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STOCHASTIC REPRESENTATION AND PATH PROPERTIES OF A FRACTIONAL COX–INGERSOLL–ROSS PROCESS

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Abstract. We consider the Cox–Ingersoll–Ross process that satisfies the stochastic differential equation $dX_t = aX_t dt + \sigma\sqrt{X_t} dB_t^H$ driven by a fractional Brownian motion B_t^H with the Hurst index exceeding $\frac{2}{3}$, where $\int_0^t \sqrt{X_s} dB_s^H$ is the pathwise integral defined as the limit of the corresponding Riemann–Stieltjes sums. We show that the Cox–Ingersoll–Ross process coincides with the square of the fractional Ornstein–Uhlenbeck process up to the first return to zero. Based on this observation, we consider the square of the fractional Ornstein–Uhlenbeck process with an arbitrary Hurst index and prove that it satisfies the above stochastic differential equation up to the first return to zero if $\int_0^t \sqrt{X_s} dB_s^H$ is understood as the pathwise Stratonovich integral. Then a natural question arises about the first visit to zero of the fractional Cox–Ingersoll–Ross process which coincides with the first visit to zero of the fractional Ornstein–Uhlenbeck process. Since the latter process is Gaussian, we use the bounds for the distributions of Gaussian processes to prove that the probability of a visit to zero over a finite time equals 1 if $a < 0$. Otherwise this probability is positive. We provide an upper bound for this probability.

1. Introduction

The standard diffusion Cox–Ingersoll–Ross process is introduced and studied in the papers [5–7] as a generalization of the Vasiček model for better modeling the evolution of interest rates. The Cox–Ingersoll–Ross process is a one-factor model that depends on a single source of market risk. This model assumes that the instantaneous value of the interest rate r_t is a solution of the stochastic differential equation

$$dr_t = a(b - r_t)\, dt + \sigma\sqrt{r_t}\, dW_t, \qquad t \geq 0,$$

where $a, b, \sigma \in \mathbb{R}^+$, $W = \{W_t, t \geq 0\}$ is a Wiener process, and $r|_{t=0} = r_0 > 0$. The parameter a corresponds to the speed of adjustment of the model, that is, to the rate of convergence to the mean value, b to the mean value, and σ to volatility. If $2ab \geq \sigma^2$, then the process assumes only positive values and does not visit zero with probability one. In contrast to the Vasiček model, where the standard deviation is constant, the standard deviation in the Cox–Ingersoll–Ross model equals $\sigma\sqrt{r_t}$ and thus depends on the values of the process.

The Cox–Ingersoll–Ross process is ergodic and has a stationary distribution. The conditional distribution of its future value r_{t+T} given r_t coincides with the noncentral χ^2 distribution, while that of the limit values r_∞ coincides with the Gamma distribution. Another application of the Cox–Ingersoll–Ross process lies in modeling the stochastic

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volatility in the Heston model. A comprehensive bibliography on this topic can be found in the papers [11, 12].

Note that many real financial models exhibit the so-called phenomenon of memory. This phenomenon means that the fluctuations of prices in a market cannot be characterized exclusively by the randomness generated by a Wiener process. A survey of financial markets with memory can be found, for example, in [1, 3, 9, 23]. It was believed until recently that the best models of the evolution of interest rates necessarily involve a fractional Brownian motion with the Hurst index $H > \frac{1}{2}$, while modern studies of markets (see, for instance, [2]) indicate that the volatility can be so irregular that the corresponding Hurst index is close to the value 0.1. This observation leads to a conclusion that a fractional Ornstein–Uhlenbeck process is a better model for the corresponding interest rates as well as for stochastic volatility. Another conclusion is that a fractional Cox–Ingersoll–Ross process can also be used succesfully for modeling since the corresponding stochastic process is nonnegative. A fractional Ornstein–Uhlenbeck process is Gaussian and thus the stochastic integration with respect to it delivers no problem (properties of fractional Ornstein–Uhlenbeck processes are described in [4]). On the other hand, there are several approaches to the integration with respect to a fractional Cox–Ingersoll–Ross process. The approach where the integral with respect to a fractional Brownian motion is understood pathwisely is considered in [18] for $H > \frac{2}{3}$. Another approach, the so-called rough-path approach, is introduced in [17]. One more approach is based on the property that a standard Cox–Ingersoll–Ross process belongs to the class of Pearson diffusions and thus it can be defined as a Cox–Ingersoll–Ross process subordinate to an inverse stable subordinator [15, 16].

In the current paper, we start with considering the stochastic differential equation

$$dX_t = aX_t\,dt + \sigma\sqrt{X_t}\,dB_t^H, \qquad t \geq 0, \tag{1}$$

where $X|_{t=0} = x_0 > 0$, $a \in \mathbb{R}$, $\sigma > 0$, and $B^H = \{B_t^H, t \geq 0\}$ is a fractional Brownian motion with the Hurst index $H \in (0, 1)$. In other words, B_t^H is a centered Gaussian process with the covariance function

$$\mathsf{E}B_t^H B_s^H = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

We deal with the Hurst index $H \in (\frac{2}{3}, 1)$, since $\int_0^t \sqrt{X_s}\,dB_s^H$ can be defined as a pathwise integral that is the limit of integral Riemann–Stieltjes sums in this case. We prove that a unique solution of equation (1) up to the first visit to zero is the square of a fractional Ornstein–Uhlenbeck process. It is clear that the uniqueness of a solution implies that the solution of equation (1) remains at zero after the first visit to zero.

Then we consider the square of a fractional Ornstein–Uhlenbeck process with an arbitrary Hurst index $H \in (0, 1)$ and prove that it satisfies equation (1) up to the first visit to zero if the stochastic integral is understood as a pathwise integral in the Stratonovich sense.

It is a natural question to ask whether or not the moment of the first visit to zero of a fractional Ornstein–Uhlenbeck process is finite. We prove that this moment is finite with probability one if $a < 0$. Otherwise, that is, if $a > 0$, this probability belongs to the interval $(0, 1]$ or to the interval $(0, 1)$ if a is sufficiently large. Using the explicit form of the covariance function of a fractional Ornstein–Uhlenbeck process we find an upper estimate of this probability.

The paper is constructed as follows. In Section 2, we consider equation (1) for the case of $H \in (\frac{2}{3}, 1)$ and with the pathwise Riemann–Stieltjes integral with respect to a fractional Brownian motion. We show that a solution of equation (1) is the square of a fractional Ornstein–Uhlenbeck process up to the first visit to zero.

A fractional Cox–Ingersoll–Ross process is introduced in Section 3 for all $H \in (0,1)$ as the square of the corresponding Ornstein–Uhlenbeck process. We prove that the process defined in this way satisfies equation (1) if the pathwise integral is understood in the Stratonovich sense.

Section 4 is devoted to the study of the probability of a visit to zero of the above process over a finite time.

Section 5 contains an auxiliary result, namely the derivation of the explicit form of the covariance function of a fractional Ornstein–Uhlenbeck process.

2. Fractional Cox–Ingersoll–Ross process with Hurst index $H \in (2/3, 1)$

Consider the stochastic differential equation

$$dX_t = \tilde{a} X_t\, dt + \tilde{\sigma}\sqrt{X_t}\, dB_t^H, \qquad t \geq 0, \tag{2}$$

where $X|_{t=0} = x_0 > 0$, $\tilde{a} \in \mathbb{R}$, and $\tilde{\sigma} > 0$.

According to Theorem 6 of the paper [18], if $H > 2/3$, then equation (2) has a unique solution until the first visit to zero if $\int_0^t \sqrt{X_s}\, dB_s^H$ is understood as a pathwise integral that is the limit of Riemann–Stieltjes sums.

This fact can be explained as follows. An integral with respect to a fractional Brownian motion (see, for example, [24] concerning conditions for the existence and properties of such kind of integrals) is well-defined as a pathwise limit of integral Riemann–Stieltjes sums if the sum of Hölder exponents of the integrand and fractional Brownian motion exceeds 1. On the other hand, if a solution exists, then the integral possesses the Hölder property up to order H (see, for example, [10]). Thus the integrand $\sqrt{X_t}$ is a Hölder function up to the order $H/2$. Therefore $H/2 + H > 1$ or $H > 2/3$ is the sufficient condition for the existence of the pathwise integral with respect to a fractional Brownian motion as the limit of integral Riemann–Stieltjes sums. We stress once more that equation (2) has a unique solution in this case and the trajectories of a solution are positive until the first visit to zero.

Let $\tau_0 := \inf\{t > 0 \colon X_t = 0\}$ and consider trajectories of the process $\{X_t, t \geq 0\}$ in the interval $[0, \tau_0)$. Changing $Y_t = \sqrt{X_t}$ and using the Itô formula for integrals with respect to a fractional Brownian motion (see [19]) we conclude that

$$dY_t = \frac{dX_t}{2\sqrt{X_t}} = \frac{\tilde{a} X_t\, dt}{2\sqrt{X_t}} + \frac{\tilde{\sigma}}{2}\, dB_t^H.$$

Denoting $a = \tilde{a}/2$ and $\sigma = \tilde{\sigma}/2$ we obtain the equation

$$dY_t = a\, Y_t\, dt + \sigma\, dB_t^H \tag{3}$$

with the initial condition $Y_0 = \sqrt{X_0}$. Thus a solution $\{X_t, t \in [0, \tau_0)\}$ of equation (2) is the square of a fractional Ornstein–Uhlenbeck process $\{Y_t, t \geq 0\}$ until the first visit to zero (a fractional Ornstein–Uhlenbeck process is introduced in [4]).

3. Generalization of a fractional Cox–Ingersoll–Ross process to the case of $H \in (0,1)$

Based on the conclusion of Section 2, we define a fractional Cox–Ingersoll–Ross process for all Hurst indices $H \in (0,1)$.

Definition 3.1. Let $H \in (0,1)$ be an arbitrary number, let $\{Y_t, t \geq 0\}$ be a fractional Ornstein–Uhlenbeck process satisfying equation (3), and let τ be the moment of the first visit to zero. Then $\{X_t, t \geq 0\}$ is called a *fractional Cox–Ingersoll–Ross process* if

$$X_t(\omega) = Y_t^2(\omega)\mathbf{1}_{\{t < \tau(\omega)\}} \tag{4}$$

for all $t \geq 0$.

A stochastic process defined in Definition 3.1 satisfies stochastic differential equation (2) if $\int_0^t \sqrt{X_s}\, dB_s^H$ is understood as a pathwise integral in the Stratonovich sense. Below is the corresponding definition.

Definition 3.2. Let $\{X_t, t \ge 0\}$ and $\{Y_t, t \ge 0\}$ be stochastic processes. *The pathwise integral* $\int_0^T X_s \circ dY_s$ *in the Stratonovich sense* over the interval $[0, T]$ is defined as the limit of sums of the form

$$\sum_{k=1}^{n} \frac{X_{t_k} + X_{t_{k-1}}}{2} \left(Y_{t_k} - Y_{t_{k-1}}\right)$$

as the diameter of a partition $0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = T$ tends to zero (provided the limit exists).

Indeed, let $\{Y_t, t \ge 0\}$ be a fractional Ornstein–Uhlenbeck process that starts from a point $\sqrt{X_0}$ and let $\tau = \inf\{s > 0\colon Y_s = 0\}$. For some $\omega \in \Omega$, consider a point t such that $t < \tau(\omega)$. Then

$$X_t = Y_t^2 = \left(\sqrt{X_0} + a\int_0^t Y_s\, ds + \sigma B_t^H\right)^2 \tag{5}$$

according to Definition 3.1.

Consider an arbitrary partition of the interval $[0, t]$:

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t.$$

Using representation (5) we obtain

$$\begin{aligned}
X_t &= \sum_{k=1}^{n} \left(X_{t_k} - X_{t_{k-1}}\right) + X_0 \\
&= \sum_{k=1}^{n} \left(\left[\sqrt{X_0} + a\int_0^{t_k} Y_s\, ds + \sigma B_{t_k}^H\right]^2 - \left[\sqrt{X_0} + a\int_0^{t_{k-1}} Y_s\, ds + \sigma B_{t_{k-1}}^H\right]^2 \right) + X_0 \\
&= \sum_{k=1}^{n} \left(2\sqrt{X_0} + a\left(\int_0^{t_k} Y_s\, ds + \int_0^{t_{k-1}} Y_s\, ds\right) + \sigma\left(B_{t_k}^H + B_{t_{k-1}}^H\right)\right) \\
&\qquad \times \left(a\int_{t_{k-1}}^{t_k} Y_s\, ds + \sigma\left(B_{t_k}^H - B_{t_{k-1}}^H\right)\right) + X_0.
\end{aligned}$$

Expanding the brackets we obtain

$$\begin{aligned}
X_t &= 2a\sqrt{X_0}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} Y_s\, ds + a^2\sum_{k=1}^{n}\left(\int_0^{t_k} Y_s\, ds + \int_0^{t_{k-1}} Y_s\, ds\right)\int_{t_{k-1}}^{t_k} Y_s\, ds \\
&\quad + a\sigma\sum_{k=1}^{n}\left(B_{t_k}^H + B_{t_{k-1}}^H\right)\int_{t_{k-1}}^{t_k} Y_s\, ds + 2\sigma\sqrt{X_0}\sum_{k=1}^{n}\left(B_{t_k}^H - B_{t_{k-1}}^H\right) \\
&\quad + a\sigma\sum_{k=1}^{n}\left(\int_0^{t_k} Y_s\, ds + \int_0^{t_{k-1}} Y_s\, ds\right)\left(B_{t_k}^H - B_{t_{k-1}}^H\right) \\
&\quad + \sigma^2\sum_{k=1}^{n}\left(B_{t_k}^H + B_{t_{k-1}}^H\right)\left(B_{t_k}^H - B_{t_{k-1}}^H\right).
\end{aligned} \tag{6}$$

Now let the diameter Δt of a partition tend to zero. The limit of the sum of the first three terms in the latter expression equals

$$\begin{aligned}
&2a\sqrt{X_0}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} Y_s\,ds + a^2\sum_{k=1}^{n}\left(\int_0^{t_k} Y_s ds + \int_0^{t_{k-1}} Y_s\,ds\right)\int_{t_{k-1}}^{t_k} Y_s\,ds \\
&\quad + a\sigma\sum_{k=1}^{n}\left(B_{t_k}^H + B_{t_{k-1}}^H\right)\int_{t_{k-1}}^{t_k} Y_s\,ds \\
&\quad \to 2a\sqrt{X_0}\int_0^t Y_s\,ds + 2a^2\int_0^t Y_s\int_0^s Y_u\,du\,ds + 2a\sigma\int_0^t B_s^H Y_s\,ds \qquad (7)\\
&\quad = 2a\int_0^t Y_s\left(\sqrt{X_0} + a\int_0^s Y_u\,du + \sigma B_s^H\right)ds = 2a\int_0^t Y_s^2\,ds \\
&\quad = 2a\int_0^t X_s\,ds = \tilde{a}\int_0^t X_s\,ds, \qquad \Delta t \to 0.
\end{aligned}$$

The limit of the sum of the last three terms is found similarly,

$$\begin{aligned}
&2\sigma\sqrt{X_0}\sum_{k=1}^{n}\left(B_{t_k}^H - B_{t_{k-1}}^H\right) + a\sigma\sum_{k=1}^{n}\left(\int_0^{t_k} Y_s ds + \int_0^{t_{k-1}} Y_s ds\right)\left(B_{t_k}^H - B_{t_{k-1}}^H\right) \\
&\quad + \sigma^2\sum_{k=1}^{n}\left(B_{t_k}^H + B_{t_{k-1}}^H\right)\left(B_{t_k}^H - B_{t_{k-1}}^H\right) \qquad (8)\\
&\quad \to 2\sigma\int_0^t\left(\sqrt{X_0} + a\int_0^s Y_u du + \sigma B_s^H\right)\circ dB_s^H \\
&\quad = \tilde{\sigma}\int_0^t \sqrt{X_s}\circ dB_s^H, \qquad \Delta t \to 0.
\end{aligned}$$

Therefore the fractional Cox–Ingersoll–Ross process defined in Definition 3.1 satisfies the stochastic differential equation

$$X_t = X_0 + \tilde{a}\int_0^t X_s ds + \tilde{\sigma}\int_0^t \sqrt{X_s}\circ dB_s^H, \qquad (9)$$

where $\int_0^t \sqrt{X_s}\circ dB_s^H$ is understood as a pathwise integral in the Stratonovich sense.

Below are some remarks concerning stochastic differential equation (9).

Remark 3.3. The limit in (8) exists, since the left-hand side of equality (6) does not depend on a partition, whence we conclude that the corresponding pathwise Stratonovich integral exists, as well.

Remark 3.4. The solution of equation (9) coincides with the solutions of equation (2) if $H > 2/3$, since the corresponding pathwise Stratonovich integral coincides in this case with the integral defined as the limit of integral Riemann–Stieltjes sums.

4. Probability of hitting zero for a fractional Ornstein–Uhlenbeck process

Our aim is to study the probability that the random moment τ, the first hitting time of zero for a fractional Ornstein–Uhlenbeck process that is a solution of stochastic differential equation (3), is finite. According to [4], the explicit form of this solution is given by

$$Y_t = e^{at}\left(Y_0 + \sigma\int_0^t e^{-as}\,dB_s^H\right), \qquad (10)$$

where the integral with respect to the fractional Brownian motion is the limit of the corresponding Riemann–Stieltjes integral sums and can be defined with the help of integration by parts:

$$J_t := \int_0^t e^{-as}\, dB_s^H := e^{-at} B_t^H + a \int_0^t e^{-as} B_s^H\, ds. \tag{11}$$

Equality (10) implies that the first hitting time of zero for the process Y_t coincides with the first hitting time of level $-Y_0/\sigma$ for the integral (11). Note that the latter integral is a normal random variable with zero mean. Since the normal distribution is symmetric, the probability of hitting a negative level $-Y_0/\sigma$ by the integral (11) is equal to the probability of hitting the positive level Y_0/σ.

Thus we need to solve a general problem of finding the probability of hitting a level $x > 0$ by the integral J_t over a finite time. It is clear that the behavior of this integral depends essentially on the sign of the parameter $a \in \mathbb{R}$.

Consider two mutually exclusive cases.

Case $a \le 0$.

Proposition 4.1. *If $a < 0$, then*

$$\mathsf{P}\left(\limsup_{t\to\infty} J_t = +\infty\right) = 1.$$

Proof. It is known that the process

$$G_t = e^{at} \int_{-\infty}^t e^{-as}\, dB_s^H$$

is Gaussian, stationary, and ergodic if $a < 0$ (see [4]). The ergodic theorem implies that, for an arbitrary $x \in \mathbb{R}$,

$$\frac{1}{n}\sum_{k=1}^n \mathbf{1}_{\{G_k > x\}} \to \mathsf{E}\mathbf{1}_{\{G_0 > x\}} = \mathsf{P}(G_0 > x) > 0 \quad \text{a.s.}, \qquad n \to \infty.$$

Hence

$$\sum_{k=1}^\infty \mathbf{1}_{\{G_k > x\}} = +\infty \quad \text{a.s.}$$

Here and in the sequel "a.s." abbreviates the expression "almost surely". This means that the random events $\{G_k > x\}$ occur infinitely often. This yields

$$\limsup_{t\to\infty} G_t = +\infty \quad \text{a.s.}$$

Then

$$\limsup_{t\to\infty} J_t = \limsup_{t\to\infty} \left(e^{-at} G_t - G_0\right) = +\infty \quad \text{a.s.} \qquad \square$$

Remark 4.2. Proposition 4.1 remains true for the case $a = 0$, as well, since $J_t = B_t^H$. More detail concerning the asymptotic behavior of the supremum and hitting probabilities for a fractional Brownian motion can be found, for example, in [8, 14, 20, 21].

Case $a > 0$. By Corollary 5.6 in Section 5,

$$V_t^2 := \operatorname{Var} J_t = H \int_0^t z^{2H-1} \left(e^{-2at+az} + e^{-az}\right) dz.$$

The derivative of V_t^2 is equal to

$$\frac{d}{dt} V_t^2 = 2H \left(t^{2H-1} e^{-at} - a e^{-2at} \int_0^t z^{2H-1} e^{az}\, dz \right).$$

Since the second term in brackets is exponentially less than the first one, there exists $t(a)$ such that the derivative is positive for all $t \geq t(a)$. Note that

$$\lim_{t\to\infty} V_t^2 = H \int_0^\infty z^{2H-1} e^{-az} dz = \frac{H\Gamma(2H)}{a^{2H}}.$$

Consider the Gaussian process

$$Z_t = J_{t/(1-t)}, \qquad t \in [0,1],$$

with $Z_1 = J_\infty$. The derivative of its variance v_t^2 is given by

$$\begin{aligned} \frac{d}{dt} v_t^2 &= \frac{d}{ds} V_s^2 \Big|_{s=t/(1-t)} \left(\frac{t}{1-t}\right)' \\ &= 2H \frac{\big(t/(1-t)\big)^{2H-1} e^{-at/(1-t)} - a e^{-2at/(1-t)} \int_0^{t/(1-t)} z^{2H-1} e^{az}\, dz}{(1-t)^2}. \end{aligned} \tag{12}$$

The derivative exists and tends to zero as $t \to 1$. Since the factors are exponential, we conclude that the second derivative also tends to zero as $t \to 1$. Now we use the covariance function obtained in Corollary 5.6 (throughout below $s < t$). After some simplification we get

$$\begin{aligned} \mathsf{E}(J_t - J_s)^2 &= V_t^2 + V_s^2 - 2\operatorname{Cov}(J_s, J_t) \\ &= He^{-2at} \int_0^t z^{2H-1} e^{-az}\, dz + H \int_0^t z^{2H-1} e^{-az}\, dz + He^{-2as} \int_0^s z^{2H-1} e^{az}\, dz \\ &\quad + H \int_0^s z^{2H-1} e^{-az}\, dz + He^{-2as} \int_0^{t-s} z^{2H-1} e^{-az}\, dz \\ &\quad - He^{-2at} \int_{t-s}^t z^{2H-1} e^{az}\, dz + H \int_s^t z^{2H-1} e^{-az}\, dz \\ &\quad - He^{-2as} \int_0^s z^{2H-1} e^{az}\, dz - 2H \int_0^t z^{2H-1} e^{-az}\, dz \\ &= He^{-2at} \int_0^{t-s} z^{2H-1} e^{az}\, dz + He^{-2as} \int_0^{t-s} z^{2H-1} e^{-az}\, dz. \end{aligned}$$

Then

$$\begin{aligned} \mathsf{E}(Z_t - Z_s)^2 &= He^{-2at/(1-t)} \int_0^{(t-s)/(1-t)(1-s)} z^{2H-1} e^{az}\, dz \\ &\quad + He^{-2as/(1-s)} \int_0^{(t-s)/(1-t)(1-s)} z^{2H-1} e^{-az}\, dz. \end{aligned}$$

Further, for every $s \in [0,1)$,

$$\frac{t-s}{(1-t)(1-s)} = \frac{t-s}{(1-s)^2} + \frac{(t-s)^2}{(1-t)(1-s)},$$

$$\begin{aligned} e^{-2at/(1-t)} &= e^{-2as/(1-s)} e^{-2a(t-s)/(1-t)(1-s)} \\ &= e^{-2as/(1-s)} \left(1 + \frac{-2a(t-s)}{(1-s)^2} + \frac{-4a(t-s)^2}{(1-s)^2} + \frac{O\left((t-s)^3\right)}{(1-s)^2}\right) \end{aligned}$$

as $t \downarrow s$. Moreover

$$\int_0^{(t-s)/(1-t)(1-s)} z^{2H-1} e^{\pm az}\, dz = \frac{1}{2H} \frac{(t-s)^{2H} + O\left((t-s)^{2H+1}\right)}{(1-s)^{2H}}$$

as $t \downarrow s$, whence

$$\mathsf{E}(Z_t - Z_s)^2 = He^{-2as/(1-s)} \left(\frac{(t-s)^{2H}}{H(1-s)^{2H}} + \frac{O\left((t-s)^{2H+1}\right)}{(1-s)^{2H}}\right) \tag{13}$$

as $t \downarrow s$. Thus

$$\limsup_{t-s\downarrow 0} \frac{\mathsf{E}(Z_t - Z_s)^2}{(t-s)^{2H}} \le H \max_{s\in[0,1]} (1-s)^{-2} e^{-2as/(1-s)} \tag{14}$$

for $s \in [0,1]$. Equalities (12) and (14) imply that

$$\mathsf{P}\left(\sup_{t\ge 0} J_t < \infty\right) = \mathsf{P}\left(\max_{t\in[0,1]} Z_t < \infty\right) = 1. \tag{15}$$

Applying Theorem D.4 of [22] together with equalities (12) and (14) we prove the following result.

Proposition 4.3. *There exists a constant C such that*

$$\mathsf{P}\left(\sup_{t\ge 0} J_t \ge x\right) = \mathsf{P}\left(\max_{t\in[0,1]} Z_t \ge x\right) \le Cx^{\frac{1}{H}-1} \exp\left(-\frac{x^2}{2v^2}\right) \tag{16}$$

for an arbitrary $x > 0$, where

$$v^2 = \sup_{t\ge 0} V_t^2 = \max_{t\in[0,1]} v_t^2 < \infty.$$

Moreover, since v_t is twice differentiable, we derive from (14) that

$$\mathrm{Cov}(Z_s/v_s, Z_t/v_t) \ge 1 - c|t-s|^{2H}$$

for some $c > 0$ and sufficiently small $t - s$. Now one can use the Slepian lemma (see, for example, [22]) to estimate the probability (16) from above by the corresponding probability for the process $v_t U_t$, where U_t is a Gaussian stationary process with zero mean and with covariance function whose behavior at zero is such that $1 - |t|^{2H} + o(|t|^{2H})$. Then Theorem D.4 of [22] yields the following result.

Proposition 4.4. *There exists a constant C_1 such that*

$$\mathsf{P}\left(\sup_{t\ge 0} J_t \ge x\right) \le C_1 x^{\frac{1}{H}-2} \exp\left(-\frac{x^2}{2v^2}\right)$$

for an arbitrary $x > 0$.

Remark 4.5. It is easy to make sure that

$$\max_{t\in[0,1]} v_t^2 = v_1^2 = V_\infty^2 = \frac{H\Gamma(2H)}{a^{2H}}.$$

Indeed, equality (12) implies that $t=1$ is a point of local maximum of the function v_t. We have $v_0^2 = V_0^2 = 0$ at $t=0$. The result desired follows if we show that the function v_t^2 does not have local extremums at points of the open interval $(0,1)$. If, by contradiction, such a point exists, then equality (12) implies that this point satisfies the equation

$$\left(\frac{t}{1-t}\right)^{2H-1} e^{-at/(1-t)} - ae^{-2at/(1-t)} \int_0^{t/(1-t)} z^{2H-1}e^{az}\,dz = 0, \qquad t\in(0,1).$$

The latter equation for the variable $s=\frac{t}{1-t}$ becomes of the form

$$e^{-2as}\left(s^{2H-1}e^{as} - a\int_0^s z^{2H-1}e^{az}\,dz\right) = 0, \qquad s>0. \tag{17}$$

Now we study the behavior of the function

$$h(s) = s^{2H-1}e^{as} - a\int_0^s z^{2H-1}e^{az}\,dz$$

for $s>0$. If $H=\frac12$, then $h(s)\equiv 1$ and thus equation (17) does not have any root. For $H\neq\frac12$, we evaluate the derivative

$$\frac{d}{dt}h(t) = (2H-1)s^{2H-2}e^{as}.$$

If $H<\frac12$, then the function h decreases and thus the left-hand side of (17) also decreases and tends to 0 as $s\to\infty$, since $a>0$. This implies that equation (17) does not have any root in the interval $(0,+\infty)$. If $H>\frac12$, then h increases and $h(0)=0$, whence we conclude that $h(s)>0$ for $s>0$ and therefore equation (17) does not have roots in the interval $(0,+\infty)$, as well.

Remark 4.6. Let $1>t>s$. Relations (13) and (14) imply that $\mathsf{E}(Z_t-Z_s)^2$ exponentially approaches zero as $s\to 1$. One can also prove that all derivatives with respect to the argument s of the expectation approach zero as $s\to1$. The same result holds for v_s^2. This explains why Theorem D.3 of [22] is not applicable in our case for finding the asymptotic behavior of the probability

$$\mathsf{P}\left(\sup_{t\geq 0} Z_t \geq x\right)$$

as $x\to\infty$. However the asymptotic behavior desired can be obtained with the help of the same methods as those used in the proof of Theorem D.3 of [22].

Now we turn back to the question on the finiteness of τ, the moment of the first hitting time of zero for a fractional Ornstein–Uhlenbeck process (3). The reasoning above together with Propositions 4.1 and 4.4 and Remark 4.2 yields the following result.

Theorem 4.7. (1) *If $a\leq 0$, then $\mathsf{P}(\tau<\infty)=1$.*
(2) *If $a>0$, then $\mathsf{P}(\tau<\infty)\in(0,1]$. Moreover*

$$\mathsf{P}(\tau<\infty) \leq C_1\left(\frac{Y_0}{\sigma}\right)^{\frac1H-2}\exp\left(-\frac{a^{2H}Y_0^2}{\sigma^2\Gamma(2H+1)}\right),$$

where $C_1>0$ is a constant.

5. Covariance function of an Ornstein–Uhlenbeck process

Consider an Ornstein–Uhlenbeck process Y_t that is a solution of equation (3) with the initial condition $Y_t = y_0 \in \mathbb{R}$. According to relations (10)–(11), this solution is given by

$$Y_t = y_0 e^{at} + a\sigma e^{at} \int_0^t e^{-as} B_s^H \, ds + \sigma B_t^H, \qquad t \geq 0. \tag{18}$$

Proposition 5.1. *Let $t \geq s \geq 0$. Then the covariance function of a fractional Ornstein–Uhlenbeck process* (18) *is given by*

$$\begin{aligned} R_H(t,s) = \frac{H\sigma^2}{2} \Bigg(-e^{at-as} \int_0^{t-s} e^{-az} z^{2H-1} \, dz + e^{-at+as} \int_{t-s}^{t} e^{az} z^{2H-1} \, dz \\ - e^{at+as} \int_s^t e^{-az} z^{2H-1} \, dz + e^{at-as} \int_0^s e^{az} z^{2H-1} \, dz \\ + 2e^{at+as} \int_0^t e^{-az} z^{2H-1} \, dz \Bigg). \end{aligned} \tag{19}$$

Proof. Using representation (18) and the explicit form of the covariance of a fractional Brownian motion, we write

$$\begin{aligned} R_H(t,s) &= \mathsf{E}\left[\left(Y_t - y_0 e^{at}\right)\left(Y_s - y_0 e^{as}\right)\right] \\ &= \mathsf{E}\left[\left(a\sigma e^{at} \int_0^t e^{-au} B_u^H \, du + \sigma B_t^H\right)\left(a\sigma e^{as} \int_0^s e^{-av} B_v^H \, dv + \sigma B_s^H\right)\right] \\ &= \frac{a\sigma^2}{2} e^{at} \int_0^t e^{-au} \left(u^{2H} + s^{2H} - |u-s|^{2H}\right) du \\ &\quad + \frac{a\sigma^2}{2} e^{as} \int_0^s e^{-av} \left(v^{2H} + t^{2H} - |v-t|^{2H}\right) dv + \frac{\sigma^2}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \\ &\quad + \frac{a^2\sigma^2}{2} e^{at+as} \int_0^t \int_0^s e^{-au-av} \left(u^{2H} + v^{2H} - |u-v|^{2H}\right) du \, dv \\ &= \frac{\sigma^2}{2} \sum_{n=1}^{10} I_n, \end{aligned}$$

where

$$I_1 = a e^{at} \int_0^t e^{-au} s^{2H} \, du, \qquad I_2 = a e^{at} \int_0^t e^{-au} u^{2H} \, du,$$

$$I_3 = -a e^{at} \int_0^t e^{-au} |u-s|^{2H} \, du, \qquad I_4 = a e^{as} \int_0^s e^{-av} t^{2H} \, dv,$$

$$I_5 = a e^{as} \int_0^s e^{-av} v^{2H} \, dv, \qquad I_6 = -a e^{as} \int_0^s e^{-av} (t-v)^{2H} \, dv,$$

$$I_7 = t^{2H} + s^{2H} - (t-s)^{2H}, \qquad I_8 = a^2 e^{at+as} \int_0^t e^{-av} \, dv \int_0^s e^{-au} u^{2H} \, du,$$

$$I_9 = a^2 e^{at+as} \int_0^s e^{-au} \, du \int_0^t e^{-av} v^{2H} \, dv,$$

$$I_{10} = -a^2 e^{at+as} \int_0^t \int_0^s e^{-au-av} |u-v|^{2H} \, du \, dv.$$

The first two integrals are equal to

$$I_1 = s^{2H}\left(e^{at} - 1\right) \quad \text{and} \quad I_2 = -e^{at} \int_0^t u^{2H} \, de^{-au} = -t^{2H} + 2He^{at} \int_0^t e^{-au} u^{2H-1} \, du,$$

respectively. Changing the variables and integrating by parts we get

$$
\begin{aligned}
I_3 &= -ae^{at}\int_0^s e^{-au}(s-u)^{2H}\,du - ae^{at}\int_s^t e^{-au}(u-s)^{2H}\,du \\
&= -ae^{at-as}\int_0^s e^{az}z^{2H}\,dz - ae^{at-as}\int_0^{t-s} e^{-az}z^{2H}\,dz \\
&= -e^{at-as}\Big(e^{as}s^{2H} - 2H\int_0^s e^{az}z^{2H-1}\,dz - e^{-a(t-s)}(t-s)^{2H} \\
&\quad +2H\int_0^{t-s} e^{-az}z^{2H-1}\,dz\Big) \\
&= -e^{at}s^{2H} + (t-s)^{2H} + 2He^{at-as}\int_0^s e^{az}z^{2H-1}\,dz - 2He^{at-as}\int_0^{t-s} e^{-az}z^{2H-1}\,dz.
\end{aligned}
$$

Similarly to the integrals I_1–I_3 we rewrite the integrals I_4–I_6:

$$
\begin{aligned}
I_4 = t^{2H}\left(e^{as}-1\right), \qquad I_5 &= -s^{2H} + 2He^{as}\int_0^s e^{-av}v^{2H-1}\,dv, \\
I_6 = -ae^{as-at}\int_{t-s}^t e^{az}z^{2H}\,dz &= -e^{as-at}\int_{t-s}^t z^{2H}\,de^{az} \\
&= -e^{as}t^{2H} + (t-s)^{2H} + 2He^{as-at}\int_{t-s}^t e^{az}z^{2H-1}\,dz.
\end{aligned}
$$

Further

$$
I_8 = e^{at+as}\left(e^{-at}-1\right)\int_0^s u^{2H}\,de^{-au} = (1-e^{at})s^{2H} - 2He^{as}(1-e^{at})\int_0^s e^{-au}u^{2H-1}\,du
$$

and analogously

$$
I_9 = (1-e^{as})t^{2H} - 2He^{at}(1-e^{as})\int_0^t e^{-av}v^{2H-1}\,dv.
$$

Finally we consider the integral I_{10}. First we represent it as the sum of two integrals, namely

$$
\begin{aligned}
I_{10} &= -a^2e^{at+as}\int_0^s\int_0^v e^{-au-av}(v-u)^{2H}\,du\,dv \\
&\quad - a^2e^{at+as}\int_0^s\int_v^s e^{-au-av}(u-v)^{2H}\,du\,dv \\
&\quad - a^2e^{at+as}\int_s^t\int_0^s e^{-au-av}(v-u)^{2H}\,du\,dv \\
&= -2a^2e^{at+as}\int_0^s\int_0^v e^{-au-av}(v-u)^{2H}\,du\,dv \\
&\quad - a^2e^{at+as}\int_s^t\int_0^s e^{-au-av}(v-u)^{2H}\,du\,dv \\
&=: I'_{10} + I''_{10}.
\end{aligned}
$$

Changing the variables $v-u=z$, then changing the order of integration, and finally integrating by parts we obtain

$$\begin{aligned}
I'_{10} &= -2a^2 e^{at+as}\int_0^s e^{-2av}\int_0^v e^{az}z^{2H}\,dz\,dv\\
&= -2a^2 e^{at+as}\int_0^s e^{az}z^{2H}\int_z^s e^{-2av}\,dv\,dz\\
&= -2a^2 e^{at+as}\int_0^s e^{az}z^{2H}\,\frac{e^{-2as}-e^{-2az}}{-2a}\,dz\\
&= ae^{at-as}\left(\int_0^s e^{az}z^{2H}\,dz-\int_0^s e^{-au}u^{2H}\,du\right)\\
&= e^{at}s^{2H}-2He^{at-as}\int_0^s e^{az}z^{2H-1}\,dz+e^{at}s^{2H}-2He^{at+as}\int_0^s e^{-au}u^{2H-1}\,du.
\end{aligned}$$

Now we consider two cases for the integral I''_{10}. If $t>2s$, then we change the variable $v-u=z$ in the inner integral, interchange the order of integrals, and integrate with respect to the variable v. We obtain

$$\begin{aligned}
I''_{10} &= -a^2 e^{at+as}\int_s^t\int_{v-s}^v e^{az-2av}z^{2H}\,dz\,dv\\
&= -a^2 e^{at+as}\left(\int_0^s\int_s^{z+s} e^{az-2av}z^{2H}\,dv\,dz+\int_s^{t-s}\int_z^{z+s} e^{az-2av}z^{2H}\,dv\,dz\right.\\
&\qquad\left.+\int_{t-s}^t\int_z^t e^{az-2av}z^{2H}\,dv\,dz\right)\\
&= -a^2 e^{at+as}\left(\int_0^s e^{az}z^{2H}\,\frac{e^{-2a(z+s)}-e^{-2as}}{-2a}\,dz\right.\\
&\quad\left.+\int_s^{t-s} e^{az}z^{2H}\,\frac{e^{-2a(z+s)}-e^{-2az}}{-2a}\,dz+\int_{t-s}^t e^{az}z^{2H}\,\frac{e^{-2at}-e^{-2az}}{-2a}\,dz\right)\\
&= \frac{a}{2}e^{at-as}\int_0^{t-s} e^{-az}z^{2H}\,dz-\frac{a}{2}e^{at+as}\int_s^t e^{-az}z^{2H}\,dz-\frac{a}{2}e^{at-as}\int_0^s e^{az}z^{2H}\,dz\\
&\quad+\frac{a}{2}e^{as-at}\int_{t-s}^t e^{az}z^{2H}\,dz.
\end{aligned}$$

Integrating by parts in each of the four integrals above and collecting similar terms we conclude that

$$\begin{aligned}
I''_{10} &= -He^{as-at}\int_{t-s}^t e^{az}z^{2H-1}\,dz\\
&\quad+He^{at-as}\int_0^s e^{az}z^{2H-1}\,dz\\
&\quad-He^{at+as}\int_s^t e^{-az}z^{2H-1}\,dz\\
&\quad+He^{at-as}\int_0^{t-s} e^{-az}z^{2H-1}\,dz\\
&\quad-e^{at}s^{2H}+e^{as}t^{2H}-(t-s)^{2H}.
\end{aligned}$$

The latter relation holds for the case $s<t<2s$, as well. The proof is analogous to the case $t>2s$.

Therefore equality (19) is proved by adding the corresponding terms. □

Remark 5.2. Proposition 5.1 applied for the case of $s = t$ yields the formula that allows one to get the precise value of the variance of the Ornstein–Uhlenbeck process Y_t, namely it is equal to

$$\operatorname{Var} Y_t = H\sigma^2 \int_0^t z^{2H-1} \left(e^{az} + e^{2at-az}\right) dz.$$

This result is proved in [13, Lemma A.1].

Remark 5.3. The covariance function of a fractional Ornstein–Uhlenbeck process is such that

$$\begin{aligned} R_H(t,s) = \frac{H\sigma^2}{2} \Bigg(& -e^{a|t-s|} \int_0^{|t-s|} e^{-az} z^{2H-1}\, dz \\ & + e^{-a|t-s|} \int_{|t-s|}^{\max\{t,s\}} e^{az} z^{2H-1}\, dz \\ & - e^{a(t+s)} \int_{\min\{t,s\}}^{\max\{t,s\}} e^{-az} z^{2H-1}\, dz \\ & + e^{a|t-s|} \int_0^{\min\{t,s\}} e^{az} z^{2H-1}\, dz \\ & + 2e^{a(t+s)} \int_0^{\max\{t,s\}} e^{-az} z^{2H-1}\, dz \Bigg) \end{aligned}$$

for all $t, s \in \mathbb{R}^+$.

Corollary 5.4. *Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, $a < 0$, and let $t > 0$ be fixed. Then the covariance function R_H of the fractional Ornstein–Uhlenbeck process admits the following asymptotic representation:*

$$\begin{aligned} R_H(t+s,t) = {} & \frac{\sigma^2 H(2H-1)}{2(-a)^{2H}} \\ & \times \Bigg(e^{as} \int_1^{-as} e^y y^{2H-2}\, dy + e^{-as} \int_{-as}^{+\infty} e^{-y} y^{2H-2}\, dy \\ & - e^{at} \Bigg[e^{-a(t+s)} \int_{-a(t+s)}^{+\infty} e^{-y} y^{2H-2}\, dy \\ & + e^{a(t+s)} \int_1^{-a(t+s)} e^y y^{2H-2}\, dy \Bigg] \Bigg) \\ & + O(e^{as}), \qquad s \to \infty. \end{aligned} \tag{20}$$

Proof. Without loss of generality we consider the case of those s such that $-as > 1$.

Changing the variable $y = -az$ we transform equality (19) as follows:

$$
\begin{aligned}
R_H(t+s,t) = \frac{H\sigma^2}{2(-a)^{2H}} \Big(& -e^{as} \int_0^1 e^y y^{2H-1}\,dy - e^{as} \int_1^{-as} e^y y^{2H-1}\,dy \\
& + e^{-as} \int_{-as}^{-a(t+s)} e^{-y} y^{2H-1}\,dy - e^{2at+as} \int_{-at}^{-a(t+s)} e^y y^{2H-1}\,dy \\
& + e^{as} \int_0^{-at} e^{-y} y^{2H-1}\,dy + 2e^{2at+as} \int_0^1 e^y y^{2H-1}\,dy \\
& + 2e^{2at+as} \int_1^{-a(t+s)} e^y y^{2H-1}\,dy \Big) \\
= \frac{H\sigma^2}{2(-a)^{2H}} \Big(& -e^{as} \int_1^{-as} e^y y^{2H-1}\,dy + e^{-as} \int_{-as}^{-a(t+s)} e^{-y} y^{2H-1}\,dy \\
& - e^{2at+as} \int_{-at}^{-a(t+s)} e^y y^{2H-1}\,dy \\
& + 2e^{2at+as} \int_1^{-a(t+s)} e^y y^{2H-1}\,dy \Big) \\
+ O(e^{as}). &
\end{aligned}
$$

Integrating by parts in the latter relation we prove (20). □

Remark 5.5. The results obtained above are well agreed with those obtained in [4]. Indeed, let $a < 0$ and $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$. It is shown in the proof of Corollary 2.5 in [4] that

$$
R_H(t+s,t) = \operatorname{Cov}(\widetilde{Y}_t^H, \widetilde{Y}_{t+s}^H) - e^{at} \operatorname{Cov}(\widetilde{Y}_0^H, \widetilde{Y}_{t+s}^H) + O(e^{as}), \qquad s \to \infty, \tag{21}
$$

where

$$
\widetilde{Y}_t^H := \sigma \int_{-\infty}^t e^{a(t-u)}\,dB_u^H.
$$

The following asymptotic representation of $\operatorname{Cov}(\widetilde{Y}_t^H, \widetilde{Y}_{t+s}^H)$ is obtained in the proof of Theorem 2.3 in [4]:

$$
\begin{aligned}
\operatorname{Cov}(\widetilde{Y}_t^H, \widetilde{Y}_{t+s}^H) = {} & \frac{\sigma^2}{2(-a)^{2H}} H(2H-1) \\
& \times \left(e^{as} \int_1^{-as} e^y y^{2H-2}\,dy + e^{-as} \int_{-as}^{\infty} e^{-y} y^{2H-2}\,dy \right) \\
& + O(e^{as})
\end{aligned} \tag{22}
$$

as $s \to \infty$. Substituting representation (22) in asymptotic equality (21) we derive an expression which coincides with that in equality (20).

Note that the integral J_t defined by (11) is equal to $e^{-at} Y_t$, where Y_t is a process defined by (18) with parameters $y_0 = 0$ and $\sigma = 1$. Therefore we proved the following result.

Corollary 5.6. *If $s \le t$, then*

$$\begin{aligned}\operatorname{Cov}(J_s, J_t) = &-\frac{H}{2} e^{-2as} \int_0^{t-s} z^{2H-1} e^{-az}\, dz + \frac{H}{2} e^{-2at} \int_{t-s}^{t} z^{2H-1} e^{az}\, dz \\ &- \frac{H}{2} \int_s^t z^{2H-1} e^{-az}\, dz + \frac{H}{2} e^{-2as} \int_0^s z^{2H-1} e^{az}\, dz \\ &+ H \int_0^t z^{2H-1} e^{-az}\, dz.\end{aligned}$$

In particular,

$$\operatorname{Var} J_t = H \int_0^t z^{2H-1} \left(e^{az-2at} + e^{-az}\right) dz.$$

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