SEMIREDUCIBLE HYPERCOMPLEX NUMBER SYSTEMS*

BY

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Introduction.

It is customary to classify hypercomplex number systems as *reducible* and *irreducible* and, owing to the theorem that a reducible system can always be built up out of irreducible ones, the latter only are enumerated. In the following paper, in which only systems with modulus are considered, the irreducible systems are further classified by means of their groups as "semireducible of the first kind" (§ 1), "semireducible of the second kind" (§ 2) and "absolutely irreducible" (§ 3).

§ 1.

If

(1) $$e_1 \cdots e_m \, e_{m+1} \cdots e_n$$

are the n units of a hypercomplex number system containing a modulus, then

$$x = \sum_{i_1=1}^{n} x_{i_1} e_{i_1}, \qquad y = \sum_{i_2=1}^{n} y_{i_2} e_{i_2}$$

are the general numbers of the system. † The product must also be a number of the system

$$x' = xy = \sum_{i_3=1}^{n} x'_{i_3} e_{i_3} = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} x_{i_1} y_{i_2} e_{i_1} e_{i_2};$$

and, since

$$e_{i_1} e_{i_2} = \sum_{i_3=1}^{n} \gamma_{i_1 i_2 i_3} e_{i_3},$$

$$\sum_{i_3=1}^{n} x'_{i_3} e_{i_3} = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \sum_{i_3=1}^{n} \gamma_{i_1 i_2 i_3} y_{i_2} x_{i_1} e_{i_3}.$$

Hence we obtain in the form

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† The scheme of subscripts employed in this paper is the following: The i's, viz., $i_1, i_2, i_3 = 1, \cdots, n$; the j's; $j_1, j_2, j_3 = 1, \cdots, m$; the k's, $k_1, k_2, k_3 = m+1, \cdots, n$.

(2) $$G \equiv x'_{i_3} = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \gamma_{i_1 i_2 i_3} y_{i_2} x_{i_1} \qquad (i_3 = 1, \cdots, n),$$

the group of the system (1),* the x's being regarded as variables, the y's as parameters.

A hypercomplex number system E in the n units $e_1, \cdots, e_n$ is said to be reducible when the system satisfies the following conditions†:

$A1)$ $E_1 \equiv e_1 \cdots e_m$ forms a system by itself; i. e.,

$$e_{j_1} e_{j_2} = \sum_{j_3} \gamma_{j_1 j_2 j_3} e_{j_3} \qquad (\gamma_{j_1 j_2 k} = 0).$$

$A2)$ $E_2 \equiv e_{m+1} \cdots e_n$ forms a system by itself; i. e.,

$$e_{k_1} e_{k_2} = \sum_{k_3} \gamma_{k_1 k_2 k_3} e_{k_3} \qquad (\gamma_{k_1 k_2 j} = 0).$$

$B1)$ $$e_j e_k = 0 \qquad (\gamma_{jki} = 0),$$

$B2)$ $$e_k e_j = 0 \qquad (\gamma_{kji} = 0);$$

otherwise the system is said to be irreducible.‡

Under these conditions the group G (2) is evidently the direct product of two groups. Two general numbers of the system

$$E \equiv E_1 E_2 \equiv e_1 \cdots e_m e_{m+1} \cdots e_n$$

are

$$x = \sum_{j_1} x_{j_1} e_{j_1} + \sum_{k_1} x_{k_1} e_{k_1}, \qquad y = \sum_{j_2} y_{j_2} e_{j_2} + \sum_{k_2} y_{k_2} e_{k_2};$$

their product is

(3) $$x' = xy = \sum_{j_1} \sum_{j_2} x_{j_1} y_{j_2} e_{j_1} e_{j_2} + \sum_{j_1} \sum_{k_2} x_{j_1} y_{k_2} e_{j_1} e_{k_2} + \sum_{k_1} \sum_{j_2} x_{k_1} y_{j_2} e_{k_1} e_{j_2} + \sum_{k_1} \sum_{k_2} x_{k_1} y_{k_2} e_{k_1} e_{k_2}.$$

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* For the conditions imposed on the constants $\gamma_{i_1 i_2 i_3}$ as well as for other details concerning the connection between linear homogeneous groups and hypercomplex number systems see LIE-SCHEFFERS, *Continuierliche Gruppen*, Chapter 21.

† The terms *reducible* and *irreducible* are due to SCHEFFERS, Mathematische Annalen, vol. 39 (1891), p. 317; *Continuierliche Gruppen*, p. 660. As Dr. H. E. HAWKES pointed out in his paper, *Estimate of Peirce's Linear Associative Algebra*, American Journal of Mathematics, vol. 24 (1902), p. 92, these are merely new names for PEIRCE's "mixed" and "pure" systems, ibid., vol. 4 (1881), p. 100.

‡ This is meant in the sense of equivalence; if the units of a given system do not fulfill all of these conditions, but if certain linear combinations with constant coefficients of the units do, then the system will still be called reducible. On the other hand if the given units do not fulfill all of these conditions, nor do any linear combinations with constant coefficients do so either, then the system is irreducible.

According to $(A1)$, $(A2)$, $(B1)$, $(B2)$ we have

$$x' = xy = \sum_{j_1}\sum_{j_2}\sum_{j_3} \gamma_{j_1 j_2 j_3} y_{j_2} x_{j_1} e_{j_3} + \sum_{k_1}\sum_{k_2}\sum_{k_3} \gamma_{k_1 k_2 k_3} y_{k_2} x_{k_1} e_{k_3}.$$

But

$$x' = xy = \sum_{j_3} x'_{j_3} e_{j_3} + \sum_{k_3} x'_{k_3} e_{k_3},$$

and therefore

$$(4) \qquad \begin{aligned} x'_{j_3} &= \sum_{j_1}\sum_{j_2} \gamma_{j_1 j_2 j_3} y_{j_2} x_{j_1} \qquad & (j_3 = 1, \cdots, m), \\ x'_{k_3} &= \sum_{k_1}\sum_{k_2} \gamma_{k_1 k_2 k_3} y_{k_2} x_{k_1} \qquad & (k_3 = m+1, \cdots, n). \end{aligned}$$

The group (4) may be written schematically

$$(4') \qquad \left.\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1m} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & \cdots & a_{mm} & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & a_{m+1m+1} & \cdots & a_{m+1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & a_{nm+1} & \cdots & a_{nn} \end{array}\right. \qquad \text{or} \qquad \begin{array}{c|c} G_{11} & 0 \\ \hline 0 & G_{22} \end{array}.$$

A group is said to be reducible* when it can be put in the form

$$(5) \qquad \left.\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1m} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & \cdots & a_{mm} & 0 & \cdots & 0 \\ \hline a_{m+1\,1} & \cdots & a_{m+1\,m} & a_{m+1\,m+1} & \cdots & a_{m+1\,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nm} & a_{n\,m+1} & \cdots & a_{nn} \end{array}\right. \qquad \text{or} \qquad \begin{array}{c|c} G_{11} & 0 \\ \hline G_{21} & G_{22} \end{array}.$$

It is therefore evident that under the PEIRCE-SCHEFFERS requirements $A1$, $A2$, $B1$, $B2$, the group of the system is the direct product of G_{11} by G_{22}, where G_{11} is the group of E_1 and G_{22} is the group of E_2.

Retaining the conditions $A1$, $A2$, but replacing $B1$, $B2$, by the less exacting conditions:

$$C1) \qquad e_j e_k = \sum_{k_3} \gamma_{jkk_3} e_{k_3}, \qquad (\gamma_{jkj_3} = 0),$$

$$C2) \qquad e_k e_j = \sum_{k_3} \gamma_{kjk_3} e_{k_3}, \qquad (\gamma_{kjj_3} = 0),$$

* LOEWY, Transactions, vol. 4, January, 1903. The present paper is the outcome of a question which Dr. H. E. HAWKES of Yale University once proposed to me in a conversation, viz., "What will be true of the number system when its group is reducible, i. e., has the form (5)"?

we obtain, by substituting in (3),

$$x' = xy = \sum_{j_1}\sum_{j_2}\sum_{j_3} \gamma_{j_1 j_2 j_3} x_{j_1} y_{j_2} e_{j_3} + \sum_{j_1}\sum_{k_2}\sum_{k_3} \gamma_{j_1 k_2 k_3} x_{j_1} y_{k_2} e_{k_3}$$

(6)

$$+ \sum_{k_1}\sum_{j_2}\sum_{k_3} \gamma_{k_1 j_2 k_3} x_{k_1} y_{j_2} e_{k_3} + \sum_{k_1}\sum_{k_2}\sum_{k_3} \gamma_{k_1 k_2 k_3} x_{k_1} y_{k_2} e_{k_3}.$$

But

$$x' = xy = \sum_{j_3} x'_{j_3} e_{j_3} + \sum_{k_3} x'_{k_3} e_{k_3},$$

and therefore,

$$x'_{j_3} = \sum_{j_1}\sum_{j_2} \gamma_{j_1 j_2 j_3}\, y_{j_2} x_{j_1}$$

(7)

$$x'_{k_3} = \sum_{j_1}\sum_{k_2} \gamma_{j_1 k_2 k_3} y_{k_2} x_{j_1} + \sum_{k_1}\sum_{j_2} \gamma_{k_1 j_2 k_3} y_{j_2}\, x_{k_1} + \sum_{k_1}\sum_{k_2} \gamma_{k_1 k_2 k_3} y_{k_2} x_{k_1}$$

$$(j_3 = 1, \cdots, m;\ k_3 = m+1, \cdots, n).$$

The group (7) has the form

$$\begin{array}{c|c} G_{11} & 0 \\ \hline G_{21} & G_{22} \end{array},$$

where G_{22} is

(8)
$$x'_{k_3} = \sum_{k_1} \{ \sum_{j_2} \gamma_{k_1 j_2 k_3} y_{j_2} + \sum_{k_2} \gamma_{k_1 k_2 k_3} y_{k_2} \}\, x_{k_1} \qquad (k_3 = 1, \cdots, m).$$

The group of E_2 is

(9)
$$x'_{k_3} = \sum_{k_1}\sum_{k_2} \gamma_{k_1 k_2 k_3} y_{k_2} x_{k_1},$$

and thus we see that while G_{11} is the group of E_1, G_{22} is not the group of E_2 unless the $\gamma_{k_1 j_2 k_3} = 0$.

It seems advisable to call the number system E *semireducible of the first kind* when the conditions $A1$, $A2$, $C1$, $C2$ are fulfilled; E is said to be reducible when the conditions $A1$, $A2$, $B1$, $B2$ are satisfied. Clearly the latter is a special case of the former.

As an example we may consider the one given by STUDY *

(10)
$$E \equiv E_1 E_2 \equiv (e_1 e_2)(e_3)$$

whose multiplication table is

	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	e_2	e_1	e_3
e_3	e_3	$-e_3$	0

* STUDY, Monatshefte für Mathematik und Physik, vol. 1 (1890), pp. 296, 336.

and whose group is

$$(11) \qquad \begin{aligned} x_1' &= y_1x_1 + y_2x_2, \\ x_2' &= y_2x_1 + y_1x_2, \\ x_3' &= y_3x_1 + y_3x_2 + (y_1 - y_2)x_3. \end{aligned}$$

As the general theory requires,

$$\begin{aligned} x_1' &= y_1x_1 + y_2x_2, \\ x_2' &= y_2x_1 + y_1x_2, \end{aligned}$$

is the group of the system $E_1 \equiv e_1 e_2$.

According to PEIRCE-SCHEFFERS the system E is not reducible since the conditions $B1$ and $B2$ are not satisfied. The system (10) is semireducible of the first kind.

§ 2.

In § 1 it was shown that if the system

$$E = E_1 E_2$$

is semireducible of the first kind its group G is reducible;

$$(7) \qquad G \equiv \frac{G_{11} \mid 0}{G_{21} \mid G_{22}};$$

G_{11} is the group of E_1 but G_{22} may or may not be the group of E_2. In the more general case where G_{22} is not the group of E_2 it is nevertheless highly probable that there always exists a system Q which has G_{22} for its group. The system Q may be called the quotient E/E_1 of the system E, E_1 since its group G_{22} is the quotient $G/\ _{11}$ of the groups G, G_{11}.* In this paper we shall study the special case where the quotient system $Q = E_2$, i. e., where G_{22} is the group of E_2 and the above existence theorem will therefore not be needed. A simple example will suffice at this point to illustrate the more general case. In the example (§ 1, 10, 11) let $y_1 - y_2 = a$, then

$$G_{22} \equiv x_3' = ax_3$$

is clearly not the group of the system $E_2 = e_3\ (e_3^2 = 0)$. There exists however a system

* G_{22} is called the quotient of G by G_{11} for the following reasons: If the linear differential equation $P_n y = 0$ is reducible: $P_n = P_{n-m} P_m$; LOEWY showed (Leipziger Berichte, Jan., 1902) that the group of $P_n y = 0$ has the form (7) where G_{11} is the group of P_m and G_{22} is the group of P_{n-m},

$$P_n = P_{n-m} P_m, \qquad G = G_{22} G_{11}.$$

Therefore we may regard G as the product of G_{22} by G_{11} just as we regard P_n as the product of P_{n-m} by P_m.

$$Q \equiv q$$

which has G_{22} for its group, namely, the system whose multiplication table is

	q
q	q

.

Although the following table anticipates § 3 to some extent, it will be easily intelligible.

Products of Units.	Conditions on γ's.	Conditions on Number System.	Name of System.	Group.
$(A1)\ e_{j_1} e_{j_2} = \sum_{j_3} \gamma_{j_1 j_2 j_3} e_{j_3}$	$\gamma_{j_1 j_2 k} = 0$	$A1, A2, C1, C2$	Irreducible but semireducible of the first kind.	G is reducible, G_{22} is not necessarily the group of E_2.
$(A2)\ e_{k_1} e_{k_2} = \sum_{k_3} \gamma_{k_1 k_2 k_3} e_{k_3}$	$\gamma_{k_1 k_2 j} = 0$			
$(B1)\ e_j e_k = 0$	$\gamma_{jki} = 0$	$A1, A2, C1, B2$	Irreducible but semireducible of the second kind.	G is reducible, G_{22} is the group of E_2.
$(B2)\ e_k e_j = 0$	$\gamma_{kji} = 0$	$A1, A2, B1, B2$	Reducible.	G is reducible and is direct product of G_{22} and G_{11}.
$(C1)\ e_j e_k = \sum_{k_3} \gamma_{jkk_3} e_{k_3}$	$\gamma_{jkj_3} = 0$			
$(C2)\ e_k e_j = \sum_{k_3} \gamma_{kjk_3} e_{k_3}$	$\gamma_{kjj_3} = 0$		Absolutely irreducible.	G is irreducible.

§ 3.

In accordance with the above table the conditions

$$A1, \quad A2, \quad C1, \quad B2$$

are less exacting than those of PEIRCE-SCHEFFERS but more exacting than $A1$, $A2$, $C1$, $C2$; systems satisfying them will be called *semireducible of the second kind.* Semireducibility of the second kind is a special case of semireducibility of the first kind and reducibility is a special case of semireducibility of the second kind.

A system will be said to be *absolutely irreducible* when its group is irreducible. *

* [August 26, 1903. As initially given the definition of absolute irreducibility (cf. also § 2 table) was erroneous; the correction involved an important change in interpretation in § 3. I am indebted to Professor E. H. MOORE for calling my attention to this. Much of the paper was rewritten as a result of his friendly criticism.]

When the system is semireducible of the second kind, the condition $B2$ makes every $\gamma_{k_1 j_2 k_3}$ vanish and consequently, G_{22} (8) and the group (9) become the same. *In semireducible number systems of the second kind,* G_{11} *is the group of* E_1 *and* G_{22} *is the group of* E_2. Or, *when conditions* $A1, A2, C1, B2$ *are satisfied, then* $E/E_1 = E_2$.

The necessary and sufficient conditions that a number system shall be semireducible of the second kind are that its group G shall be reducible, that G_{11} shall be the group of E_1 and G_{22} the group of E_2.

We proceed to consider the cases in which the system

$$(12)\quad E = E_1 E_2 \cdots E_{p-1} E_p \equiv e_1 \cdots e_{m_1} e_{m_1+1} \cdots e_{m_2} \cdots e_{m_{p-2}+1} \cdots e_{m_{p-1}} e_{m_{p-1}+1} \cdots e_n$$

has the following properties: by a suitable choice of the units it is possible to decompose E in such a way that

$$E_1 \cdots E_h E_{h+1} \qquad (h = 1, \cdots, p-1)$$

is semireducible of the second kind, E_{h+1} being absolutely irreducible. By a linear transformation

$$(13)\qquad \bar{e}_{i_1} = \sum_{i_2} \alpha_{i_1 i_2} e_{i_2}, \qquad |\alpha_{i_1 i_2}| \neq 0,$$

the system $E = E_1 \cdots E_p$ may be similarly decomposed in the form $\overline{E} \equiv \overline{E}_1 \cdots \overline{E}_q$. The systems E and $\overline{E}$ are said to be of the same type * and are not regarded as essentially different.

Let the group of E be G and that of $\overline{E}$ be $\overline{G}$. Since E is semireducible of the second kind G has the form

$$G \equiv \begin{matrix} G_{11} & 0 & 0 & \cdots & 0 \\ G_{21} & G_{22} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ G_{p1} & G_{p2} & G_{p3} & \cdots & G_{pp} \end{matrix}$$

where G_{11} is the group of E_1, G_{22} is the group of E_2, $\cdots$, G_{pp} is the group of E_p.

The group of $\overline{G}$ of $\overline{E}$ is obtained from G by a transformation of variables.† G and $\overline{G}$ are thus similar, $\overline{G}$ has the form

$$\overline{G} \equiv \begin{matrix} \overline{G}_{11} & 0 & 0 & \cdots & 0 \\ \overline{G}_{21} & \overline{G}_{22} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \overline{G}_{q1} & \overline{G}_{q2} & G_{q3} & \cdots & G_{qq} \end{matrix}$$

* *Continuierliche Gruppen*, pp. 642–3.

† Loc. cit., p. 643.

According to LOEWY's fundamental theorem * $q = p$ and

$$G_{11},\ G_{22},\ \cdots,\ G_{pp}$$

are similar in some order to

$$\overline{G}_{11},\ \overline{G}_{22},\ \cdots,\ \overline{G}_{pp};$$

let us say that $G_{\alpha\alpha}$ is similar to $\overline{G}_{\beta_\alpha\beta_\alpha}$ $(\alpha = 1, \cdots, p)$.

Thus it is evident that $\overline{E}$ *decomposes into the same number of subsystems as* $E (q = p)$. Furthermore, owing to the similarity of $G_{\alpha\alpha}$ with $\overline{G}_{\beta_\alpha\beta_\alpha}$, the subsystems E_α and $\overline{E}_\beta$ must be similar $(\alpha = 1, \cdots, p)$. Hence, in analogy with LOEWY's theorem for reducible linear homogeneous groups, we have proved for number systems the following theorem:

If a system E is, according to two different choices of the units, decomposable once into the sequence of p absolutely irreducible systems $E_1, \cdots, E_p$, where the system

$$E_1 \cdots E_h E_{h+1}$$

is semireducible of the second kind $(h = 1, 2, \cdots, p-1)$, and again similarly decomposable into the sequence of q systems $\overline{E}_1 \cdots \overline{E}_q$, then $q = p$ and the subsystems $\overline{E}_1, \cdots, \overline{E}_q$ are similar to the subsystems $E_1, \cdots, E_p$, apart from the order. In other words, *similar systems being regarded as not different, the absolutely irreducible subsystems obtained by a linear transformation* (13) *are equivalent, apart from the arrangement, to the subsystems obtained by any other such transformation.*

The preceding theorem includes, of course, the particular case in which semireducibility of the second kind is the reducibility in the sense of PEIRCE-SCHEFFERS. Then the group is the direct product of the groups of the component systems.

THE UNIVERSITY OF CHICAGO,
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* LOEWY, Transactions, vol. 4, January, 1903, pp. 46–47.