ON NILPOTENT ALGEBRAS*

BY

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Part I. Introduction.

Benjamin Peirce first pointed out † that with respect to any idempotent basis i of an algebra all its numbers may be divided into four classes (11), (12), (21), (22) characterized by the following multiplicative relations:

$$i(11) = (11) = (11)i; \quad i(12) = (12), \ (12)i = 0;$$

$$i(22) = 0 = (22)i; \qquad i(21) = 0, \ (21)i = (21);$$

Then these classes give when multiplied into each other the results shown in the following table:

	11	12	21	22
11	11	12	0	0
12	0	0	11	12
21	21	22	0	0
22	0	0	21	22

that is, $(g'h')(g''h'')$ is $(g'h'')$ or 0 according as h' is or is not the same as g''. Each class forms a sub-algebra of the algebra, the class (11) containing the idempotent basis. The class (22) may or may not contain an idempotent. If (11) contains two distinct idempotents, the process may be repeated, giving classes which we may represent by (11), (12), (13), (21), (22), (23), (31), (32), (33), with the analogous multiplication table. This division can be carried on in the same manner, giving multiplication tables of the form of perfect quadrates. The process stops when all independent idempotents have been found and isolated, so that every class on the main diagonal contains at most one idempotent, and otherwise only nilpotents. The classes not appearing on the main diagonal contain *skew* numbers only, and their squares vanish. Some of the

* Presented to the Society (Chicago) January 2, 1903. Received for publication May 30, 1903.

† *Linear Associative Algebra*, American Journal of Mathematics, vol. 4 (1884), p. 111.

classes on the main diagonal may contain no idempotent, in which case all the numbers of the class are nilpotent. The problem of determining the skew numbers is simple, and easily solved when the *direct* numbers of the classes (11), (22) $\cdots$ (*kk*) are known. Thus the basal problem of linear associative algebra is to determine all nilpotent algebras. With these at hand we may build up algebras of any type by combining the nilpotent algebras and adding the proper skew units. The present paper is devoted to the consideration of this basal problem.

In part II, it is shown that in a *canonical* form, the units of any nilpotent algebra are expressible in the forms,

$$i_1,\ i_2,\ i_3,\ \cdots,\ i_{s_0},\ j,\ i_1 j,\ i_2 j,\ \cdots,\ i_{s_0-r} j,\ j^2,\ i_1 j^2,\ \cdots,$$

where $j^m = 0$. In this order, the product of any two units is expressible linearly in terms of units which *follow* both factors. Second, it is shown that a set of units homologous to the units $i_1, i_2, \cdots, i_{s_0}$ may be chosen, which form a sub-algebra, the products being isomorphic with the products of $i_1, i_2, \cdots i_{s_0}$, so far as concerns terms involving only $i_1, i_2, \cdots, i_{s_0}$. This sub-algebra is the *base*, the unit j being *adjoined* to this base. From any base an increasing system of nilpotent algebras may be determined, each algebra in turn yielding others. Finally, in part III, certain applications are given to exemplify the method.

Part II.

§ 1. It has been shown in a previous paper * that any linear associative algebra can be brought to a form in which we may express any number thus:

$$\phi = \Sigma\, a_{ijk} \lambda_{ijk},$$

with the following conditions:

(1) for any given value of k, $i = 1, \cdots, s_k$; $j = 1, \cdots, s_0$;

(2) $$s_0 \geqq s_1 \geqq s_2 \cdots \geqq s_P;$$

(3) the number k is called the *weight* of the term to which it is attached; also, for λ's that have the subscripts i and j equal, there is in each case a maximum weight, called the *multiplicity* of the λ's corresponding, and represented by μ_i for the terms λ_{iik}; thus the terms λ_{iik} are $\lambda_{ii0}, \lambda_{ii1}, \lambda_{ii2}, \cdots, \lambda_{ii\mu_i}$; we have then for λ_{ijk}, $k \geqq 0$, and $\mu_i > k > \mu_i - \mu_j - 1$;

(4) finally

$$\lambda_{ijk} \cdot \lambda_{i'j'k'} = c\delta_{ji'} \lambda_{ij'k+k'},$$

where $\delta_{ji'} = 0$ if $i' \neq j$, $\delta_{ji'} = 1$ if $i' = j$; and further if $\mu_i > k + k' > \mu_i - \mu_{j'} - 1$, $c = 1$, while if $k + k' \geqq \mu_i$, $c = 0$.

* *Theory of linear associative algebra*, Transactions of the American Mathematical Society, vol. 4 (1903), pp. 251–287. This paper is cited hereafter as *Theory*.

(5) The numbers ϕ satisfy an equation which consists of factors of determinant form, of orders $w_1, w_2, \cdots, w_e$. These numbers are the *widths* of the factors, and w_r represents the number of multiplicities of a certain value μ_r, which are equal. The factor corresponding is a determinant of order w_r, affected with the exponent μ_r. The equation is then

$$\Pi \begin{vmatrix} a_{i_r i_r 0} - \phi, & a_{i_r i_r+1 0}, & \cdots & a_{i_r i_r+w_r-1 0} \\ a_{i_r+1 i_r 0}, & a_{i'_r+1 i_r+1 0} - \phi, & \cdots & a_{i_r+1 i_r+w_r-1 0} \\ \cdot & \cdot & \cdot & \cdot \\ a_{i_r+w_r-1 i_r 0}, & a_{i_r+w_r-1 i_r 0}, & \cdots & a_{i_r+w_r-1 . i_r+w_r-1 0} - \phi \end{vmatrix}^{\mu_r} = 0$$

$$(i_r = w_1\mu_1 + \cdots w_{r-1}\mu_{r-i};\ r = 1, 2, \cdots e).$$

The *degree* of this equation is the *degree of the algebra.*

It is also evident that if n is the order of the algebra

$$w_1\mu_1 + w_2\mu_2 + \cdots w_e\mu_e = n \quad \text{or} \quad n + 1,$$

according as we have or have not a *modulus.* When the factors are all linear the algebra is *non-quaternionic* in SCHEFFERS's notation, of type $(1, 1, \cdots, 1)$ in the nomenclature used here; when there is a factor of determinant order two, it is *quaternionic*, or of type $(2, \cdots)$, and so for higher types.* The equation of the algebra determines some of the *units* that define the algebra, but in general there will be units not determined by the equation. If we cut out of the algebra the units which are determined by the characteristic equation, those which are left must form a *nilpotent* algebra, and for every number of such algebra we have $\phi^m = 0$, $(m \leqq n + 1)$.

It was further shown, l. c., p. 275 that if we operate by ϕ on a certain set of n units which define the domain of the algebra, indicated by

$$\phi_{11}, \phi_{21}, \cdots, \phi_{s_0 1}, \phi_{12}, \phi_{22}, \cdots, \phi_{s_1 2}, \cdots, \cdots, \phi_{1\mu_1} - \phi_{w_1\mu_1}, (\mu_1 > \mu_2 \cdots > \mu_e),$$

then

$$\phi \cdot \phi_{ij} = a_{1i0}\phi_{1j} + a_{2i0}\phi_{2j} + \cdots + a_{1i1}\phi_{1j+1} + \cdots + a_{1i2}\phi_{1j+2} + \cdots.$$

In the nilpotent algebra, ϕ_{11} is an idempotent modulus which is used simply to enable us to express the units in a matricular form. We may choose the notation so that $a_{ij0} = 0$ if $j \geqq i$.

§ 2. Since only for $\phi = \phi_{11}$ does $\phi_{ij}\phi$ contain ϕ_{ij}, from the equation last given; since $\phi_{ij}\phi_{11}$ contains ϕ_{ij}, whatever i and j are, and since from the formation of

* *Cf. with reference to this paper and the preceding:* E. CARTAN: *Les Groupes Bilinéaires et les Systemes de Nombres Complexes*, Annales de la Faculté des Sciences de Toulouse, vol. 12 (1898), pp. 81–99.

the original matrical forms for the study of associative algebras,* there is a modulus, in the algebra of order $n+1$, therefore ϕ_{11} is this modulus, and therefore $\phi_{11} = \lambda_{110} + \lambda_{220} + \lambda_{330} + \cdots \lambda_{s_0 s_0 0}$. The algebra *with* ϕ_{11} has the equation

$$(\phi - x_{11}\phi_{11})^m = 0 \qquad (m \leqq n+1).$$

The nilpotent algebra formed by the omission of ϕ_{11} has the equation

$$\phi^m = 0 \qquad (m \leqq n+1).$$

§ 3. Now because ϕ_{11} is the modulus

$$\phi_{ij}\phi_{11} = a_{110}\phi_{11} + a_{210}\phi_{21} + \cdots + a_{111}\phi_{12} + \cdots + a_{112}\phi_{13} + \cdots = \phi_{ij}.$$

Therefore

$$a_{i1j-1} = 1, \quad \text{and} \quad a_{i1s} = 0 \quad \text{if} \quad s \mp j-1.$$

Hence

$$\phi_{ij} = \lambda_{i1j-1} + \phi'_{ij}$$

and ϕ'_{ij} has no constituent of the form $a_{g1h}\lambda_{g1h}$. Obviously no number ϕ of the algebra contains a multiple of ϕ_{11}.

§ 4. It follows at once from the form of ϕ_{ij}, that since λ_{i1j-1} must occur in one number at least, and since

$$\lambda_{i1j-1}\lambda_{111} = \lambda_{i1j},$$

therefore

$$\phi_{ij}\phi_{12} = \phi_{ij+1} + \text{other terms possibly.}$$

But these other terms could arise only from the presence in ϕ_{ij} of terms having the form λ_{r1t}, and there are none such, by the equation above. Hence

$$\phi_{ij}\phi_{12} = \phi_{ij+1}.$$

Therefore the n units of the algebra are expressed completely by the non-vanishing products in the list:

$$\phi_{21},\ \phi_{31},\ \cdots,\ \phi_{s_0 1}; \qquad \phi_{12},\ \phi_{21}\phi_{12},\ \cdots,\ \phi_{s_0 1}\phi_{12}; \qquad \phi_{12}^2,\ \phi_{21}\phi_{12}^2,\ \cdots.$$

§ 5. If we express $\phi_{21}, \phi_{31}, \cdots, \phi_{s_0 1}$ in terms of the λ's, there will be certain terms of weight zero, others of a greater weight. If all those whose weight exceeds zero be cut off from these expressions, it is obvious that the expressions left, which we may represent by $\phi_{210}, \phi_{310}, \cdots, \phi_{s_0 10}$, form a nilpotent algebra of order $s_0 - 1$. By choosing the coefficients properly, it is evident from the form of the expression of the general number of such algebra, ϕ_0, that this nilpotent algebra may be made to be *any* nilpotent algebra of order $s_0 - 1$. Hence, by adjoining to any nilpotent algebra of order $s_0 - 1$, an additional unit,

* *Theory*, p. 254, § 2.

$\phi_{12}, \phi_{12}^{p+1} = 0$, and also annexing to the product of any two units, ϕ_{i10}, ϕ_{j10}, terms of the form $a\phi_{i1} \cdot \phi_{12}^r$ (which is equivalent to changing ϕ_{i10} into ϕ_{i1}, ϕ_{j10} into ϕ_{j1}), we arrive at a new nilpotent algebra of a higher order, the order depending on the number of non-vanishing products $\phi_{i1}\phi_{12}^r$. The original algebra we call the *base*. It is *primary* when it is not derivable from any lower base. The only primary algebras are those of the form

$$\phi_{12}, \phi_{12}^2, \cdots, \phi_{12}^{s_0-1}, \phi_{12}^{s_0} = 0.$$

These algebras are all of those whose *order* equals the *degree* less one.

The adjunction is most simply shown by an example. Suppose the base is the algebra composed of i, j, where $i^2 = j$, $i^3 = 0$. This takes the form

$$i = \phi_{210} = \lambda_{210} + \lambda_{320}, \qquad j = \phi_{310} = \lambda_{310}.$$

The adjoined unit ϕ_{12} will then give us, to fix our ideas, for all nilpotent algebras of order four, on this base, the frames:*

$$(a)\ \phi = a_{210}\lambda_{210} + a_{310}\lambda_{310} + a_{210}\lambda_{320} + a_{111}\lambda_{111} + a_{121}\lambda_{121} + a_{211}\lambda_{211} + a_{221}\lambda_{221},$$

$$(b)\ \phi = a_{210}\lambda_{210} + a_{310}\lambda_{310} + a_{210}\lambda_{320} + a_{111}\lambda_{111} + a_{112}\lambda_{112} + a_{122}\lambda_{122}.$$

In (a),

$$i = \phi_{21} = \lambda_{210} + \lambda_{320} + a\lambda_{121} + b\lambda_{221},$$

$$j = \phi_{31} = \lambda_{310} + a'\lambda_{121} + b'\lambda_{221},$$

$$k = \phi_{12} = \lambda_{111} + c\lambda_{221},$$

$$l = \phi_{21} \cdot \phi_{12} = \lambda_{211}.$$

We now have $i^2 = \lambda_{310} + a\lambda_{111} + a\lambda_{221} + b\lambda_{211} = j + ak + bl$, and $a' = 0$, $a(c-1) + b' = 0$.

In (b),

$$i = \phi_{21} = \lambda_{210} + \lambda_{320} + a\lambda_{122},$$

$$j = \phi_{31} = \lambda_{310} + b\lambda_{122},$$

$$k = \phi_{12} = \lambda_{111} + c\lambda_{122},$$

$$l = \phi_{12}^2 = \lambda_{112}.$$

We have, therefore, $i^2 = \lambda_{310} + a\lambda_{112} = j + al$, $b = 0$, $ki = cl$.

It is to be observed that if

$$\phi_{i10} \cdot \phi_{j10} = a\phi_{k10} + a'\phi_{k'10} + \cdots,$$

* *Theory*, p. 275, § 1.

we may have

$$\phi_{i1}\phi_{j1} = a\phi_{k1} + a'\phi_{k'1} + \cdots + b\phi_{21}\phi_{12} + \cdots + c\phi_{31}\phi_{12} + \cdots.$$

§ 6. We can reduce the multiplication table of any given nilpotent algebra to this canonical form by choosing as the first unit ϵ_1, say, any one of the numbers, preferably that one which has the highest power that does not vanish. Let its successive powers be the next units, ϵ_1^2, ϵ_1^3, $\cdots$. We may now choose from the remaining independent numbers one such that its product into ϵ_{11} shall be free from powers of ϵ_1, say ϵ_2,

$$\epsilon_2\epsilon_1 = \epsilon_2',$$

and

$$\epsilon_2'\epsilon_1 = \epsilon_2'',$$

etc. A third may be so chosen, and so on, until the domain is exhausted. This will give the column of products of ϵ_1 by each unit, in a canonical form, and the number ϵ_1 may be taken as ϕ_{12}. The others are then

$$\phi_{21}, \quad \phi_{21}\phi_{12}, \quad \cdots,$$

$$\phi_{31}, \quad \phi_{31}\phi_{12}, \quad \cdots.$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Thus, let the algebra be PEIRCE's (u_5)

	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	k	0	0	0	k
m	$j + rl$	$(1 + r)k$	0	rk	$j + k + (2r - 1)l$

$$r^3 = -1, \qquad r^2 - r + 1 = 0.$$

Put $m' = m - (1 + r)i$, $l' = rl - rj$; then the first column is

	i
i	j
j	k
k	0
l'	0
m'	l'

According to the general principles worked out in the preceding paper,* the algebra must be a form of

$$\phi = a_{210}\lambda_{210} + a_{111}\lambda_{111} + a_{211}\lambda_{211} + a_{221}\lambda_{221} + a_{112}\lambda_{112} + a_{122}\lambda_{122} + a_{113}\lambda_{113} + a_{123}\lambda_{123}.$$

In fact,

$$\begin{aligned} \phi_{12} &= i = \lambda_{111} - r^2\lambda_{221} - r\lambda_{122}, \\ \phi_{12}^2 &= j = \lambda_{112} - (1+r)\lambda_{123}, \\ \phi_{12}^3 &= k = \lambda_{113}, \\ \phi_{21} &= m' = \lambda_{210} + (r^2-1)\lambda_{221} - (1-r)\lambda_{122} + \lambda_{123}, \\ \phi_{21}\phi_{12} &= l' = \lambda_{211} + r\lambda_{123}. \end{aligned}$$

Whence

$$m = \lambda_{210} - r\lambda_{122} + \lambda_{123} + (1+r)\lambda_{111},$$

$$l = -r^2\lambda_{211} + \lambda_{112} - r\lambda_{123}.$$

In any case, it is obvious that we may take any number as adjunct. It and its powers being removed from the domain, the remaining part of the domain can be represented by $\phi_{21}, \phi_{21}\phi_{12}, \phi_{21}\phi_{12}^2, \cdots \phi_{31}, \phi_{31}\phi_{12}, \cdots$. We then select as *base*, $\phi_{210}, \phi_{310}, \cdots$ differing from $\phi_{21}, \phi_{31}, \cdots$ only in the fact that their products do not contain the terms $\phi_{21}\phi_{12}, \cdots, \phi_{31}\phi_{12}, \cdots$. This base must then be expressed in the form (which is always possible and feasible),

$$\begin{aligned} & a_{210}\lambda_{210} \\ & + a_{310}\lambda_{310} + a_{320}\lambda_{320} \\ & + \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ & + a_{s_010}\lambda_{s_010} + a_{s_020}\lambda_{s_020} + \cdots + a_{s_0s_0-1\,0}\lambda_{s_0s_0-1\,0}. \end{aligned}$$

§ 7. Another theorem is necessary to complete this statement of the possible forms of nilpotent numbers.

It is obvious from the λ forms, (since no one of them can be of the form λ_{iir}) that, if we choose any $n-1$ independent numbers (defining the field), say

$$\phi_1, \phi_2, \cdots, \phi_{n-1},$$

the products

$$\phi_r\phi_s \qquad (r = 1 \cdots n-1;\ s = 1 \cdots n-1),$$

are expressible in terms of at most $n-2$ independent numbers, let us say in terms of $n-1-h_1$.

*Cf. especially pp. 275–276, § 1.

The products $\phi_r \phi_s$ must then form an algebra of $n - 1 - h_1$ units. Let that part of the domain of the original algebra which is excluded from this sub-algebra, be defined by the numbers

$$\epsilon_1', \epsilon_2', \cdots, \epsilon_h'.$$

Let the sub-algebra be denoted by

$$\{\phi''\}.$$

Then $$\epsilon_r' \epsilon_s' = \{\phi''\},$$

i. e., the product of any two units ϵ_r' is in the sub-algebra which excludes the units ϵ_r'.

All triple products $\phi_r \phi_s \phi_t$ also form a sub-algebra of at most $n - 1 - h_1 - 1$ units, say $n - 1 - h_1 - h_2$. Let the domain of $\{\phi''\}$ excluded from this sub-algebra $\{\phi'''\}$ be defined by

$$\epsilon_1'', \epsilon_2'', \cdots, \epsilon_{h2}''.$$

Then we have the equations

$$\epsilon_r'\{\phi''\} = \{\phi'''\}, \qquad \{\phi''\}\epsilon_r' = \{\phi'''\},$$

$$\epsilon_r'\epsilon_s' = \{\epsilon''\phi'''\}, \qquad \epsilon_r'\epsilon_s'' = \{\phi'''\}, \qquad \epsilon_s''\epsilon_r' = \{\phi'''\}.$$

We may so continue separating the field into classes of units, which give the table below, wherein

$$(\epsilon') = (\epsilon_1', \epsilon_2', \cdots, \epsilon_{h1}'),$$

$$(\epsilon'') = (\epsilon_1'', \epsilon_2'', \cdots, \epsilon_{h2}''),$$

	(ϵ')	(ϵ'')	(ϵ''')	$\cdots$
(ϵ')	$\{\phi''\}$	$\{\phi'''\}$	$\{\phi^{iv}\}$	$\cdots$
(ϵ'')	$\{\phi'''\}$	$\{\phi^{iv}\}$	$\cdots$	
(ϵ''')	$\{\phi^{iv}\}$	$\cdots$		

As a corollary* we may say that any nilpotent algebra may be written so that its table is

	1	2	3	$\cdots$	$n-2$	$n-1$
1	$(2 \cdots n-1)$	$(3 \cdots n-1)$	$(4 \cdots n-1)$	$\cdots$	$(n-1)$	0
2	$(3 \cdots n-1)$	$(4 \cdots n-1)$	$\cdots$	$\cdots$	0	0
3	$(4 \cdots n-1)$	. . . .	. . . .	. . . .	. . . .	. . . .
$\vdots$						
$n-2$	$(n-1)$	0	. . . .	. . . .	. . . .	. . . .
$n-1$	0	0	. . . .	. . . .	. . . .	. . . .

* CARTAN, loc. cit., pp. 13–33.

A further corollary: If the order is n, then for any $n+1$ numbers of the algebra,

$$\phi_1 \phi_2 \phi_3 \cdots \phi_{n+1} = 0 .$$

Consequently, in the ϕ notation,

	ϕ_{21}	ϕ_{31}	$\cdots$	ϕ_{m1}	ϕ_{12}	$\phi_{21}\phi_{12}$	$\cdots$
ϕ_{21}	$(\phi_{31}\cdots)$	$(\phi_{41}\cdots)$	$\cdots$	$(\phi_{12}\cdots)$	$(\phi_{21}\phi_{12}\cdots)$	$(\phi_{31}\phi_{12}\cdots)$	$\cdots$
ϕ_{31}	$(\phi_{41}\cdots)$	. . .	. . .	. . .	. . .	. . .	. . .
$\vdots$							
ϕ_{m1}	$(\phi_{12}\cdots)$	. . .	. . .	. . .	. . .	. . .	. . .
ϕ_{12}	$(\phi_{21}\phi_{12}\cdots)$	. . .	. . .	. . .	. . .	. . .	. . .
$\phi_{21}\phi_{12}$	$(\phi_{31}\phi_{12}\cdots)$	. . .	. . .	. . .	. . .	. . .	. . .
$\vdots$							

This is equivalent to the statement that for any term λ_{ijk} occurring in the representation of $\phi_{g1}\cdot\phi_{12}^t$, we must have

$$(1) \qquad k \geqq t,$$

and

$$(2) \quad i > g, \qquad \text{if} \quad k = t,$$

$$(3) \quad i \leqq g, \qquad \text{only when} \quad k > t.$$

For instance, if the frame is

210			
310	320		
410	420	430	
111	121		
211	221		
311	321		
112	122	132	142

then the possible forms are

$$\begin{aligned}\phi_{21} = (210) &+ a_{320}(320) + a_{420}(420) + a_{430}(430)\\ &+ a_{121}(121) + a_{221}(221) + a_{321}(321)\\ &+ a_{122}(122) + a_{132}(132) + a_{142}(142)\end{aligned}$$

$$\phi_{31} = (310) + a'_{420}(420) + a''_{430}(430) + \cdots$$

$$\phi_{41} = (410) + a''_{121}(121) + \cdots$$

$$\phi_{12} = (111) + a'''_{221}(221) + \cdots$$

$$\phi_{21}\phi_{12} = (211) + a^{\text{iv}}_{321}(321) + a^{\text{iv}}_{122}(122) + a^{\text{iv}}_{132}(132) + a^{\text{iv}}_{142}(142),$$

$$\phi_{31}\phi_{12} = (311) + a^{\text{v}}_{122}(122) + a^{\text{v}}_{132}(132) + a^{\text{v}}_{142}(142),$$

$$\phi^2_{12} = (112).$$

Since $\phi^2_{12} = [(111) + a'''_{221}(221) + \cdots]^2 = (112) + a'''_{121}a'''_{221}(122)$ it follows that $a'''_{121} = 0$ or $a'''_{221} = 0$, etc.

We are enabled to reduce the possible forms quite rapidly.

§ 8. It shonld also be remarked that the base is itself subject to the same laws as any other nilpotent algebra, and may be expressed in the same general forms.

Part III. Applications.

§ 1. As the first application of the preceding theory, let us consider *all nilpotent algebras for which the degree = order = n.*

There must then be at least one expression ϕ for which

$$\phi,\ \phi^2,\ \phi^3,\ \cdots,\ \phi^{n-1},$$

are all independent and do not vanish. Let this expression be ϕ_{12}, then the other unit gives a *base* of the form

	ϕ_{21}
ϕ_{21}	0

Hence the form of all such algebras is given by

$$\phi_{21} = \lambda_{210} + a\lambda_{12n-1}, \qquad \phi_{12} = \lambda_{111} + b\lambda_{12n-1}.$$

We need to note that

$$\phi^2_{21} = a\lambda_{11n-1} = a\phi^{n-1}_{12}, \qquad \phi_{21}\phi_{12} = 0, \qquad \phi_{12}\phi_{21} = b\phi^{n-1}_{12}.$$

These results agree with Peirce.*

§ 2. We can extend the applications by considering next the class of *all nilpotent algebras for which degree = order* $- 1 = n - 1$.

The base may be of order one or order two.

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*American Journal of Mathematics, vol. 4 (1884), p. 116.

If the base is of order two, there are two cases, since the two units may combine in two ways. These two algebras are of the forms† corresponding to the two equations

$$\phi^3 = 0, \qquad \text{and} \qquad \phi^2 = 0.$$

In the first case the algebra is

$$\phi_{12}, \phi_{12}^2.$$

In the second case it is ϕ_{21}, ϕ_{12}. The base has the indefinite form

$$\begin{matrix} 210 & \\ 310 & 320 \end{matrix}$$

and

$$(1) \quad \phi_{12} = 210 + 320, \qquad \phi_{12}^2 = 310.$$

$$(2) \quad \phi_{12} = 310, \qquad \phi_{21} = 210.$$

The first case is represented by the forms

$$\phi_{21} = \lambda_{210} + \lambda_{320} + a\lambda_{12n-2} + b\lambda_{13n-2},$$

$$\phi_{21}^2 = \phi_{31} = \lambda_{310} + a\lambda_{11n-2} + b\lambda_{12n-2},$$

$$\phi_{12} = \lambda_{111} + c\lambda_{12n-2} + d\lambda_{13n-2},$$

$$\phi_{21}\phi_{12} = 0 = \phi_{31}\phi_{12}.$$

Since

$$\phi_{12}\phi_{21} = c\lambda_{11n-2} + d\lambda_{12n-2} = c\phi_{12}^{n-2} + c\lambda_{11n}.$$

therefore

$$d = 0, \qquad \phi_{12}\phi_{31} = 0.$$

In a table, these results would appear as follows:

	i	i^2	$\cdots$	i^{n-2}	j	j^2
i	i^2	i^3	$\cdots$	0	ci^{n-2}	0
i^2	i^3	i^4	$\cdots$	0	0	0
$\vdots$	$\vdots$			$\vdots$	$\vdots$	.
i^{n-2}	0	0		0	0	0
j	0	0		0	j^2	0
j^2	0	0		0	0	0

In the second case,

$$\phi_{21} = \lambda_{210} + a\lambda_{12n-2} + b\lambda_{13n-2},$$

$$\phi_{31} = \lambda_{310} + c\lambda_{12n-2} + d\lambda_{13n-2},$$

$$\phi_{12} = \lambda_{111} + e\lambda_{12n-2} + f\lambda_{13n-2}.$$

†American Journal of Mathematics, vol. 4 (1884), p. 121.

Therefore,

$$\phi_{21}^2 = a\phi_{12}^{n-2}, \qquad \phi_{31}^2 = d\phi_{12}^{n-2}, \qquad \phi_{21}\phi_{12} = 0 = \phi_{31}\phi_{12},$$

$$\phi_{12}\phi_{21} = e\phi_{12}^{n-2}, \qquad \phi_{12}\phi_{31} = f\phi_{12}^{n-2}.$$

Tabulated, we have

	i	i^2	i^3	$\cdots$	i^{n-2}	j	k
i	i^2	i^3	i^4	$\cdots$	0	ei^{n-2}	fi^{n-2}
i^2	i^3	i^4	i^5	$\cdots$	0	0	0
i^3	i^4	i^5	i^6	$\cdots$	0	0	0
$\vdots$	$\vdots$						
i^{n-2}	0	0	0	$\cdots$	0	0	0
j	0	0	0	$\cdots$	0	0	0
k	0	0	0	$\cdots$	0	0	0

If the base is of order one,

$$\phi_{21} = \lambda_{210} + a\lambda_{221} + b\lambda_{12n-3} + c\lambda_{12n-2},$$

$$\phi_{12} = \lambda_{111} + a'\lambda_{221} + b'\lambda_{12n-3} + c'\lambda_{12n-2},$$

$$\phi_{21}\phi_{12} = \lambda_{211} + a'b\lambda_{12n-2}.$$

From these equations,

$$\phi_{21}^2 = a\lambda_{211} + b\lambda_{11n-3} + c\lambda_{11n-2} + ab\lambda_{12n-2}.$$

This must be equal to $a\phi_{21}\phi_{12} + b\phi_{12}^{n-3} + c\phi_{12}^{n-2}$.

We have to consider separately the cases $n = 4$ and $n = 5$. *Let us suppose* $n > 5$; then

$$a\lambda_{211} + b\lambda_{11n-3} + c\lambda_{11n-2} + ab\lambda_{12n-2} = a\lambda_{211} + b\lambda_{11n-3} + c\lambda_{11n-2} + aa'b\lambda_{12n-2}.$$

Hence $ab(a'-1) = 0$. Again,

$$\phi_{12}\phi_{21} = a'\lambda_{211} + b'\lambda_{11n-3} + c'\lambda_{11n-2} + ab'\lambda_{12n-2}$$

$$= a'(\lambda_{211} + a'b\lambda_{12n-2}) + b'\phi_{12}^{n-3} + c'\phi_{12}^{n-2};$$

therefore

$$a'^2 b = ab'.$$

Combining this with the preceding, we have

$$a'b = 0, \qquad \text{if} \qquad a = 0;$$

$$ab' = 0, \qquad \text{if} \qquad b = 0;$$

$$b = ab', \qquad \text{if} \qquad a' = 1.$$

The sub-cases are,

(1) $a = 0 = b$, giving $\phi_{21} = \lambda_{210} + c\lambda_{12n-2}$.

(2) $a = 0 = a'$ giving $\phi_{12} = \lambda_{111} + b'\lambda_{12n-3} + c'\lambda_{12n-2}$, $\phi_{21} = \lambda_{210} + b\lambda_{12n-3} + c\lambda_{12n-2}$.

(3) $b = 0 = b'$, giving $\phi_{12} = \lambda_{111} + a'\lambda_{221} + c'\lambda_{12n-2}$, $\phi_{21} = \lambda_{210} + a\lambda_{221} + c\lambda_{12n-2}$.

(4) $a' = 1$, $b = ab'$, giving

$$\phi_{12} = \lambda_{111} + \lambda_{221} + b'\lambda_{12n-3} + c'\lambda_{12n-2},$$

$$\phi_{21} = \lambda_{210} + a\lambda_{221} + ab'\lambda_{12n-3} + c\lambda_{12n-2}.$$

When $n = 4$, we have

$$\phi_{21} = \lambda_{210} + a\lambda_{221} + b\lambda_{121} + c\lambda_{122},$$

$$\phi_{12} = \lambda_{111} + a'\lambda_{221} + c'\lambda_{122},$$

$$\phi_{21}\phi_{12} = \lambda_{211} + a'b\lambda_{122},$$

$$\phi_{12}\phi_{21} = a'\lambda_{211} + c'\lambda_{112} + b\lambda_{122} = a'(\lambda_{211} + a'b\lambda_{122}) + c'\lambda_{112}.$$

Therefore,

$$b(a' + 1)(a' - 1) = 0,$$

$$\phi_{21}^2 = a\lambda_{211} + b\lambda_{111} + c\lambda_{112} + ab\lambda_{122}$$

$$= a(\lambda_{211} + a'b\lambda_{122}) + b(\lambda_{111} + a'\lambda_{221} + c'\lambda_{122}) + c\lambda_{112}.$$

Therefore,

$$aa'b + bc' = ab, \qquad ba' = 0, \qquad b = 0.$$

Hence

$$\phi_{21} = \lambda_{210} + a\lambda_{221} + c\lambda_{122} = j,$$

$$\phi_{12} = \lambda_{111} + a'\lambda_{221} + c'\lambda_{122} = i,$$

$$\phi_{21}\phi_{12} = \lambda_{211},$$

$$\phi_{12}\phi_{21} = a'\lambda_{211} + c'\lambda_{112},$$

or

	i	i^2	j	ji
i	i^2	0	$a'ji + c'i^2$	0
i^2	0	0	0	0
j	ji	0	$aji + ci^2$	0
ji	0	0	0	0

This type embraces a number of forms.*

* See PEIRCE's l_4, n_4, o_4, p_4, q_4, r_4, *loc. cit.*

When $n = 5$, we have

$$\phi_{21} = \lambda_{210} + a\lambda_{221} + b\lambda_{122} + c\lambda_{123},$$

$$\phi_{12} = \lambda_{111} + a'\lambda_{221} + b'\lambda'_{122} + c'\lambda_{123},$$

$$\phi_{21}\phi_{12} = \lambda_{211} + a'b\lambda_{123};$$

$$\phi_{21}^2 = a\lambda_{211} + b\lambda_{112} + c\lambda_{113} + ab\lambda_{123},$$

$$= a(\lambda_{211} + a'b\lambda_{123}) + b(\lambda_{112} + b'\lambda_{123} + a'b'\lambda_{123}) + c\lambda_{113}.$$

Therefore,

$$aa'b + bb' + a'bb' = ab.$$

Again,

$$\phi_{12}\phi_{21} = a'\lambda_{211} + b'\lambda_{112} + c'\lambda_{113} + b\lambda_{123} + ab'\lambda_{123}$$

$$= a'(\lambda_{211} + a'b\lambda_{123}) + b'(\lambda_{112} + b'\lambda_{123} + a'b'\lambda_{123}) + c'\lambda_{113}.$$

Therefore,

$$a'^2b + b'^2 + a'b'^2 = b + ab'.$$

(1) If $b = 0$,

$$b'^2(a' + 1) = ab'.$$

(1_1) If also $b' = 0$, both equations are satisfied. [Case 5_1].

(1_2) If $b' \neq 0$, $b'(a' + 1) = a$. [Case 5_2].

(2) If $b \neq 0$, $aa' + b' + a'b' - a = 0$, or $a = b'\dfrac{a' + 1}{1 - a'}$, if $a' \neq 1$.

(2_1) $a' \neq 1$, $b'^2(a' + 1) = (1 - a')(a'^2 - 1)b + b'^2(a' + 1)$, hence $a' + 1 = 0$, or $a' = -1$, and $a = 0$. [Case 5_3].

(2_2) If $a' = 1$, $bb' = 0$, $b' = 0$. [Case 5_4].

We have then the following cases:

5_1.

$$\phi_{21} = \lambda_{210} + a\lambda_{221} + c\lambda_{123},$$

$$\phi_{12} = \lambda_{111} + a'\lambda_{221} + c'\lambda_{123},$$

$$\phi_{21}\phi_{12} = \lambda_{211}.$$

5_2.

$$\phi_{21} = \lambda_{210} + b'(a' + 1)\lambda_{221} + c\lambda_{123},$$

$$\phi_{12} = \lambda_{111} + a'\lambda_{221} + b'\lambda_{122} + c'\lambda_{123},$$

$$\phi_{21}\phi_{12} = \lambda_{221}.$$

5_3.

$$\phi_{21} = \lambda_{210} + b\lambda_{122} + c\lambda_{123},$$

$$\phi_{12} = \lambda_{111} - \lambda_{221} + b'\lambda_{122} + c'\lambda_{123},$$

$$\phi_{21}\phi_{12} = \lambda_{211} - b\lambda_{123}.$$

5_4.

$$\phi_{21} = \lambda_{210} + a\lambda_{221} + b\lambda_{122} + c\lambda_{123},$$
$$\phi_{12} = \lambda_{111} + \lambda_{221} + c'\lambda_{123},$$
$$\phi_{21}\phi_{12} = \lambda_{211} + b\lambda_{123}.$$

Or,* if $\phi_{12} = i$, $\phi_{21} = j$,

5_1	i	i^2	i^3	j	ji
i	i^2	i^3	0	$a'ji + c'i^3$	0
i^2	i^3	0	0	0	0
i^3	0	0	0	0	0
j	ji	0	0	$aji + ci^3$	0
ji	0	0	0	0	0

5_2	i	i^2	i^3	j	ji
i	i^2	i^3	0	$a'ji + b'i^2 + c'i^3$	$b'i^3$
i^2	i^3	0	0	$(a'+1)b'i^3$	0
i^3	0	0	0	0	0
j	ji	0	0	$(a'+1)b'ji + ci^3$	0
ji	0	0	0	0	0

5_3	i	i^2	i^3	j	ji
i	i^2	i^3	0	$-ji + b'i^2 + c'i^3$	$b'i^3$
i^2	i^3	0	0	0	0
i^3	0	0	0	0	0
j	ji	0	0	$bi^2 + ci^3$	bi^3
ji	0	0	0	$-bi^3$	0

5_4	i	i^2	i^3	j	ji
i	i^2	i^3	0	$ji + c'i^3$	0
i^2	i^3	0	0	0	0
i^3	0	0	0	0	0
j	ji	0	0	$aji + bi^2 + ci^3$	bi^3
ji	0	0	0	bi^3	0

§ 3. The next case would obviously be that in which

$$\phi_{12}^{n-3} = 0.$$

The discussion, however, must be postponed to a later paper. It involves no special difficulties.

§ 4. The problem of finding all algebras which satisfy a given equation of degree m, i. e. such that

$$\phi^m = 0,$$

is an interesting one. The simplest case is that in which

$$\phi^2 = 0.$$

The adjoined unit must be simply λ_{111}. The base units ϕ_i will then give either $\phi_i\lambda_{111} = 0$, or $\phi_i\lambda_{111} \neq 0$.

* These four types represent PEIRCE's t_5 to av_5 inclusive.

In the first case

$$\phi_i = \lambda_{i10} + L_i + a_2^{(i)}\lambda_{121} + a_3^{(i)}\lambda_{131} + \cdots,$$
$$\phi_{i'} = \lambda_{i'10} + L_{i'} + a_2^{(i')}\lambda_{121} + a_b^{(i')}\lambda_{131} + \cdots,$$

where L_i and $L_{i'}$ are the groups of terms in each of the units ϕ_i, $\phi_{i'}$, of the form λ_{rs0} where $s > 1$.

Hence

$$\phi_i\phi_{i'} = L_i\lambda_{i'10} + L_iL_{i'} + a_{i'}^{(i)}\lambda_{111},$$
$$\phi_{i'}\phi_i = L_{i'}\lambda_{i10} + L_{i'}L_i + a_i^{(i')}\lambda_{111}.$$

But for any associative numbers which all satisfy the law $\phi^2 = 0$, we must have

$$\phi_i\phi_{i'} + \phi_{i'}\phi_i = 0.$$

Therefore,

$$a_{i'}^{(i)} + a_i^{(i')} = 0, \qquad \text{or} \qquad a_{i'}^{(i)} = -a_i^{(i')}.$$

In words, ϕ_i cannot contain $a\lambda_{1i'1}$ unless $\phi_{i'}$ contain $-a\lambda_{1i1}$.

It is evident the adjunction does not change the form of the base units in any way, unless for every term of weight unity, $a\lambda_{1j1}$, added to ϕ_{i10}, we add $-a\lambda_{1j1}$ to ϕ_{j10}. Hence we have the process of building up all algebras of this class, those of order $n+1$ from those of order n.

For examples, we have algebras as follows:

Order 1,

$$i = \lambda_{111}.$$

From (1), order 2,

$$i = \lambda_{210},\ j = \lambda_{111}.$$

From (2), order 3,

$$(a),\quad i = \lambda_{210},\ j = \lambda_{310},\ k = \lambda_{111}.$$
$$(b),\quad i = \lambda_{210} + a\lambda_{131},\ j = \lambda_{310} - a\lambda_{121},\ k = \lambda_{111}.$$

From (3a), order 4,

$$(a),\quad i = \lambda_{210},\ j = \lambda_{310},\ k = \lambda_{410},\ l = \lambda_{111}.$$
$$(b),\quad i = \lambda_{210} + a\lambda_{131},\ j = \lambda_{310} - a\lambda_{121},\ k = \lambda_{410},\ l = \lambda_{111},$$
$$(c),\quad i = \lambda_{210} + a\lambda_{131} + b\lambda_{141},\ j = \lambda_{310} - a\lambda_{121} + b'\lambda_{141},$$
$$k = \lambda_{410} - b\lambda_{121} - b'\lambda_{131},\ l = \lambda_{111}.$$

From (3*b*),

(b'), $i=\lambda_{210}+a\lambda_{430}$, $j=\lambda_{310}-a\lambda_{420}$, $k=\lambda_{410}$, $l=\lambda_{111}$,

(d), $i=\lambda_{210}+a\lambda_{430}+b\lambda_{131}$, $j=\lambda_{310}-a\lambda_{420}-b\lambda_{121}$, $k=\lambda_{410}$, $l=\lambda_{111}$.

The forms (b) and (b') are essentially the same.

From (4*a*), order 5,

(a), $i=\lambda_{210}$, $j=\lambda_{310}$, $k=\lambda_{410}$, $l=\lambda_{510}$, $m=\lambda_{111}$.

(b), $i=\lambda_{210}+a\lambda_{151}$, $j=\lambda_{310}$, $k=\lambda_{410}$, $l=\lambda_{510}-a\lambda_{121}$, $m=\lambda_{111}$.

(c), $i=\lambda_{210}+a\lambda_{151}$, $j=\lambda_{310}+b\lambda_{141}$, $k=\lambda_{410}-b\lambda_{131}$, $l=\lambda_{510}-a\lambda_{121}$, $m=\lambda_{111}$.

(d), $i=\lambda_{210}+a\lambda_{151}+a'\lambda_{141}$, $j=\lambda_{310}+b\lambda_{141}+b'\lambda_{151}$,
$k=\lambda_{410}-a\lambda_{121}-b\lambda_{131}$, $l=\lambda_{510}-a\lambda_{121}-b'\lambda_{131}$, $m=\lambda_{111}$.

(e), $i=\lambda_{210}+a\lambda_{151}+a'\lambda_{141}+a''\lambda_{131}$, $j=\lambda_{310}+b\lambda_{141}+b'\lambda_{151}-a''\lambda_{121}$
$k=\lambda_{410}-a\lambda_{121}-b\lambda_{131}$, $l=\lambda_{510}-a\lambda_{121}-b'\lambda_{131}$, $m=\lambda_{111}$.

From (4*b*),

(f), $i=\lambda_{210}+a\lambda_{430}+b\lambda_{151}$, $j=\lambda_{310}-a\lambda_{420}+c\lambda_{151}$, $k=\lambda_{410}$,
$l=\lambda_{510}-b\lambda_{121}-c\lambda_{131}$, $m=\lambda_{111}$.

(g), $i=\lambda_{210}+a\lambda_{430}+b\lambda_{151}+c\lambda_{131}$, $j=\lambda_{310}-a\lambda_{420}+d\lambda_{151}-c\lambda_{121}$,
$k=\lambda_{410}$, $l=\lambda_{510}-b\lambda_{121}-c\lambda_{131}$, $m=\lambda_{111}$.

From (4*c*),

(h), $i=\lambda_{210}+a\lambda_{530}+b\lambda_{540}$, $j=\lambda_{310}-a\lambda_{520}+b'\lambda_{540}$,
$k=\lambda_{410}-b\lambda_{520}-b'\lambda_{530}$, $l=\lambda_{510}$, $m=\lambda_{111}$.

(i), $i=\lambda_{210}+a\lambda_{530}+b\lambda_{540}+c\lambda_{151}$, $j=\lambda_{310}-a\lambda_{520}+b'\lambda_{540}+d\lambda_{151}$,
$k=\lambda_{410}-b\lambda_{520}-b'\lambda_{530}+e\lambda_{151}$, $l=\lambda_{510}-c\lambda_{121}-d\lambda_{131}-e\lambda_{141}$,
$m=\lambda_{111}$.

(j), $i=\lambda_{210}+a\lambda_{530}+b\lambda_{540}+c\lambda_{151}+c'\lambda_{141}+c''\lambda_{131}$,
$j=\lambda_{310}-a\lambda_{520}+b'\lambda_{540}+d\lambda_{151}+d'\lambda_{141}-c''\lambda_{121}$,
$k=\lambda_{410}-b\lambda_{520}-b'\lambda_{530}+e\lambda_{151}-d'\lambda_{131}-c'\lambda_{121}$,
$l=\lambda_{510}-c\lambda_{121}-d\lambda_{131}-e\lambda_{141}$,
$m=\lambda_{111}$.

From $4d$,

$$(k),\quad i = \lambda_{210} + a\lambda_{430} + b\lambda_{530} + c\lambda_{131} + c'\lambda_{141} + c''\lambda_{151},$$
$$j = \lambda_{310} - a\lambda_{420} - b\lambda_{520} + d\lambda_{141} + d'\lambda_{151} - c\lambda_{121},$$
$$k = \lambda_{410} + e\lambda_{151} - c'\lambda_{121} - d\lambda_{131},$$
$$l = \lambda_{510} - c''\lambda_{121} - d'\lambda_{131} - e\lambda_{141},$$
$$m = \lambda_{111}.$$

In the other class, when $\phi_i\phi_{12} \neq 0$, then

$$\phi_i = \lambda_{i10} + L_i + a_{21}^{(i)}\lambda_{121} + \cdots + a_{22}^{(i)}\lambda_{221} + \cdots$$

$$\phi_i\phi_{12} = \lambda_{i11} + \lambda_{i10}L_i + \cdots = -\phi_{12}\phi_i = -a'_{ii}\lambda_{i11} + \cdots.$$

Therefore,

$$a'_{ii} = -1, \qquad a'_{ij} = 0, \qquad i \neq j.$$

Hence

$$\phi_{12} = \lambda_{111} - \delta_1\lambda_{221} - \delta_2\lambda_{331}\cdots, \qquad \delta_1, \delta_2, \cdots = 1 \text{ or } 0.$$

As before, ϕ_i contains $a\lambda_{1i'1}$ only if $\phi_{i'}$ contains $-a\lambda_{1i1}$.

For example,

$$i = \lambda_{210} + \lambda_{431}, \qquad j = \lambda_{310} - \lambda_{421}, \qquad k = \lambda_{410},$$

$$l = \lambda_{111} - \lambda_{221} - \lambda_{331} - \lambda_{441}, \qquad m = \lambda_{211}, \qquad n = \lambda_{311}, \qquad p = \lambda_{411}.$$

Further developments on this line must be deferred. The next problem would evidently be the determination of the forms for the class of algebras satisfying the equation $\phi^3 = 0$. The construction of algebras with idempotents from the nilpotent forms is also a large part of the further development.