ON THE STRUCTURE OF HYPERCOMPLEX NUMBER SYSTEMS*

BY

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In this paper theorems regarding linear associative algebras are enunciated which are analogous to the JORDAN-HÖLDER theorems concerning the quotient groups and indices of composition in the theory of finite groups (cf. BURNSIDE, *Theory of Groups*, chapter 7); and of the corresponding VESSIOT-ENGEL theorems in the theory of continuous groups (*Annales de l'École Normale Supérieure*, vol. 9 (1892), p. 203; LIE-ENGEL, *Transformationsgruppen*, vol. 3, pp. 704, 765).

The methods used throughout are rational and hence the results apply to hypercomplex number systems in which the coefficients are restricted to be marks of a given field, finite or infinite.

§ 1.

We consider the linear associative algebra

$$(1) \quad E \equiv e_1 \cdots e_m e_{m+1} \cdots e_r e_{r+1} \cdots e_n,$$

where

$$(2) \quad e_{i_1} e_{i_2} = \sum_{i_3} \gamma_{i_1 i_2 i_3} e_{i_3} \quad (i = 1, \cdots, n).$$

From the associativity property $(e_{i_1} e_{i_2}) e_{i_3} = e_{i_1} (e_{i_2} e_{i_3})$ and the linear independence of the units are deduced the relations

$$(3) \quad \sum_{i_3} (\gamma_{i_1 i_2 i_3} \gamma_{i_3 i_4 i_5} - \gamma_{i_1 i_3 i_5} \gamma_{i_2 i_4 i_3}) = 0 \quad (i_1, i_2, i_4, i_5 = 1, \cdots, n).$$

Addition of complexes. If $X_1, \cdots, X_m$ are any m independent numbers of the system E , all numbers linearly dependent on the $X_1, \cdots, X_m$ and only on these numbers are said to form a complex of order m . We may write $E_1 = X_1 \cdots X_m$. If, similarly, $E_2 = X'_1 \cdots X'_{m'}$, the complex formed of all members linearly dependent on $X_1, \cdots, X_m, X'_1, \cdots, X'_{m'}$ is called the sum of E_1 and E_2 . The order of $E_1 + E_2$ is not greater than the sum of the orders of E_1 and E_2 but may be less; thus if $E_1 = e_1 \cdots e_m \cdots e_r$, $E_2 = e_{m+1} \cdots e_r \cdots e_{m'}$, the sum $E_1 + E_2$ is of order m' . According to this convention, $E + E_1 = E$, $E + E_2 = E$, $E + E = E$.

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*Multiplication of complexes.** When E_1 and E_2 are defined as above, the product $E_1 E_2$ is defined as the complex of all numbers linearly dependent on $X_p X'_q$ ($p = 1, \dots, m; q = 1, \dots, m'$). It is easily shown from the definition of multiplication of complexes that the process is associative. Thus if we consider the system

$$\begin{array}{cc|cc} & & e_1 & e_2 \\ e_1 & & e_1 & e_2 \\ e_2 & & e_2 & 0 \end{array}$$

and let $E_1 \equiv e_1$, $E_2 \equiv e_2$, then $E_1 E_2 = e_1 e_2 = e_2$. It can easily be seen that if the main system E contains a modulus (say e_1) then $EE(=E^2) = E$; for, the modulus (unit) e_1 multiplied by $e_1 \dots e_n$ on the right gives $e_1 \dots e_n$ and therefore E^2 contains all the units of E . If E does not contain a modulus it may happen that in E^2 some of the units $e_1, \dots, e_n$ do not appear and therefore $E^2 \subsetneq E$. The condition that E be a system, closed under multiplication, is $E^2 \supseteq E$.

Semireducibility. The algebra E is said to be semireducible† when by a proper choice of units the following conditions are fulfilled :

$$\left. \begin{array}{l} (4) \quad X \cdot {}_2X = {}_2X' \\ (5) \quad {}_2X \cdot X = {}_2X'' \end{array} \right\} \text{for every } X, {}_2X,$$

the general number of E being

$$X = {}_1X + {}_2X' = \sum_{j=1}^m x_j e_j + \sum_{k=m+1}^n x_k e_k.$$

It follows at once that the complex $E_2 = e_{m+1} \dots e_n$ is closed under multiplication. The conditions (4) and (5) may be written

$$(4') \quad E \cdot E_2 \supseteq E_2,$$

$$(5') \quad E_2 E \supseteq E_2,$$

and, therefore, of course, $E_2 E_2 = E_2^2 \supseteq E_2$. Under these conditions the sub-algebra E_2 is said to be invariant in E .‡

THEOREM. E_2 cannot contain the modulus of E among its numbers. For, from (4) and (5) it is impossible that any number ${}_2I$ of E_2 shall have the property that ${}_2I \cdot X = X \cdot {}_2I = X$ for every number X of E .

Accompanying system, § complementary system. If

$$E = E_1 + E_2 \equiv e_1 \dots e_m e_{m+1} \dots e_n,$$

* FROBENIUS, Berliner Sitzungsberichte, 1895, p. 164.

† Transactions, vol. 4 (1903), p. 440.

‡ CARTAN, Annales de Toulouse, vol. 12 (1898).

§ MOLIER'S Begleitendes System, Mathematische Annalen, vol. 41 (1893), p. 93.

we may write, in well known notation, $E = E_1(\text{mod } E_2)$, $E = E_2(\text{mod } E_1)$; similarly, if $X = {}_1X + {}_2X$, where X belongs to E , ${}_1X$ to E_1 , ${}_2X$ to E_2 , we may write $X = {}_1X(\text{mod } E_2)$ or $X = {}_2X(\text{mod } E_1)$. Regarding as equal* any two numbers of E which are equal, modulo E_2 , we get, when E_2 is an invariant subsystem, a hypercomplex number system K which is said to accompany E and to be complementary to the invariant subsystem E_2 with respect to E . It is evidently sufficient to show that the system so defined is associative. If ${}_1X_\alpha$, ${}_1X_\beta$, ${}_1X_\gamma$ are three numbers of E_1 and if

$${}_1X_\alpha \cdot {}_1X_\beta = X_{\alpha\beta} = {}_1X_{\alpha\beta} + {}_2X_{\alpha\beta}, \quad {}_1X_\beta \cdot {}_1X_\gamma = X_{\beta\gamma} = {}_1X_{\beta\gamma} + {}_2X_{\beta\gamma},$$

then

$$({}_1X_\alpha \cdot {}_1X_\beta) \cdot {}_1X_\gamma = {}_1X_{\alpha\beta} \cdot {}_1X_\gamma = {}_1X_{(\alpha\beta)\gamma} + {}_2X',$$

if $E_2 E_1 \subseteq E_2$. Similarly

$${}_1X_\alpha \cdot ({}_1X_\beta \cdot {}_1X_\gamma) = {}_1X_{\alpha(\beta\gamma)} + {}_2X''.$$

Now by the associative law and the linear independence of the units we have

$${}_1X_{(\alpha\beta)\gamma} \equiv {}_1X_{\alpha(\beta\gamma)},$$

and therefore

$$({}_1X_\alpha \cdot {}_1X_\beta) \cdot {}_1X_\gamma \equiv {}_1X_\alpha \cdot ({}_1X_\beta \cdot {}_1X_\gamma) (\text{mod } E_2).$$

The above may also be deduced from a consideration of the $\gamma_{i_1 i_2 i_3}$'s. Using (3) it is seen that

$$(3_j) \quad \sum_{i_3=1}^n (\gamma_{j_1 j_2 i_3} \gamma_{i_3 j_4 j_5} - \gamma_{j_1 i_3 j_5} \gamma_{j_2 j_4 i_3}) = 0 \quad (j_1, j_2, j_4, j_5 = 1, \dots, m).$$

Now if

$$E_2 E_1 \subseteq E_2, \quad E_1 E_2 \subseteq E_2,$$

or,

$${}_2X \cdot {}_1X = {}_2X', \quad {}_1X \cdot {}_2X = {}_2X'',$$

that is to say, if every $\gamma_{kjj} = 0$ and every $\gamma_{jkk} = 0$, we may write (3_j) in the form

$$(3'_j) \quad \sum_{j_3=1}^m (\gamma_{j_1 j_2 j_3} \gamma_{j_3 j_4 j_5} - \gamma_{j_1 j_3 j_5} \gamma_{j_2 j_4 j_3}) = 0. \dagger$$

§ 2.

By $E_1 \cap E_2$ is meant the common complex of E_1 and E_2 . Thus if in (1), $E_1 \equiv e_1 \dots e_m \dots e_r$, $E_2 \equiv e_{m+1} \dots e_r \dots e_n$, then $E_1 \cap E_2 \equiv e_{m+1} \dots e_r$. $E_1 \cap E_2 = 0$ states that E_1 and E_2 have no numbers in common.

* FROBENIUS, l. c., p. 634.

† According to MOLIEN, loc. cit., p. 92, it is necessary that $EE_2 \subseteq E_2$, $E_2 E \subseteq E_2$. Evidently, from the text, we can have an accompanying system with the milder conditions $E_2 E_1 \subseteq E_2$, $E_1 E_2 \subseteq E_2$.

It may be remarked at this point that the γ_{ijk} are the structure constants of K .

Reducibility. According to PEIRCE* an algebra E is said to be mixed (or reducible) if new units can be introduced such that $E = E_1 + E_2$, where $E_1^2 \equiv E_1$, $E_2^2 \equiv E_2$, $E_1 E_2 \equiv E_1 \sim E_2$, $E_2 E_1 \equiv E \sim E_2$.†

THEOREM I. *If E_1 is a maximal invariant subalgebra of E and if there exists a second invariant subalgebra E'_1 of E , then either E is reducible or else E'_1 is a subalgebra E_1 .*

Since E_1 and E'_1 are invariant we have (a) $EE_1 \equiv E_1$, (b) $E_1 E'_1 \equiv E_1$, (c) $EE'_1 \equiv E'_1$, (d) $E'_1 E \equiv E'_1$. We have now that $E_1 + E'_1$ is an invariant subalgebra of E for $E(E_1 + E'_1) \equiv E_1 + E'_1$, $(E_1 + E'_1)E \equiv E_1 + E'_1$ and since E_1 is maximal by hypothesis $E_1 + E'_1 = E$ or else $E_1 + E'_1 = E_1$. If $E_1 + E'_1 = E$ the algebra is easily seen to be reducible. First, $E_1^2 \equiv E_1$ by (a), $E_1'^2 \equiv E'_1$ by (c), $E_1 E'_1 \equiv E_1$ by (b) and $E'_1 E_1 \equiv E'_1$ by (d) and therefore $E_1 E'_1 \equiv E_1 \sim E'_1$. Similarly by (b) and (d), $E'_1 E_1 \equiv E_1 \sim E'_1$. In the second case, where $E_1 + E'_1 = E_1$, it is evident that E'_1 is a subalgebra of E_1 .

THEOREM II. *If E_1 and E'_1 are maximal invariant subalgebras of E , and if $E_1 \sim E'_1 = F \neq 0$, then F is a maximal invariant subalgebra of both E_1 and E'_1 .*

By theorem I, $E = E_1 + E'_1$. Let

$$\begin{aligned} E &= E_0 + E_1 & (E_0 \sim E_1 = 0), \\ &= E'_0 + E'_1 & (E'_0 \sim E'_1 = 0). \end{aligned}$$

Evidently then $E = E_0 + E'_0 + F$. Because the products $E_1 F$, $F E_1$, $E'_1 F$, $F E'_1$ must be contained in E_1 and in E'_1 it follows that

$$E_1 F \equiv F, \quad F E_1 \equiv F, \quad E'_1 F \equiv F, \quad F E'_1 \equiv F,$$

and therefore that F is invariant in both E_1 and E'_1 .

Suppose now that F is not maximal in E_1 and let $G + F$ (where $G \sim F = 0$) be a maximal invariant subsystem of E_1 , i. e.,

$$E_1(G + F) \equiv G + F, \quad (G + F)E_1 \equiv G + F.$$

Consider the complex $G + E'_1$. Clearly $G \sim E'_1 = 0$ and

$$G E'_1 \equiv F, \quad E'_1 G \equiv F,$$

since the products must both lie in E_1 and E'_1 . Therefore, since

$$E_1 G \equiv G + F, \quad E_1 E'_1 \equiv F, \quad E'_1 G \equiv F, \quad E_1'^2 \equiv E'_1,$$

we have

$$E(G + E'_1) = (E_1 + E'_1)(G + E'_1) \equiv G = E'_1.$$

It is shown in a similar manner that $(G + E'_1)E \equiv G + E'_1$. Thus, if F is not a maximal invariant subalgebra of E_1 , E'_1 will not be a maximal invariant

* American Journal, vol. 4 (1881), p. 100.

† In SCHEFFERS's definition, *Mathematische Annalen*, vol. 39 (1891), p. 317, $E_1 \sim E_2 = 0$.

subalgebra of E (since $G + E'_1$ has been proved invariant and $G \cap E'_1 = 0$), which is contrary to the hypothesis. It is shown in exactly the same way that F is maximal in E'_1 .

THEOREM III. *Let $E, E_1, E_2, \dots$ be a normal series of subalgebras of E (i. e., E_r is a maximal invariant subalgebra of E_{r-1} , $E_0 = E$) and let $K_1, K_2, \dots$ be a series of complementary algebras, such that K_r accompanies E_{r-1} and is complementary to E_r . Under these assumptions the series $K_1, K_2, \dots$ is, apart from the order, independent of the choice of the series $E, E_1, E_2, \dots$. In other words, if $E, E'_1, E'_2, \dots$ is any other normal series, the complementary series of algebras $K'_1, K'_2, \dots$ which it defines is the same as the series $K_1, K_2, \dots$, apart from the sequence.*

According to the demonstration of theorem 1, we have (if $E_1 \neq E'_1$)

$$(6) \quad E = E_1 + E'_1.$$

Let $E_1 \cap E'_1 = F_1$, then

$$(7) \quad E_1 = F_1 + D_1 \quad (F_1 \cap D_1 = 0),$$

$$E'_1 = F_1 + D'_1 \quad (F_1 \cap D'_1 = 0),$$

and it is evident that

$$(8) \quad E = F_1 + D_1 + D'_1.$$

By theorem 2, F_1 is a maximal invariant subalgebra of E_1 and E'_1 .

If F_1, F_2, F_3 is a normal series of F_1 , then $E, E_1, F_1, F_2, \dots$ and $E, E'_1, F_1, F_2, \dots$ are two new normal series of E . We will now have to prove that the proposition is true for these two series. For this purpose it is merely necessary to prove that the complementary algebras defined by E, E_1, F_1 and E, E'_1, F_1 are the same. If

$$E \equiv e_1 \cdots e_{d_1} e_{d_1+1} \cdots e_t e_{t+1} \cdots e_n,$$

$$E_1 \equiv e_1 \cdots e_{d_1} e_{d_1+1} \cdots e_t,$$

$$E'_1 \equiv e_{d_1+1} \cdots e_t e_{t+1} \cdots e_n,$$

then

$$E_1 \cap E'_1 = F_1 \equiv e_{d_1+1} \cdots e_t,$$

$$D_1 \equiv e_1 \cdots e_{d_1}, \quad D'_1 \equiv e_{t+1} \cdots e_n.$$

Now the complementary algebra of $E_1 = F_1 + D_1$ with respect to E is defined by the $(n - t)^3$ characteristic constants $\gamma_{p_1 p_2 p_3}$ where

$$(9) \quad \sum_{p_3} (\gamma_{p_1 p_2 p_3} \gamma_{p_3 p_4 p_5} - \gamma_{p_1 p_3 p_5} \gamma_{p_2 p_4 p_3}) = 0 \quad (p = t + 1, t + 2, \dots, n).$$

Furthermore, the complementary algebra of F_1 with respect to $E_1 (E_1 = F_1 + D_1)$ is defined by the d_1^3 characteristic constants $\gamma_{q_1 q_2 q_3}$ where

$$(10) \quad \sum_{q_3} (\gamma_{q_1 q_2 q_3} \gamma_{q_3 q_4 q_5} - \gamma_{q_1 q_3 q_5} \gamma_{q_2 q_4 q_3}) = 0 \quad (q = 1, \dots, d_1).$$

Now it is seen in the same manner that the complementary algebra of E'_1 with respect to E is defined by (10) and the complementary algebra of F'_1 with respect to E'_1 is defined by (9), and therefore the normal complementary series defined by E, E_1, F_1 is identical, apart from the sequence, with the normal complementary series defined by E, E'_1, F_1 .

It is now necessary to show that the normal series $E, E_1, E_2 \dots, E, E_1, F_1, F_2, \dots$ define the same complementary series, and similarly for $E, E'_1, E'_2, \dots, E, E'_1, F_1, F_2, \dots$. This amounts to proving the theorem for the algebras E_1 and E'_1 , which involves a finite number of repetitions of the above proof.

§ 3.

Let $E, E_1, E_2, \dots$ be a normal series of E , let a_s be the order (number of units) of E_s and let l_s be the difference between a_{s-1} and the maximal order of a subalgebra* of E_{s-1} which contains E_s ($E_0 = E$).

THEOREM IV. *The numbers $l_1, l_2, \dots$, apart from their sequence, are independent of the choice of the normal series.*

This theorem is an immediate consequence of theorem 3, the integer l_s being regarded as the difference between the order of the complementary algebra K_s of E_s with respect to E_{s-1} and the order of a maximal subalgebra of K_s ($s = 1, 2, \dots$).

Similarly it follows also that the series of integers $k_s = a_{s-1} - a_s$ is independent of the choice of the normal series.

Simple algebras. An algebra which contains no invariant subalgebra is said to be simple.

THEOREM V. *The complementary algebras $K_1, K_2, \dots$ defined by the normal series $E, E_1, E_2 \dots$ are all simple.†*

In order to prove this, consider the algebra $A = B + A_1$ (where $B \cap A_1 = 0$) wherein A_1 is a maximal invariant subalgebra, and let K be the complementary system of A_1 with respect to A . Suppose now that K has an invariant subalgebra $K_1, K = K' + K_1$ (where $K' \cap K_1 = 0$). This says that

$$(11) \quad KK_1 \subseteq K_1, \quad K_1K \subseteq K_1.$$

From the correspondence between K and B ‡ it is seen that we can write $B = B' + B_1$ (where $B' \cap B_1 = 0$) where, by means of (11),

$$(12) \quad BB_1 \subseteq B_1 + A_1, \quad B_1B \subseteq B_1 + A_1.$$

Consider now the complex $B_1 + A_1$. We have, since A_1 is invariant in A , and by means of (12),

* i. e., the order of the subalgebra containing the largest number of independent units.

† This theorem is implicitly contained in MOLIER's paper, loc. cit., p. 96.

‡ The γ_{jk} of K are obtained by writing every $\gamma_{jk} = 0$ in the γ_{jk} of B .

$$A(B_1 + A_1) = (B + A_1)(B_1 + A_1) \equiv B_1 + A_1;$$

and similarly

$$(B_1 + A_1)A \equiv B + A_1.$$

Thus $B_1 + A_1$ is an invariant subalgebra of A and since $B_1 \cap A_1 = 0$, it follows that $B_1 + A_1 > A_1$ which is impossible since A_1 is maximal by hypothesis. Therefore K is simple.

The chief series of an algebra. The series $E, P_1, P_2, \dots$ is said to be a chief or principal series (chief composition series) when P_i is a maximal subalgebra of P_{i-1} which is invariant in E ($P_0 = E$). For the chief series it is shown in exactly the same manner as for the normal series that *the system of complementary algebras $C_1, C_2, C_3, \dots$ is invariant, apart from the sequence (C_i being complementary to P_i with respect to P_{i-1}).*

We can now define a series of chief indices of composition analogous to the indices of composition above, and show that for a given algebra E , *the system of chief indices of composition is independent of the choice of the chief series, apart from the sequence.*

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