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MULTI-CHANNEL QUEUEING NETWORKS WITH INTERDEPENDENT INPUT FLOWS IN HEAVY TRAFFIC

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Abstract. A service process in a multi-channel stochastic network with interdependent input flows is considered. Such a model is used when analyzing computer or communication networks as well as in medicine and particle physics (high-energy physics). Under the assumption of the critical load, theorems on diffusion approximations are proved. The local characteristics of the diffusion process are expressed in terms of parameters of the network.

1. Introduction

Modern information and telecommunication networks are becoming more and more complex. This trend is explained by the growing requirements concerning the reliability of a network, rate of transmission as well as that of processing the information in the network, and also by wider branching of the networks. Methods of the theory of stochastic processes are an effective tool for studying the networks of transmitting information, computer networks, systems of collective access, etc. These methods allow one to estimate the capability of a network and to find reserve loads as well as to control information flows in an optimal way. Also they allow one to choose the capacity of buffer memory of service nodes in the case of packet switching in the network (see, for example, [1]).

Stochastic networks (known also as queueing networks) are an adequate model for various real-life networks. The structure of the processes in such networks is described by the probability characteristics of input flows, disciplines of service, and commutation schemes for customer packages. The process of servicing the customers in a stochastic network (which is the main object of interest in this paper) can be viewed as a vector of a high dimension determined by a complex system of stochastic relations. Earlier papers (see [2,3] and [4,5]) deal with networks whose input flows are independent. The dependence between components of the network occurs only if the customer trajectories intersect at nodes of a network during the service process. We remove this restriction in the current paper and this leads to an essential complication of the model.

Since our models are complicated, we follow the method of functional limit theorems for studying the multi-dimensional service process. This method allows us to find the main characteristics of the service process, to construct the corresponding approximation process, and to evaluate the distribution of important functionals that estimate the service efficiency.

A model of a network of a parallel structure is considered in Section 2. Each node of a such a system is a multi-channel queueing system. The input traffic in the system

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consists of two types of flows, namely of autonomous flows (each node of the system has its own autonomous input flow) and of a common flow of customers arriving in packages. The size of a package equals the number of nodes in the system. Each customer in a package arrives at every node simultaneously for a parallel service. Input flows of such a type are considered in the papers [6, 7].

We study a multi-dimensional service process in such a system functioning in the overloaded mode. The convergence of the normalized service process to a diffusion process is proved in the uniform topology.

A more complex model of type $[G\,|M|\,\infty]^r$ is considered in Sections 3 and 4 for multi-channel stochastic networks. Neither assumption is imposed on the structure of the input process. The service time at nodes of the network has an exponential distribution. The traffic within the stochastic network is described by the routing matrix.

We prove the convergence in the uniform topology of the normalized service process to a Gaussian process under the assumptions that the functional central limit theorem holds for the multi-dimensional input flow and that the service intensity is low at every node of the network. The limit process is the sum of two independent Gaussian processes and each of them has its own description in terms of the stochastic network. It is also shown that the limit Gaussian process is a multi-dimensional diffusion.

A $[G\,|M|\,\infty]^r$ stochastic network is studied in Section 4 for a quasistationary mode. In addition to the critical load, there exists an initial load in the network and it is asymptotically large. Based on the main limit theorem of Section 3, we prove the convergence of the normalized service process to a multi-dimensional Ornstein–Uhlenbeck process.

2. Multi-channel queueing network with multi-dimensional Poisson input flow

Consider a model consisting of r nodes $E_1, \dots, E_r$. Each node is functioning as an $M/M/\infty$ queueing system. This means that the arrival times of customers at every node E_i are described by Poisson processes $y_i(t)$ with rates $\lambda_i > 0$, $i = 1, 2, \dots, r$ (in other words, the distribution function of interarrival times is exponential). The service of every customer arriving according to the process $y_i(t)$ starts immediately after arrival by one of the service units. Note that every node is equipped with an infinite number of service units. The service time at a node E_i, $i = 1, 2, \dots, r$, has an exponential distribution with parameter $\mu_i > 0$, $i = 1, 2, \dots, r$. Every customer is served at only one node and leaves the system immediately after the service is complete. In addition, there is a common flow of packages in the system described by a Poisson process $y(t)$ with the rate $b > 0$. Each package contains r customers that simultaneously are distributed among r nodes for a parallel service.

Therefore the input process in this system is, in fact, a multi-dimensional Poisson process

$$(\nu_1(t), \dots, \nu_r(t))$$

with parameters $\lambda_i > 0$, $i = 1, 2, \dots, r$, and $b > 0$. According to the procedure described above, every component $\nu_i(t)$ admits the following representation:

$$\nu_i(t) = y_i(t) + y(t).$$

Models of the type described above can be used when scheduling the functioning of modern computer networks and allow one to increase the speed of processing the information and to save a considerable amount of time and other resources.

Denote by $\mathcal{Q}(t) = (\mathcal{Q}_1(t), \dots, \mathcal{Q}_r(t))'$, $t \in [0, T]$, the total number of customers at nodes of the system at a moment t. Our aim is to study the transition mode for the multi-dimensional service process $\mathcal{Q}(t)$ in the overloaded mode. This means that the parameters $\mu_i > 0$, $i = 1, 2, \ldots, r$, of the exponential distribution of the service time depend on the number of series n (that is, $\mu_i = \mu_i(n)$) and that

1) $\lim_{n\to\infty} n\mu_i(n) = \mu_i \neq 0$, $i = 1, 2, \ldots, r$.

Denote by $\mathcal{Q}^{(n)}(nt) = \big(\mathcal{Q}_1^{(n)}(nt), \ldots, \mathcal{Q}_r^{(n)}(nt)\big)'$, $t \in [0, T]$, the total number of customers at nodes of the queueing system in the scale of transformed time nt. We also assume that the system is empty at the initial moment, that is,

2) $\mathcal{Q}_i^{(n)}(0) = 0$, $i = 1, 2, \ldots, r$.

We prove the convergence of the normalized service process

$$\xi^{(n)}(t) = \left(\xi_1^{(n)}(t), \ldots, \xi_r^{(n)}(t)\right)' = n^{-1/2}\left(\mathcal{Q}^{(n)}(nt) - \alpha(t)n\right)$$

to a diffusion process in the uniform topology, where $\alpha(t) = (\alpha_1(t), \ldots, \alpha_r(t))'$ and $\alpha_i(t) = (\lambda_i + b)(1 - e^{-\mu_i t})$, $i = 1, 2, \ldots, r$.

Theorem 1. *Let conditions* 1) *and* 2) *hold. Then the sequence of service processes* $\xi^{(n)}(t) = \big(\xi_1^{(n)}(t), \ldots, \xi_r^{(n)}(t)\big)'$ *weakly converges in the uniform topology on an arbitrary interval* $[0, T]$ *to a diffusion process* $\xi(t)$ $\big(\xi(0) = \xi_0 = (0, \ldots, 0)'\big)$ *with the drift vector* $\alpha(x) = \big(\alpha_1(x), \ldots, \alpha_r(x)\big)' = \big(-\mu_1 x_1, \ldots, -\mu_r x_r\big)'$ *and diffusion matrix*

$$B = \begin{pmatrix} (\lambda_1 + b)(2 - e^{-\mu_1 t}) & b & \cdots & b \\ b & (\lambda_2 + b)(2 - e^{-\mu_2 t}) & b & b \\ \vdots & b & \ddots & b \\ b & \cdots & b & (\lambda_r + b)(2 - e^{-\mu_r t}) \end{pmatrix}.$$

There is a long history of studying queueing systems functioning in an overloaded mode. Several approaches are known and each of them is directed toward a specific class of systems. We prove Theorem 1 by following the local approach, which is the most useful in the case of models under consideration in this paper. Some more detail concerning this approach can be found in [8].

Proof. According to the local approach, the finite-dimensional distributions of the process $\xi^{(n)}(t)$ converge to those of a diffusion process $\xi(t)$ with a drift vector $\alpha(x)$ and diffusion matrix B if

$$\xi^{(n)}(0) \overset{d}{\Rightarrow} \xi_0, \qquad n \to +\infty,$$

$$\delta_1(n) = \sum_{k=0}^{d_n - 1} \mathsf{E}\left|\mathsf{E}\left\{\Delta\xi_{n_k} \mid F_{n_k}\right\} - \alpha\big(\xi_{n_k}\big)\Delta t_{n_k}\right| \to 0, \qquad n \to +\infty,$$

$$\delta_2(n) = \sum_{k=0}^{d_n - 1} \mathsf{E}\left|\mathsf{E}\left\{\big(\Delta\xi_{n_k}, z\big)^2 \;\middle|\; F_{n_k}\right\} - \big(Bz, z\big)\Delta t_{n_k}\right| \to 0, \qquad n \to +\infty, \qquad z \in \mathbb{R}^r,$$

$$\delta_3(n) = \sum_{k=0}^{d_n - 1} \mathsf{P}\left\{\left|\Delta\xi_{n_k}\right| \geq \varepsilon\right\} \to 0, \qquad n \to +\infty, \qquad \text{for all } \varepsilon > 0 \text{ and } z \in \mathbb{R}^r.$$

Here the symbol $\overset{d}{\Rightarrow}$ stands for the weak convergence of random vectors, $\Delta\xi_{n_k} = \xi_{n_{k+1}} - \xi_{n_k}$, $\xi_{n_k} = \xi^{(n)}(t_{n_k})$,

$$\{t_{n_k}, k = 0, 1, \ldots, d_n\}$$

is a sequence of partitions of the interval $[0, T]$ by points

$$0 = t_{n_0} < t_{n_1} < \cdots < t_{n_{d_n}} = T$$

such that $\max \Delta t_{n_k} \to 0$, $F_{n_k} = F(t_{n_k})$, and $\{F_n(t), t \in [0,T]\}$ is the sequence of families of the increasing σ-algebras (that is, $F_n(t_1) \subseteq F_n(t_2)$ if $t_1 < t_2$) generated by the processes $\xi^{(n)}(u)$, $u \leqslant t$.

For simplicity, we prove the convergence on the unit interval

$$[0,T] = [0,1].$$

Put $t_{n_k} = k/d_n$, $n = 1,2,\dots$, $k = 1,\dots,d_n$, where $d_n \to \infty$, $n \to \infty$, is a sequence of positive integer numbers such that $\lim_{n\to\infty} d_n/n = 0$. Let $F_n(t)$ be the σ-algebra generated by the family of random variables $\left\{\xi^{(n)}(\tau), \tau \leqslant t\right\}$. Then the above conditions for the process

$$\mathcal{Q}^{(n)}(nt) = \left(\mathcal{Q}_1^{(n)}(nt), \dots, \mathcal{Q}_r^{(n)}(nt)\right)'$$

can be rewritten as follows:

$$\xi_n(0) \overset{d}{\Rightarrow} \xi_0, \qquad n \to \infty, \tag{1}$$

(2)
$$\sum_{k=0}^{d_n-1} \mathsf{E}\left|\mathsf{E}\left\{\Delta\xi_i^{(n)}(t_{n_k}) \;\middle|\; F_{n_k}\right\} + \mu_i \xi_i^{(n)}(t_{n_k})\Delta t_{n_k}\right| \to 0, \qquad n \to +\infty, \qquad i = 1,2,\dots,r,$$

$$\sum_{k=0}^{d_n-1} \mathsf{E}\left|\mathsf{E}\left\{\left(\Delta\xi_i^{(n)}(t_{n_k})\right)^2 \;\middle|\; F_{n_k}\right\} - (\lambda_i + b)(2 - e^{-\mu_i t})\Delta t_{n_k}\right| \to 0, \tag{3}$$
$$n \to +\infty, \qquad i = 1,2,\dots,r,$$

$$\sum_{k=0}^{d_n-1} \mathsf{E}\left|\mathsf{E}\left\{\left(\Delta\xi_i^{(n)}(t_{n_k})\Delta\xi_j^{(n)}(t_{n_k})\right) \;\middle|\; F_{n_k}\right\} - b\Delta t_{n_k}\right| \to 0, \qquad n \to +\infty, \qquad i \neq j, \tag{4}$$

$$\sum_{k=1}^{d_n-1} \mathsf{P}\left\{\left|\Delta\xi_{n_k}\right| \geq \varepsilon\right\} \to 0,\ n \to +\infty, \quad \text{for all } \varepsilon > 0. \tag{5}$$

Since $\left(\xi_1^{(n)}(0), \dots, \xi_r^{(n)}(0)\right) = (0,\dots,0)$, condition (1) holds for

$$\left(\xi_1^{(0)}(0), \dots, \xi_r^{(0)}(0)\right) = (0,\dots,0).$$

Conditions (2)–(4) mean that the process $\xi^{(n)}(t)$ converges to a diffusion process if the characteristics of the prelimit process are asymptotically close to the corresponding characteristics of the diffusion process.

In order to check conditions (2)–(4) one needs to evaluate the conditional expectations

$$\mathsf{E}\left\{\Delta\xi_i^{(n)}(t_{n_k}) \;\middle|\; F_{n_k}\right\},$$
$$\mathsf{E}\left\{\left(\Delta\xi_i^{(n)}(t_{n_k})\right)^2 \;\middle|\; F_{n_k}\right\},$$
$$\mathsf{E}\left\{\left(\Delta\xi_i^{(n)}(t_{n_k})\Delta\xi_j^{(n)}(t_{n_k})\right) \;\middle|\; F_{n_k}\right\},$$
$$i \neq j, \qquad i,j = 1,2,\dots,r.$$

These conditional expectations can be evaluated for processes of the specific form we are dealing with if we are able to calculate the following conditional expectations for the process $\mathcal{Q}(t)$:

$$\mathsf{E}\left\{\mathcal{Q}_i(t) \mid \mathcal{Q}_i(0) = m_i\right\}, \qquad \mathsf{E}\left\{\mathcal{Q}_i(t)^2 \mid \mathcal{Q}_i(0) = m_i\right\}, \qquad i = 1,2,\dots,r,$$
$$\mathsf{E}\left\{\mathcal{Q}_i(t)\mathcal{Q}_j(t) \mid \mathcal{Q}_i(0) = m_i, \mathcal{Q}_j(0) = m_j\right\}, \qquad i \neq j, \qquad i,j = 1,2,\dots,r.$$

First of all we obtain the explicit form of the following conditional generating function:

$$\varphi(z_i, z_j, t) = M\left\{z_i^{\mathcal{Q}_i(t)} z_j^{\mathcal{Q}_j(t)} \;\middle|\; \mathcal{Q}_i(0) = m_i, \mathcal{Q}_j(0) = m_j\right\}, \qquad i \neq j,\ i,j = 1,2,\dots,r.$$

To solve this problem we use the explicit representation of the components of the vector $(\mathcal{Q}_i(t), \mathcal{Q}_j(t))$ in the form of a sum of random indicators along trajectories of the input Poisson flows. Every indicator describes the service process of a separate customer arriving at the system.

Introduce three families of independent two-dimensional random vectors, $\{\chi_k^i(t)\}_{k=1}^{\infty}$, $\{\chi_k^j(t)\}_{k=1}^{\infty}$, and $\{\chi_k(t)\}_{k=1}^{\infty}$, whose distributions do not depend on the index k. The random vector $\chi_k^i(t)$ assumes the value $(1,0)$ with probability $e^{-\mu_i t}$ or the value $(0,0)$ with probability $1 - e^{-\mu_i t}$. The random vector $\chi_k^j(t)$ assumes the value $(0,1)$ with probability $e^{-\mu_j t}$ or the value $(0,0)$ with probability $1 - e^{-\mu_j t}$.

Finally, the random vector $\chi_k(t)$ assumes four values, namely $(1,1)$ with probability $e^{-(\mu_i+\mu_j)}t$, $(1,0)$ with probability $e^{-\mu_i t}\big(1 - e^{\mu_j t}\big)$, $(0,1)$ with probability

$$e^{-\mu_j t}\big(1 - e^{\mu_i t}\big),$$

or $(0,0)$ with probability $\big(1 - e^{\mu_i t}\big)\big(1 - e^{\mu_j t}\big)$.

Then the vector constituted by the ith and jth components of the service process admits the representation

$$\big(\mathcal{Q}_i(t), \mathcal{Q}_j(t)\big) \stackrel{d}{=} \sum_{k=1}^{m_i} \chi_k^i(t) + \sum_{k=1}^{m_j} \chi_k^j(t) + \sum_{k=1}^{y_i(t)} \chi_k^i\big(t - t_k^i\big) + \sum_{k=1}^{y_j(t)} \chi_k^j\big(t - t_k^j\big) + \sum_{k=1}^{y(t)} \chi_k\big(t - t_k\big),$$

where the symbol $\stackrel{d}{=}$ means the equality of conditional distributions for fixed

$$m_i, \qquad m_j, \qquad t_k^i, \qquad t_k^j, \qquad t_k, \qquad k = 1,2,\dots,\ i \neq j,\ i,j = 1,2,\dots,r.$$

The symbols above denote the sequential arrival times of customers according to the independent Poisson flows $y_i(t)$, $y_j(t)$, $y(t)$, $i \neq j$, $i,j = 1,2,\dots,r$, correspondingly.

Using the well-known properties of conditional expectations we find

$$\begin{aligned}
\varphi\big(z_i, z_j, t\big) = {} & \big(1 - e^{-\mu_i t} + z_i e^{-\mu_i t}\big)^{m_i} \big(1 - e^{-\mu_j t} + z_j e^{-\mu_j t}\big)^{m_j} \\
& \times \mathsf{E}\Bigg\{ \prod_{k=1}^{y_i(t)} \Big(\big(1 - e^{-\mu_i (t - t_k^i)}\big) + z_i e^{-\mu_i (t - t_k^i)}\Big) \\
& \times \prod_{k=1}^{y_j(t)} \Big(\big(1 - e^{-\mu_j (t - t_k^j)}\big) + z_j e^{-\mu_j (t - t_k^j)}\Big) \\
& \times \prod_{k=1}^{y(t)} \Big\{\big(1 - e^{-\mu_i (t - t_k)}\big)\big(1 - e^{-\mu_j (t - t_k)}\big) + z_i e^{-\mu_i (t - t_k)}\big(1 - e^{-\mu_j (t - t_k)}\big) \\
& + z_j e^{-\mu_j (t - t_k)}\big(1 - e^{-\mu_i (t - t_k)}\big) + z_i z_j e^{-(\mu_i + \mu_j)(t - t_k)}\Big\}\Bigg\}.
\end{aligned}$$

The latter expression allows one to obtain the explicit form of the conditional generating functions. To derive this explicit form, we rewrite $\varphi\big(z_i, z_j, t\big)$ as the conditional expectation given $y_i(t)$, $y_j(t)$, and $y(t)$:

$$\begin{aligned}\varphi(z_i, z_j, t) = \big(1 - e^{-\mu_i t} + z_i e^{-\mu_i t}\big)^{m_i} \big(1 - e^{-\mu_j t} + z_j e^{-\mu_j t}\big)^{m_j} \\ \times \mathsf{E}\Bigg\{\mathsf{E}\Bigg\{\prod_{k=1}^{y_i(t)} \Big(\big(1 - e^{-\mu_i t}\big) + z_i e^{-\mu_i (t - t_k^i)}\Big) \\ \times \prod_{k=1}^{y_j(t)} \Big(\Big(1 - e^{-\mu_j (t - t_k^j)}\Big) + z_j e^{-\mu_j (t - t_k^j)}\Big) \\ \times \prod_{k=1}^{y(t)} \Big\{\Big(1 - e^{-\mu_i (t - t_k)}\Big)\Big(1 - e^{-\mu_j (t - t_k)}\Big) \\ + z_i e^{-\mu_i (t - t_k)}\Big(1 - e^{-\mu_j (t - t_k)}\Big) \\ + z_j e^{-\mu_j (t - t_k)}\Big(1 - e^{-\mu_i (t - t_k)}\Big) \\ + z_i z_j e^{-(\mu_i + \mu_j)(t - t_k)}\Big\} \,\Bigg|\, y_i(t),\, y_j(t),\, y(t)\Bigg\}\Bigg\}.\end{aligned}$$

It is well known from the theory of random flows of customers that the moments of jumps given the number of jumps in the Poisson flow have the same distribution as order statistics constructed from the uniform distribution in the interval $[0, t]$. This property of the Poisson flow yields the explicit form of the generating function of the random vector $\big(\mathcal{Q}_i(t),\ \mathcal{Q}_j(t)\big)$:

$$\begin{aligned}\varphi(z_i, z_j, t) = \big(1 - e^{-\mu_i t} + z_i e^{-\mu_i t}\big)^{m_i} \big(1 - e^{-\mu_j t} + z_j e^{-\mu_j t}\big)^{m_j} \\ \times \mathsf{E}\left\{A_1(z_i, t)\right\}^{y_i(t)} \mathsf{E}\left\{A_2(z_j, t)\right\}^{y_j(t)} \mathsf{E}\left\{A_3(z_i, z_j, t)\right\}^{y(t)}.\end{aligned}$$

The generating functions $A_1(z_i, t)$, $A_2(z_j, t)$, and $A_3(z_i, z_j, t)$ admit the following representations:

$$\begin{aligned}A_1(z_i, t) &= \frac{1}{t}\int_0^t \big(1 - e^{-\mu_i (t-u)} + z_i e^{-\mu_i (t-u)}\big)\,du, \\ A_2(z_j, t) &= \frac{1}{t}\int_0^t \big(1 - e^{-\mu_j (t-u)} + z_j e^{-\mu_j (t-u)}\big)\,du, \\ A_3(z_i, z_j, t) &= \frac{1}{t}\int_0^t \Big(1 - \Big(e^{-\mu_i (t-u)} - e^{-(\mu_i+\mu_j)(t-u)}\Big)(1 - z_i) \\ &\quad - \Big(e^{-\mu_j (t-u)} - e^{-(\mu_i+\mu_j)(t-u)}\Big)(1 - z_j) \\ &\quad - e^{-(\mu_i+\mu_j)(t-u)}(1 - z_i z_j)\Big)\,du.\end{aligned}$$

Note that the factor $1/t$ appears in the above representations of the generating functions $A_1(z_i, t)$, $A_2(z_j, t)$, and $A_3(z_i, z_j, t)$, since it is equal to the density of the uniform distribution in the interval $[0, t]$.

Finally, the generating functions $A_1(z_i,t)$, $A_2(z_j,t)$, and $A_3(z_i,z_j,t)$ can be written as

$$A_1(z_i,t) = 1 - \frac{1}{\mu_i t}\big(1 - e^{-\mu_i t}\big)(1 - z_i),$$
$$A_2(z_j,t) = 1 - \frac{1}{\mu_j t}\big(1 - e^{-\mu_j t}\big)(1 - z_j),$$
$$\begin{aligned} A_3(z_i,z_j,t) = 1 &- \left(\frac{1}{\mu_i t}\big(1 - e^{-\mu_i t}\big) - \frac{1}{\big(\mu_i+\mu_j\big)t}\big(1 - e^{-(\mu_i+\mu_j)t}\big)\right)(1 - z_i)\\ &- \left(\frac{1}{\mu_j t}\big(1 - e^{-\mu_j t}\big) - \frac{1}{\big(\mu_i+\mu_j\big)t}\big(1 - e^{-(\mu_i+\mu_j)t}\big)\right)(1 - z_j)\\ &- \left(\frac{1}{\big(\mu_i+\mu_j\big)t}\big(1 - e^{-(\mu_i+\mu_j)t}\big)\right)\big(1 - z_i z_j\big). \end{aligned}$$

Substituting these expressions into $\varphi\big(z_i,z_j,t\big)$ we obtain the explicit form of the generating function,

$$\begin{aligned} \varphi\big(z_i,z_j,t\big) = \big(1 - e^{-\mu_i t} + z_i e^{-\mu_i t}\big)^{m_i}\big(1 - e^{-\mu_j t} + z_j e^{-\mu_j t}\big)^{m_j}&\\ \times \exp\Bigg\{ -\Big(\frac{\lambda_i + b}{\mu_i}\big(1 - e^{-\mu_i t}\big) - \frac{b}{\mu_i+\mu_j}\big(1 - e^{-(\mu_i+\mu_j)t}\big)\Big)(1 - z_i)&\\ - \left(\frac{\lambda_j + b}{\mu_j}\big(1 - e^{-\mu_j t}\big) - \frac{b}{\mu_i+\mu_j}\big(1 - e^{-(\mu_i+\mu_j)t}\big)\right)(1 - z_j)&\\ - \frac{b}{\mu_i+\mu_j}\Big(1 - e^{-(\mu_i+\mu_j)t}\Big)\big(1 - z_i z_j\big)\Bigg\},&\\ i \neq j, \qquad i,j = 1,2,\ldots,r.& \end{aligned}$$

Differentiating the conditional generating functions we evaluate the conditional expectations,

$$\mathsf{E}\left\{\mathcal{Q}_i(t) \mid \mathcal{Q}_i(0) = m_i\right\} = m_i e^{-\mu_i t} + \frac{\lambda_i + b}{\mu_i}\big(1 - e^{-\mu_i t}\big), \qquad i = 1,2,\ldots,r, \tag{6}$$

$$\begin{aligned} &\mathsf{E}\left\{\mathcal{Q}_i(t)^2 \mid \mathcal{Q}_i(0) = m_i\right\}\\ &\quad = m_i\big(m_i - 1\big)e^{-2\mu_i t} + 2m_i e^{-\mu_i t}\frac{\lambda_i + b}{\mu_i}\big(1 - e^{-\mu_i t}\big)\\ &\qquad + \frac{\big(\lambda_i + b\big)^2}{\mu_i^2}\big(1 - e^{-\mu_i t}\big)^2 + m_i e^{-\mu_i t} + \frac{\lambda_i + b}{\mu_i}\big(1 - e^{-\mu_i t}\big),\\ &\qquad\qquad i = 1,2,\ldots,r, \end{aligned} \tag{7}$$

$$\begin{aligned} &\mathsf{E}\left\{\mathcal{Q}_i(t)\mathcal{Q}_j(t) \mid \mathcal{Q}_i(0) = m_i, \mathcal{Q}_j(0) = m_j\right\}\\ &\quad = m_j e^{-\mu_j t}\left\{m_i e^{-\mu_i t} + \frac{\lambda_i + b}{\mu_i}\big(1 - e^{-\mu_i t}\big)\right\}\\ &\qquad + m_i e^{-\mu_i t}\frac{\lambda_j + b}{\mu_j}\big(1 - e^{-\mu_j t}\big) + \frac{b}{\mu_i+\mu_j}\big(1 - e^{-(\mu_i+\mu_j)t}\big)\\ &\qquad + \frac{(\lambda_i + b)(\lambda_j + b)}{\mu_i\mu_j}\big(1 - e^{-\mu_i t}\big)\big(1 - e^{-\mu_j t}\big), \qquad i \neq j,\ i,j = 1,2,\ldots,r. \end{aligned} \tag{8}$$

Now we are in position to check conditions (2)–(4) for the sequence of stochastic processes $\xi^{(n)}(t)$. First we use expression (6) to evaluate the following conditional expectation $\mathsf{E}\{\xi^i_{n_{k+1}} - x \mid \xi^i_{n_k} = x\}$:

$$
\begin{aligned}
&\mathsf{E}\left\{\xi^i_{n_{k+1}} - x \,\middle|\, \xi^i_{n_k} = x\right\} \\
&\quad = \mathsf{E}\left\{\xi^i_{n_{k+1}} - x \mid \mathcal{Q}^n_i(nt_{nk}) = \sqrt{n}x + \frac{\lambda_i + b}{\mu_i}\big(1 - e^{-\mu_i t_{n_k}}\big)\right\} \\
&\quad = n^{-1/2}\left\{\left(\sqrt{n}x + \frac{\lambda_i + b}{\mu_i}\big(1 - e^{-\mu_i t_{n_k}}\big)\right) e^{-\mu_i \Delta t_{n_k}} + \frac{\lambda_i + b}{\mu_i} n\big(1 - e^{-\mu_i \Delta t_{n_k}}\big)\right\} \\
&\qquad - x \\
&\quad = -x\big(1 - e^{-\mu_i \Delta t_{n_k}}\big),
\end{aligned} \tag{9}
$$

where $\xi^i_{n_k} = \xi^{(n)}_i(t_{n_k})$ and $\xi^i_{n_{k+1}} = \xi^{(n)}_i(t_{n_{k+1}})$.

Then equality (7) implies the uniform boundedness of the second moment of the process $\xi^{(n)}(t)$,

$$
\sup_{t\in[0,T]} \mathsf{E}\left|\xi^{(n)}(t)\right|^2 < L_0, \tag{10}
$$

since the exponential function is uniformly bounded in the interval $[0,T]$ and since the second moment of the number of customers at nodes of the network is finite (recall that it has the Poisson distribution).

Using (9) and (10) we conclude that

$$
\begin{aligned}
&\sum_{k=0}^{d_n-1} \mathsf{E}\left|-\xi^i_{n_k}\big(1 - e^{-\mu_i \Delta t_{n_k}}\big) + \mu_i \xi^i_{n_k} \Delta t_{n_k}\right| \\
&\quad = \sum_{k=0}^{d_n-1} \mathsf{E}\left|\xi^i_{n_k}\right| \left(e^{-\mu_i \Delta t_{n_k}} - 1 + \mu_i \Delta t_{n_k}\right) \leqslant \sum_{k=0}^{d_n-1} \sqrt{\mathsf{E}\big(\xi^i_{n_k}\big)^2}\left(e^{-\mu_i \Delta t_{n_k}} - 1 + \mu_i \Delta t_{n_k}\right) \\
&\quad \leqslant \sqrt{L_0} \sum_{k=0}^{d_n-1} \left(e^{-\mu_i \Delta t_{n_k}} - 1 + \mu_i \Delta t_{n_k}\right) = \sqrt{L_0}\, d_n \left(e^{-\mu_i \frac{1}{d_n}} - 1 + \mu_i \frac{1}{d_n}\right) \xrightarrow[n\to\infty]{} 0.
\end{aligned}
$$

The latter relation proves condition (2) for the process $\xi^{(n)}(t)$. Using the explicit expressions (7) and (8) for the second moments one can similarly check conditions (3) and (4).

The convergence of the process $\xi^{(n)}(t)$ in the uniform topology in the case of a network is similar to that in the case of a single system $M/M/\infty$ (see [12, p. 158]). This indeed is the case, since

$$
\mathsf{P}\left\{\left|\xi^{(n)}(t) - \xi^{(n)}(t_1)\right| \geqslant \varepsilon,\ \left|\xi^{(n)}(t) - \xi^{(n)}(t_2)\right| \geqslant \varepsilon\right\} \leqslant \frac{1}{\varepsilon^{2r}}\big(H(t_2) - H(t_1)\big)^{2\alpha} \tag{11}
$$

for $t_1 \leqslant t \leqslant t_2$, $t_1, t_2, t \in [0,T]$, $n \geqslant 1$, $\gamma = 2$, $\alpha = \frac{3}{4}$, $\varepsilon > 0$, and $H(t) = ct$. Theorem 1 is proved. □

Theorem 1 allows one to use functionals of a diffusion process when evaluating functionals of a compound jump service process. This is the case, for example, in the problem of evaluating the total profit of the service of customers in a stochastic network.

3. Stochastic networks of the type $[G\,|M|\,\infty]^r$. Transient mode

More complex models of multi-channel networks are considered in this section. We assume that the network consists of r service nodes. Customers arrive at an ith node at times $\tau_k^{(i)}$, $k = 1, 2, \ldots$, and let $\nu_i(t)$ be the total number of customers arriving at the system during the interval $[0, t]$. Each of the r nodes is a multi-channel queueing system and each of them begins servicing a customer immediately after its arrival. The service time for a node i has an exponential distribution with parameter μ_i, $i = 1, 2, \ldots, r$. The traffic inside the network is described by the routing matrix $P = \|p_{ij}\|_1^r$. For every $i = 1, 2, \ldots, r$, the number $p_{ir+1} = 1 - \sum_{j=1}^r p_{ij}$ means the probability of exit from the network for a customer whose service is completed by the node i.

We view the service process in a $[G|M|\infty]^r$ network as an r-dimensional process,

$$Q'(t) = (Q_1(t), \ldots, Q_r(t)), \qquad t \ge 0,$$

where $Q_i(t)$ denotes the total number of customers in the node i at time t.

Our main aim is to study the service process $Q(t)$ in an overloaded mode. The overloaded mode means that the parameters of the network depend on n (the number of a series) in such a way that conditions 1) and 2) of Section 2 hold and the input flow is close to a Brownian motion. More precisely,

3) there exist constants $\lambda_i \ge 0$, $i = 1, 2, \ldots, r$, such that $\lambda_1 + \cdots + \lambda_r \ne 0$ and

$$n^{-1/2}\left(\nu_1^{(n)}(nt) - \lambda_1 nt, \ldots, \nu_r^{(n)}(nt) - \lambda_r nt\right) \underset{n\to\infty}{\overset{U}{\Longrightarrow}} W(t)' = \big(W_1(t), \ldots, W_r(t)\big),$$

where $W(t)$ is an r-dimensional Brownian motion with zero mean, $\mathsf{E}\,W(1) = 0$, and correlation matrix

$$\mathsf{E}\,W(1)W'(1) = \sigma^2 = \|\sigma_{ij}^2\|_1^r.$$

Here the symbol $\overset{U}{\Longrightarrow}$ stands for the weak convergence in the uniform topology.

Note that all other parameters of the $[G|M|\infty]^r$ network do not depend on n.

We consider the sequence of stochastic processes

$$\xi^{(n)}(t) = n^{-1/2}\big(Q^{(n)}(nt) - nq(t)\big), \qquad t \ge 0,$$

for an open $[G|M|\infty]^r$ network, where $q'(t) = (q_1(t), \ldots, q_r(t)) = (\theta/\mu)'(I - P(t))$ and $\big(\theta/\mu\big)' = \big(\theta_1/\mu_1, \ldots, \theta_r/\mu_r\big)$. Here

$$\theta' = \big(\theta_1, \ldots, \theta_r\big) = \lambda'\big(I - P\big)^{-1}$$

is a solution of the balance equation for the $[G|M|\infty]^r$ network, $\lambda' = \big(\lambda_1, \ldots, \lambda_r\big)$,

$$P(t) = \|p_{ij}(t)\|_1^r = \exp\left[\Delta(\mu)(P - I)t\right],$$

and $\Delta(\mu) = \|\delta_{ij}\mu_i\|_1^r$ is a diagonal matrix.

We introduce two independent Gaussian stochastic processes

$$\xi^{(1)\prime}(t) = \left(\xi_1^{(1)}(t), \ldots, \xi_r^{(1)}(t)\right) \quad \text{and} \quad \xi^{(2)\prime}(t) = \left(\xi_1^{(2)}(t), \ldots, \xi_r^{(2)}(t)\right)$$

to construct the limit process for the sequence $\xi^{(n)}(t)$.

The process $\xi^{(1)}(t)$ is determined by its mean values

$$\mathsf{E}\,\xi^{(1)}(t) = 0$$

and correlation matrix

$$R^{(1)}(t) = \mathsf{E}\,\xi^{(1)}(t)\xi^{(1)\prime}(t) - \mathsf{E}\,\xi^{(1)}(t)\,\mathsf{E}\,\xi^{(1)\prime}(t) = \int_0^1 P'(u)\sigma^2 P(u)\,du,$$

$$P^{(1)}(s,t) = \mathsf{E}\,\xi^{(1)}(s)\xi^{(1)}(t) - \mathsf{E}\,\xi^{(1)}(s)\,\mathsf{E}\,\xi^{(1)\prime}(t) = R^{(1)}(s)P(t-s), \qquad s > t.$$

The process $\xi^{(2)}(t)$ is such that

$$\mathsf{E}\,\xi^{(2)}(t) = 0,$$

$$R^{(2)}(t) = \sum_{m=1}^{r} \lambda_m \int_0^1 \big(\Delta[p_m(u)] - p_m(u)p'_m(u)\big)\,du,$$

$$R^{(2)}(s,t) = R^{(2)}(s)P(t-s), \qquad s < t,$$

where $p'_m(u) = \big(p_{m1}(u), \dots, p_{mr}(u)\big)$ is the mth row of the matrix $P(u)$, while

$$\Delta[p_m(u)] = \|p_{mi}(u)\delta_{ij}\|_1^r$$

is a diagonal matrix.

The process $\xi^{(1)}(t) + \xi^{(2)}(t)$ describes the asymptotic behavior of the sequence of stochastic processes $\xi^{(n)}(t)$.

Theorem 2. *Let conditions* 1)–3) *of Section* 2 *hold for a stochastic network of the type* $[G|M|\infty]^r$. *Assume that the spectral radius of the routing matrix* P *is strictly less than* 1. *Then the sequence of stochastic processes* $\xi^{(n)}(t)$ *weakly converges as* $n \to \infty$ *in the uniform topology to* $\xi^{(1)}(t) + \xi^{(2)}(t)$ *on every finite interval* $[0, T]$.

The convergence of finite-dimensional distributions is a corollary of the following two auxiliary results.

Lemma 1. *The finite-dimensional distributions of* $\int_0^1 dW'(u)P(t-u)$ *coincide with those of the Gaussian process* $\xi^{(1)}(t)$.

Lemma 1 follows directly from properties of stochastic integrals (see, for example, [9] concerning the appropriate properties of a stochastic integral).

Note that the trajectory of a customer arriving at a $[G|M|\infty]^r$ network through a node m can be described by the Markov chain $\eta^{(m)}(t) \in \{1, 2, \dots, r, r+1\}$, $t \ge 0$, with the infinitesimal matrix $\|q_{ij}\|_1^{r+1}$,

$$q_{ij} = \begin{cases} -\mu_i(1 - p_{ii}), & i = j = 1, 2, \dots, r, \\ \mu_i p_{ij}, & i \ne j,\ i = 1, 2, \dots, r,\ j = 1, 2, \dots, r, r+1, \\ 0, & i = r+1, j = 1, 2, \dots, r, r+1. \end{cases}$$

The initial distribution of the Markov chain is given by

$$\mathsf{P}\big(\eta^{(m)}(0) = i\big) = \delta_{mi}, \qquad i = 1, 2, \dots, r+1.$$

We associate an r-dimensional indicator process

$$\chi^{(m)}(t) = \Big(\chi_1^{(m)}(t), \dots, \chi_r^{(m)}(t)\Big)', \qquad t \ge 0,\ m = 1, \dots, r,$$

to the Markov chain $\eta^{(m)}(t)$ as follows:

$$\chi^{(m)}(t) = \begin{cases} e_j, & \eta^{(m)}(t) = j,\ j = 1, \dots, r, \\ e_0, & \eta^{(m)}(t) = r+1, \end{cases}$$

where e_j is an r-dimensional vector whose jth component is equal to 1, while all others are equal to 0; e_0 is the zero r-dimensional vector.

Given a positive natural number N and

$$z(j) = \big(z_1(j), \dots, z_r(j)\big)', \qquad j = 1, 2, \dots, N, \qquad \big|z(j)\big| \le 1,$$

the joint generating function of the random vectors $\chi^{(m)}(t_1), \dots, \chi^{(m)}(t_N)$, $0 < t_1 < \dots < t_N$, is denoted by $\Phi = \big(\Phi^{(1)}, \dots, \Phi^{(r)}\big)'$ and let

$$\Phi^{(m)} = \Phi^{(m)}\big(t_1, \dots, t_N, z(1), \dots, z(N)\big).$$

Lemma 2. *Given $N = 1, 2, \dots$ and $0 < t_1 < \dots < t_N$,*

$$\Phi = \overline{1} + \sum_{j=1}^{N} P(\Delta t_1)\Delta\,[z(1)] \cdots P(\Delta t_{j-1})\Delta\,[z(j-1)]\, P(\Delta t_j)(z(j) - \overline{1}), \tag{12}$$

where $\overline{1}$ is the r-dimensional vector whose components are equal to 1, $\Delta t_i = t_i - t_{i-1}$ ($t_0 = 0$), and

$$\Delta[z(i)] = \|z_k(i)\delta_{km}\|_1^r$$

is a diagonal matrix.

The proof of equality (12) uses the induction with respect to the parameter N.

In addition to the convergence of finite-dimensional distributions, one can prove that

$$\lim_{\Delta\to\infty} \overline{\lim_{n\to\infty}}\, \mathsf{P}\big(\omega_\Delta(\xi^{(n)}) > \delta\big) = 0 \tag{13}$$

for all $\delta > 0$, where

$$\omega_\Delta(x) = \sup_{\substack{|t-u|\le\Delta \\ 0\le t,u\le T}} \big|x(t) - x(u)\big|.$$

The proof of (13) uses condition 3) and is based on the representation of the service process as the sum of indicators $\chi^m(\cdot)$ defined on the trajectory of the input flow.

Also one can check that the limit Gaussian process in Theorem 2 is a diffusion.

The Markov property for Gaussian processes in the one-dimensional case can be derived from the corresponding criteria (see [10, p. 115]). The principal condition of the criteria is given in terms of covariances and is close to the characteristic property of the exponential function. This property is rather easy to check. The case of a higher dimension is more complicated, since there is no such criteria in this case. However sufficient conditions are obtained in [11] for Gaussian processes to possess the Markov property in the multi-dimensional case. Using these conditions we obtain the following result.

Corollary 1. *If the spectral radius of the routing matrix P is strictly less than 1, then the limit Gaussian process $\xi^{(1)}(t) + \xi^{(2)}(t)$ is an r-dimensional diffusion with the drift vector $A(x) = Q'x$ and diffusion matrix*

$$B(t) = \Delta\,[q'(t)Q] - Q'\Delta\,[q(t)] - \Delta\,[q(t)]\,Q + \sigma^2,$$

where $Q = \Delta(\mu)(P - I)$ and $\Delta(x)$ is the diagonal matrix whose principal diagonal is constituted by the vector x.

Now the diffusion approximation can be obtained from Theorem 2. Note however that it contains more information concerning the structure of the limit process. The first term $\xi^{(1)}(t)$ is related to the fluctuations of the input flow, while the second one, $\xi^{(2)}(t)$, to those of the service time at nodes of the network.

4. $[G|M|\infty]^r$ networks in a quasistationary mode

In this section, we use condition

4) $Q_i^{(n)}(0) = \left[n\theta_i/\mu_i + \sqrt{n}\eta_i^0\right]$, $i = 1, 2, \ldots, r$, $\eta^0 = (\eta_1^0, \ldots, \eta_r^0) \in \mathbb{R}^r$

instead of 2) and study the sequence of stochastic processes

$$\eta^{(n)}(t) = n^{-1/2}\big(Q^{(n)}(nt) - n\theta/\mu\big), \qquad n \ge 1.$$

If condition 4) holds, then the process $\eta^{(n)}(t)$ is functioning in a quasistationary mode and this changes some properties of the limit process.

To approximate $\eta^{(n)}(t)$, we consider one more Gaussian stochastic process $\xi^{(3)}(t)$ which is independent of $\xi^{(1)}(t)$ and $\xi^{(2)}(t)$ and is determined by the vector of mean values

$$\mathsf{E}\,\xi^{(3)}(t) = P'(t)\eta^0$$

and correlation matrix

$$R^{(3)}(t) = \Delta\left[(\theta/\mu)'P(t)\right] - P'(t)\Delta(\theta/\mu)P(t),$$
$$R^{(3)}(s,t) = R^{(3)}(s)P(t-s), \qquad s < t.$$

The quasistationary mode is approximated by a sum of Gaussian processes in the following way.

Theorem 3. *Assume that conditions* 1), 3), *and* 4) *hold for a* $[G|M|\infty]^r$ *stochastic network. If the spectral radius of the routing matrix* P *is strictly less than* 1, *then the sequence of stochastic processes* $\eta^{(n)}(t)$, $n \ge 1$, *weakly converges as* $n \to \infty$ *to* $\xi^{(1)}(t) + \xi^{(2)}(t) + \xi^{(3)}(t)$ *in the uniform topology on an arbitrary finite interval* $[0, T]$.

The extra term $\xi^{(3)}(t)$ in the limit process is explained by fluctuations of the service time of the customers that are in the nodes of the network at the initial time.

The limit process $\xi^{(1)}(t) + \xi^{(2)}(t) + \xi^{(3)}(t)$ can be represented as a diffusion.

Corollary 2. *If the spectral radius of the routing matrix* P *is strictly less than* 1, *then the limit Gaussian process* $\xi^{(1)}(t) + \xi^{(2)}(t) + \xi^{(3)}(t)$ *is an* r*-dimensional Ornstein–Uhlenbeck diffusion process* $\eta(t)$, $\eta(0) = \eta^0$, *with the drift vector* $A(x) = Q'x$ *and diffusion matrix* $B = \Delta(\theta)(I - P) + (I - P')\Delta(\theta) - \Delta(\lambda) + \sigma^2$.

Note that Theorems 2 and 3 generalize the results on the diffusion approximation for multi-channel networks obtained in [12], since there is no restriction imposed on the input flow in Theorems 2 and 3.

5. Concluding remarks

Theorem 3 and Corollary 2 deal with the overloaded quasistationary mode, since condition 4) is used rather than condition 2). The initial distribution of the limit process $\eta(t)$ is degenerate since $\eta(0) = \eta^0$ with probability one and does not coincide with the stationary distribution of an Ornstein–Uhlenbeck process. Therefore the limit process $\eta(t)$ is functioning in the transient mode in view of the approximation of the quasistationary mode.

The components of the routing matrix do not depend on the series parameter n in the case under consideration. If one omits this restriction and puts

$$P = P_n = P_0 + n^{-1}B_o + \left(n^{-1}\right),$$

where $P_0 = \left\|\delta_{\alpha\beta}P^{(\alpha)}\right\|_1^{r_0}$ and $P^{(\alpha)} = \left\|p_{ij}^{(\alpha)}\right\|_{i,j\in I_\alpha}$ are nondecomposable stochastic matrices, then the phenomenon of the merging of nodes of the initial network can be seen. In other words, $r \to r_0$, $r_0 \leqslant r$. The nodes of the subset I_α are merged in a node with the number α.

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