alpha+1}{4}} - \lambda_i^{\frac{\alpha+1}{4}}\right)^2\left(\lambda_{i+1}^{\frac{\alpha+1}{4}} + \lambda_i^{\frac{\alpha+1}{4}}\right)^2}{(\lambda_i\lambda_{i+1})^{\alpha+1}}\, d\lambda \\
&\le \int_{\lambda_i}^{\lambda_{i+1}} \lambda_i^2 t^2 \frac{(\lambda_{i+1}-\lambda_i)^{\frac{\alpha+1}{2}} 4\lambda_{i+1}^{\frac{\alpha+1}{2}}}{(\lambda_i\lambda_{i+1})^{\alpha+1}}\, d\lambda \le \frac{4t^2(\lambda_{i+1}-\lambda_i)^{\frac{\alpha}{2}+\frac{3}{2}}}{\lambda_i^{\frac{3\alpha}{2}-\frac{1}{2}}}.
\end{aligned} \tag{10}
$$

Therefore inequalities (8)–(10) imply that

$$
\Sigma_{\cos} \le 2\left(t^2\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^3}{3\lambda_i^{\alpha+1}} + 2^{2-\frac{3}{2}\alpha} t^{\frac{3}{2}\alpha}\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^{\alpha+2}}{\lambda_i^{\frac{\alpha}{2}+2}}\right) \tag{11}
$$

for $\alpha \le 1$, or that

$$
\Sigma_{\cos} \le 2t^2\left(\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^3}{3\lambda_i^{\alpha+1}} + \sum_{i=1}^{M-1}\frac{4(\lambda_{i+1}-\lambda_i)^{\frac{\alpha}{2}+\frac{3}{2}}}{\lambda_i^{\frac{3\alpha}{2}-\frac{1}{2}}}\right) \tag{12}
$$

for $1 < \alpha < 2$.

Therefore,

$$
\begin{aligned}
&\mathsf{E}\big(W_\alpha(t,[\lambda_1,\Lambda]) - S_M(t,\Lambda)\big)^2 \\
&\le \begin{cases}
\frac{2A^2}{\pi}\left(2t^2\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^3}{3\lambda_i^{\alpha+1}} + \left(1+2^{2-\frac{3}{2}\alpha}\right)t^{\frac{3}{2}\alpha}\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^{\alpha+2}}{\lambda_i^{\frac{\alpha}{2}+2}}\right), & \alpha \le 1; \\
\frac{4A^2t^2}{\pi}\left(\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^3}{3\lambda_i^{\alpha+1}} + 4\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^{\frac{\alpha}{2}+\frac{3}{2}}}{\lambda_i^{\frac{3\alpha}{2}-\frac{1}{2}}}\right), & 1 < \alpha < 2.
\end{cases}
\end{aligned}
$$

The second moment $\mathsf{E}(X_M(t,\Lambda))^2$ is estimated as follows:

$$
\begin{aligned}
\mathsf{E}\big(X_M(t,\Lambda)\big)^2 \le \frac{2A^2}{\pi}&\left(\frac{t^2\lambda_1^{2-\alpha}}{2(2-\alpha)} + \frac{1}{\alpha\Lambda^\alpha} + 2t^2\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^3}{3\lambda_i^{\alpha+1}}\right. \\
&\left. + (1+2^{2-\frac{3}{2}\alpha})t^{\frac{3}{2}\alpha}\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^{\alpha+2}}{\lambda_i^{\frac{\alpha}{2}+2}}\right), \qquad \alpha \le 1,
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
\mathsf{E}\big(X_M(t,\Lambda)\big)^2 \le \frac{2A^2}{\pi}&\left(\frac{t^2\lambda_1^{2-\alpha}}{2(2-\alpha)} + \frac{1}{\alpha\Lambda^\alpha} + 2t^2\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^3}{3\lambda_i^{\alpha+1}}\right. \\
&\left. + 4t^2\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^{\frac{\alpha}{2}+\frac{3}{2}}}{\lambda_i^{\frac{3\alpha}{2}-\frac{1}{2}}}\right), \\
&1 < \alpha < 2.
\end{aligned} \tag{14}
$$

Finally, we conclude from bounds (13)–(14) that condition (1) holds if

$$\int_0^T \left(\frac{2A^2}{\pi}\left(\frac{t^2\lambda_1^{2-\alpha}}{2(2-\alpha)} + \frac{1}{\alpha\Lambda^\alpha} + 2t^2\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^3}{3\lambda_i^{\alpha+1}} + \left(1+2^{2-\frac{3}{2}\alpha}\right)t^{\frac{3}{2}\alpha}\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^{\alpha+2}}{\lambda_i^{\frac{\alpha}{2}+2}}\right)\right)^{p/2} dt < \varepsilon^p\cdot\min\left\{\frac{1}{p^{\frac{p}{2}}}, \frac{1}{(-2\ln\frac{\delta}{2})^{p/2}}\right\}, \qquad \alpha\le 1, \tag{15}$$

and

$$\int_0^T \left(\frac{2A^2}{\pi}\left(\frac{t^2\lambda_1^{2-\alpha}}{2(2-\alpha)} + \frac{1}{\alpha\Lambda^\alpha} + 2t^2\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^3}{3\lambda_i^{\alpha+1}} + 4t^2\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^{\frac{\alpha}{2}+\frac{3}{2}}}{\lambda_i^{\frac{3\alpha}{2}-\frac{1}{2}}}\right)\right)^{p/2} dt < \varepsilon^p\cdot\min\left\{\frac{1}{p^{\frac{p}{2}}}, \frac{1}{\left(-2\ln\frac{\delta}{2}\right)^{p/2}}\right\}, \qquad 1<\alpha<2. \tag{16}$$

Therefore if conditions (15)–(16) hold, then Theorem 3.1 is proved. □

Corollary 3.1. *A model S_M approximates the process W_α with a given reliability $1-\delta$, $0<\delta<1$, and accuracy $\varepsilon>0$ in the space $L_2([0,T])$ if*

$$\frac{2A^2}{\pi}\left(\frac{T^3\lambda_1^{2-\alpha}}{6(2-\alpha)} + \frac{T}{\alpha\Lambda^\alpha} + \frac{2T^3}{9}\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^3}{\lambda_i^{\alpha+1}} + \frac{(1+2^{2-\frac{3}{2}\alpha})T^{\frac{3}{2}\alpha+1}}{\frac{3}{2}\alpha+1}\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^{\alpha+2}}{\lambda_i^{\frac{\alpha}{2}+2}}\right) < \frac{\varepsilon^2}{2}\cdot\min\left\{1, \left(-\ln\frac{\delta}{2}\right)^{-1}\right\}$$

in the case of $\alpha\in(0,1]$, or if

$$\frac{2A^2}{\pi}\left(\frac{T^3\lambda_1^{2-\alpha}}{6(2-\alpha)} + \frac{T}{\alpha\Lambda^\alpha} + \frac{2T^3}{9}\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^3}{\lambda_i^{\alpha+1}} + \frac{4T^3}{3}\sum_{i=1}^{M-1}\frac{(\lambda_{i+1}-\lambda_i)^{\frac{\alpha}{2}+\frac{3}{2}}}{\lambda_i^{\frac{3\alpha}{2}-\frac{1}{2}}}\right) < \frac{\varepsilon^2}{2}\cdot\min\left\{1, \left(-\ln\frac{\delta}{2}\right)^{-1}\right\}$$

in the case of $\alpha\in(1,2)$, where $M\in\mathbb{N}$ and $0=\lambda_0<\lambda_1<\dots<\lambda_M=\Lambda$ is a partition of the interval $[0,\Lambda]$.

Proof. If $p=2$, then

$$\varepsilon^p\cdot\min\left\{\frac{1}{p^{\frac{p}{2}}}, \frac{1}{(-2\ln\frac{\delta}{2})^{p/2}}\right\} = \frac{\varepsilon^2}{2}\cdot\min\left\{1, \left(-\ln\frac{\delta}{2}\right)^{-1}\right\}.$$

Further, substituting $p=2$ into the integrands in the assumptions of Theorem 3.1 and then integrating over $t\in[0,T]$ we prove Corollary 3.1. □

4. A procedure for simulation in the space $L_2([0,1])$

Note that

$$\Delta\lambda = \lambda_{i+1} - \lambda_i = \frac{\Lambda}{M}, \qquad \lambda_1 = \frac{\Lambda}{M}, \qquad \lambda_i = \frac{i\Lambda}{M}$$

for a uniform partition of the interval $[0,\Lambda]$. Corollary 3.1 allows one to easily obtain conditions for evaluating the parameters Λ and M of the model for the space $L_2([0,1])$. For example, if $\alpha \in (0,1]$, then

$$\frac{2A^2}{\pi}\left(\frac{\Lambda^{2-\alpha}}{6(2-\alpha)M^{2-\alpha}} + \frac{1}{\alpha\Lambda^\alpha} + \frac{2\Lambda^{2-\alpha}}{9M^{2-\alpha}}\sum_{i=1}^{M-1}\frac{1}{i^{\alpha+1}} + \frac{\left(1+2^{2-\frac{3}{2}\alpha}\right)\Lambda^{\frac{\alpha}{2}}}{\left(\frac{3}{2}\alpha+1\right)M^{\frac{\alpha}{2}}}\sum_{i=1}^{M-1}\frac{1}{i^{\frac{\alpha}{2}+2}}\right) < \frac{\varepsilon^2}{2}\cdot\min\left\{1, \left(-\ln\frac{\delta}{2}\right)^{-1}\right\}.$$

The trajectories of some models of the fractional Brownian motion are depicted in Figures 1–4 for the reliability $1-\delta = 0.95$ and accuracy $\varepsilon = 0.05$ for several values of the parameter α.

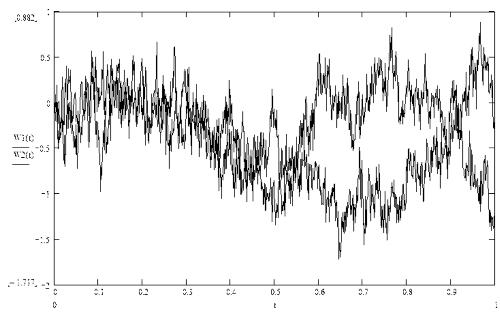

FIGURE 1. $\alpha = 0.4$

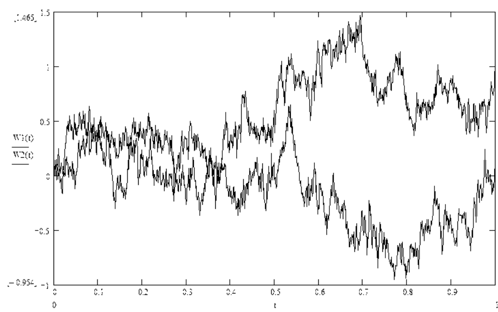

FIGURE 2. $\alpha = 0.6$

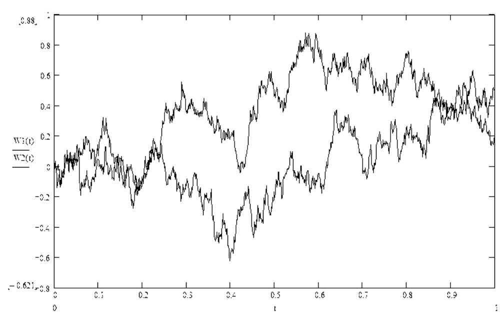

FIGURE 3. $\alpha = 0.8$

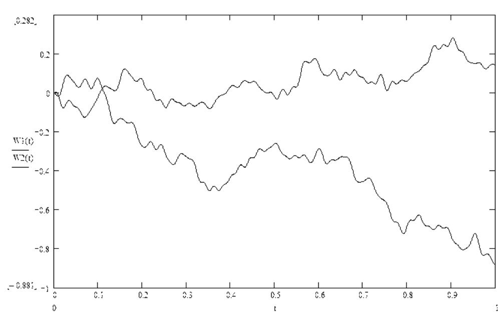

FIGURE 4. $\alpha = 1.2$

As expected, a larger value of α results in a smoother trajectory of a model of the fractional Brownian motion.

The procedure for simulation of a fractional Brownian motion presented in the paper [26] is based on a representation of this process in the form of a random series [5]. This procedure requires enormous preliminary work needed to evaluate the zeros of a Bessel function with a given accuracy. Instead, the procedure presented in the current paper is based on the spectral representation of the fractional Brownian motion and is more effective as far as the running time is concerned.

5. Concluding remarks

A new procedure for simulation of a fractional Brownian motion with a given reliability and accuracy in the space $L_p([0,T])$ is proposed in the paper. The general results obtained for $p \geq 1$ are used for a particular example for simulation of the fractional Brownian motion in the space $L_2([0,1])$.

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