ON A GENERAL METHOD FOR TREATING TRANSMITTED MOTIONS AND ITS APPLICATION TO INDIRECT PERTURBATIONS*

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The mathematical treatment of any physical problem demands the construction of an ideal problem in which the conditions are different from those of the actual problem. It is assumed that the same general laws hold for the actual and ideal problems, but the complexity of the circumstances surrounding the former makes simplifications of some kind necessary in order that the analysis should not be unreasonably tedious. The ideal problem is therefore usually constructed by neglecting at first some of the influences which form a part of it but which are assumed to affect the results to a much smaller extent than those we retain. The simplified problem, which I call problem A, is then solved. The second step consists in finding what changes are necessary in the solution of problem A when some or all of the neglected influences are included; this second problem I call B. The question under consideration here is the deduction of the solution of B when that of A has been found.

One of the methods for solving B is that known as the *Variation of Arbitrary Constants*. In the deduction of the solution of A, arbitrary constants a arise, and numerical values, depending on the given conditions of the problem, can be assigned to them. In order to obtain the solution of B, it is assumed that the solution of A is still available provided that a different set of values be assigned to the constants a. But the new set of values will usually be different at different times and will therefore depend on the time; in other words, the constants a become variables in problem B.

This last fact gives us another view of the relation between A and B. Suppose that the position of the system which constitutes B is defined by variables x, and that we desire to change from the variables x to other variables p in order to simplify the treatment of B. Then one p-set, namely, the a-set, can be obtained by supposing that the x-set and the p-set are connected by the same equations which connect the x-set and the a-set in the solution of problem A.

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Thus A is no longer looked upon as a dynamical problem but as a collection of relations which give certain transformations useful for the treatment of B.

But we can go further than this. There is no special necessity imposed on us to use any of the a-sets thus found. We may be able to discover another set of variables, say a p'-set, allied to the x-set by relations different from those which connect the x-set and the a-set — relations suggested indeed by the latter and perhaps possessing similar properties, but possessing also additional properties more useful for the purpose in hand.

It is proposed to investigate here one such p'-set. The relations between the x- and a-sets usually contain a certain number of other constants α which were present in the original equations of problem A. It is assumed that the x- and p'-sets are connected by relations of the same *form* as those which connect the x- and a-sets, but that the α have given *variable* instead of given *constant* values assigned to them in the p'-set. Certain advantages are gained in the application of the method to the consideration of the 'indirect inequalities' in the motion of the moon, i. e., in the consideration of those perturbations of the lunar motion which arise from the action of a planet on the earth's motion and which are transmitted to the moon on account of the predominating influence of the earth on the moon. I have given some indications of this p'-set in a former paper.* Unfortunately, the main idea was obscured by the method adopted to develop it, and some statements concerning the applicability of it to the lunar problem require further limitations than those previously imposed. The present method of treatment also brings to light a theorem (art. 10 below) which I believe is new and which is of considerable importance in the discussion of the indirect inequalities.

1. *The method of Jacobi.* The equations of a dynamical system are supposed to have been expressed in the canonical form

$$(1) \qquad \frac{dx_i}{dt}=\frac{\partial\phi}{\partial y_i},\qquad \frac{dy_i}{dt}=-\frac{\partial\phi}{\partial x_i}\qquad [\phi=\phi(x_i,\,y_i,\,t)],$$

where, to fix ideas, the n variables x_i may be thought as the coördinates defining the position of the system, the n variables y_i as the corresponding momenta, and ϕ as the sum of the kinetic and potential energies; these interpretations are nevertheless not necessary.

Let the variables† x, y be changed to $2n$ new variables p, q, connected with x, y by the relations

* *On the variation of the arbitrary and given constants in dynamical equations*, Transactions of the American Mathematical Society, vol. 4 (1903), pp. 333–350.

† The suffixes will be omitted whenever no confusion is thereby caused. In all cases i, j take the values $1, 2, \cdots, n$.

$$y_i = \frac{\partial S}{\partial x_i}, \qquad q_i = \frac{\partial S}{\partial p_i} \qquad [S = S(x, p, t)], \tag{2}$$

then JACOBI has shown that the equations satisfied by the new variables are

$$\frac{dq_i}{dt} = \frac{\partial}{\partial p_i}\left(\phi + \frac{\partial S}{\partial t}\right), \qquad \frac{dp_i}{dt} = -\frac{\partial}{\partial q_i}\left(\phi + \frac{\partial S}{\partial t}\right), \tag{3}$$

in which, after $\partial S/\partial t$ has been formed from S expressed in its original form as a function of t and the x, p, the function $\phi + \partial S/\partial t$ is supposed to be expressed in terms of p, q, t by means of (2). No assumption is made regarding the form of S, but in order to secure the independence of the x, y and also that of the p, q, it is necessary that the determinant

$$\left|\frac{\partial^2 S}{\partial x_i \partial p_j}\right| \qquad (i, j = 1, 2, \cdots, n),$$

shall not be zero.

Suppose that the x, y receive two sets of independent arbitrary variations δx, δy and $\delta' x$, $\delta' y$, inducing corresponding variations in the p, q. Equations (2) give

$$\Sigma_i(\delta x_i \delta' y_i - \delta y_i \delta' x_i) = \Sigma_i(\delta q_i \delta' p_i - \delta p_i \delta' q_i).$$

Whence, if the x, y are expressed in terms of t and the p, q, and if we adopt the usual notation:

$$[\alpha, \beta] = \Sigma_i\left[\frac{dy_i}{d\alpha}\frac{dx_i}{d\beta} - \frac{dx_i}{d\alpha}\frac{dy_i}{d\beta}\right], \tag{4}$$

we obtain the following relations:

$$\begin{aligned} &[p_i, q_i] = 1, \qquad [p_i, q_j] = 0 \qquad (i \neq j), \\ &[p_i, p_j] = 0, \qquad [q_i, q_j] = 0 \qquad (i = \text{ or } \neq j). \end{aligned} \tag{5}$$

We also have

$$\frac{d\phi}{dt} = \frac{\partial}{\partial t}\phi(x, y, t).$$

2. *Particular forms of S.* Suppose first that S is so chosen as to satisfy the equation $\phi + \partial S/\partial t = 0$. The p, q are then constants and they may be considered as the arbitraries of the solution. Such forms of S are solutions, containing n arbitrary constants exclusive of that additive to S, of the partial differential equation

$$\phi\left(x, \frac{\partial S}{\partial x}, t\right) + \frac{\partial S}{\partial t} = 0.$$

Next suppose that S is so chosen as to satisfy the equation

$$(6) \qquad \phi + \frac{\partial S}{\partial t} = B,$$

where B is a function of the p only; this is a form much more convenient in many of the applications. Then the p and therefore the q are constants. Let us put

$$(7) \qquad p_i = c_i, \qquad q_i = w_i = b_i t + \epsilon_i, \qquad b_i = -\frac{\partial B}{\partial c_i},$$

the arbitrary constants being c, ϵ, and B being a function of the c only; at present B is undetermined but it may be determined so as to satisfy future conditions as to the forms of x, y, when the latter are expressed in terms of c, ϵ, t.

I shall suppose that this problem has been solved and that this second form for S has been used so that the solution is expressed in terms of c_i, w_i, t, all of which are independent. On account of (2), we have,

$$(8) \qquad y_i = \frac{\partial S}{\partial x_i}, \qquad w_i = \frac{\partial S}{\partial c_i} \qquad [S = S(x, c, t)],$$

$$(9) \qquad \begin{aligned} &[c_i, w_i] = 1, \qquad [c_i, w_j] = 0 \qquad (i \neq j), \\ &[c_i, c_j] = 0, \qquad [w_i, w_j] = 0 \qquad (i = \text{ or } \neq j). \end{aligned}$$

3. *The constants present in ϕ.* Suppose that ϕ contains m constants α_h to which numerical values have not been assigned. Some or all of these constants will be also present in every form of S. I propose to investigate in this article certain properties of S and ϕ with relation to these constants. It is not necessary to suppose that the α_h are all the constants present in ϕ. Such of them as are not specified in the α_h can be supposed to have had numerical values assigned to them, either explicit or implicit, so that they do not enter into the discussion.

In order to make the following results more general, I suppose that the m constants α_h have been replaced by m variables or constants u_h connected with the α_h by the m independent relations

$$(10) \qquad f_k(u_h, \alpha_h, t) = 0 \qquad (h, k = 1, 2, \cdots, m),*$$

so that the u_h are functions of the α_h and t only.

The function S was originally a function of the x, c, α, t and it now becomes a function of the x, c, u, t; denote it by S' when it is expressed in the latter form. We have

$$\frac{\partial S}{\partial x_i} = \frac{\partial S'}{\partial x_i}, \qquad \frac{\partial S}{\partial c_i} = \frac{\partial S'}{\partial c_i}, \qquad \frac{\partial S}{\partial t} = \Sigma_h \frac{\partial S'}{\partial u_h} \dot{u}_h + \frac{\partial S'}{\partial t}.$$

* The values $1, 2, \cdots, m$ are given to h, k in all cases.

Put

$$U_h = \frac{\partial S'}{\partial u_h}.$$

Then

$$\frac{dU_h}{dt} = \Sigma_i \frac{\partial^2 S'}{\partial u_h \partial x_i} x_i + \Sigma_k \frac{\partial^2 S'}{\partial u_h \partial u_k} u_k + \frac{\partial^2 S'}{\partial u_h \partial t}. \tag{11}$$

Also denoting by Q' the expression of any function Q in terms of the x, c u and t, and putting

$$\frac{\partial' \phi}{\partial u_h} = \frac{\partial}{\partial u_h} \phi(x, y, u, t),$$

we obtain

$$\frac{\partial' \phi}{\partial u_h} = \frac{\partial \phi'}{\partial u_h} - \Sigma_i \frac{\partial \phi'}{\partial y_i} \frac{\partial y_i'}{\partial u_h} = - \frac{\partial}{\partial u_h} \left(\frac{\partial S}{\partial t} + B \right)' - \Sigma_i \dot{x}_i \frac{\partial^2 S'}{\partial u_h \partial x_i}$$

$$= - \frac{\partial}{\partial u_h} \left(\Sigma_k \frac{\partial S'}{\partial u_k} u_k + \frac{\partial S'}{\partial t} + B \right) - \Sigma_i x_i \frac{\partial^2 S'}{\partial u_h \partial x_i}$$

$$= - \Sigma_k \left(\frac{\partial^2 S'}{\partial u_h \partial u_k} u_k + U_k \frac{\partial u_k}{\partial u_h} \right) - \frac{\partial^2 S'}{\partial u_h \partial t} - \frac{\partial B}{\partial u_h} - \Sigma_i \frac{\partial^2 S'}{\partial u_h \partial x_i} x_i.$$

Combining with (11) we find

$$\frac{dU_h}{dt} = - \frac{\partial' \phi}{\partial u_h} - \Sigma_k U_k \frac{\partial u_k}{\partial u_h} - \frac{\partial B}{\partial u_h}, \tag{12}$$

an equation which enables us to find U_h directly.

Again

$$U_h = \frac{\partial S'}{\partial u_h} = \frac{dS}{du_h} - \Sigma_j y_j \frac{dx_j}{du_h},$$

where S and the x are expressed in terms of the c, w, u, t. Hence if U_h be similarly expressed,

$$\frac{dU_h}{dc_i} = \frac{d^2 S}{dc_i du_h} - \frac{d}{dc_i} \Sigma_j y_j \frac{dx_j}{du_h}. \tag{13}$$

But since the c, u, w are independent,

$$\frac{d^2 S}{dc_i du_h} = \frac{d^2 S}{du_h dc_i} = \frac{d}{du_h} \left(\frac{\partial S}{\partial c_i} + \Sigma_j \frac{\partial S}{\partial x_j} \frac{dx_j}{dc_i} \right)$$

$$= \frac{d}{du_h} \left(w_i + \Sigma_j y_j \frac{dx_j}{dc_i} \right) = \frac{d}{du_h} \Sigma_j y_j \frac{dx_j}{dc_i}.$$

Substituting in (13), performing the differentiations with respect to u_h, and remembering the notation (4), we find

$$(14) \qquad \frac{dU_h}{dc_i} = [u_h, c_i].$$

In an exactly similar manner it can be shown that

$$(15) \qquad \frac{dU_h}{dw_i} = [u_h, w_i],$$

$$\frac{dU_h}{du_k} - \frac{dU_k}{du_h} = [u_h, u_k].$$

The U_h can therefore be determined directly from the expressions for x, y when the solution has been completed.

4. *Two limitations.* In the remaining portion of this paper, I suppose

1°. That t is not present explicitly in $\phi(x, y, u, t)$, so that if t was present explicitly in the original expression for ϕ, one or more of the u must be variable.

2°. That t is not present explicitly in the x, y when the latter have been expressed in terms of the c, w, u, and t.

These assumptions do not very much limit the scope of the results and they render the exposition rather more simple.

We have now

$$(16) \qquad \frac{\partial S'}{\partial t} = 0, \qquad \frac{\partial S}{\partial t} = \Sigma_h U_h u_h, \qquad \phi + \Sigma_h U_h u_h = -B.$$

5. *Variation of arbitrary constants.* The usual method consists in sup posing that, after the equations (1) have been solved in terms of the c, w, and t, a new function $R(x, y, t)$ is added to ϕ; account is taken of R by supposing that the x, y retain the same *forms* when expressed in terms of the c, w, and t, so that the c, ϵ are no longer constants. This amounts to nothing more than a transformation from the old variables x, y to the new variables c, w, and the results of art. 1 can therefore be used.

In fact we now have

$$\Sigma_j \left(\frac{dx_i}{dc_j} \dot{c}_j + \frac{dx_i}{dw_j} \dot{w}_j + \frac{\partial x}{\partial t} \right) = \frac{\partial}{\partial y_i}(\phi + R),$$

$$\Sigma_j \left(\frac{dy_i}{dc_j} \dot{c}_j + \frac{dy_i}{dw_j} \dot{w}_j + \frac{\partial y}{\partial t} \right) = -\frac{\partial}{\partial x_i}(\phi + R),$$

which, on account of (9), give

$$\frac{dc_j}{dt} = \frac{\partial Q}{\partial w_j}, \qquad \frac{dw_j}{dt} = -\frac{\partial Q}{\partial c_j},$$

where $Q = \phi + R + \partial S/\partial t$ is supposed to be expressed by means of (8) in terms of the c, w, and t. But ϕ being a function of the x, y, and t, and $\partial S/\partial t$ a function of the x, c, and t, and neither ϕ nor $\partial S/\partial t$ containing total derivatives with respect to t, $\phi + \partial S/\partial t$ must retain the same form whatever be the values given to the c, w. Hence we still have $\phi + \partial S/\partial t = -B$, a function of the c only, and

$$Q = -B + R.$$

Suppose that R is due to changing one of the constants α present in ϕ to a variable. Then

$$Q = -B + \frac{\partial'\phi}{\partial\alpha}\delta\alpha + \tfrac{1}{2}\frac{\partial'^2\phi}{\partial\alpha^2}(\delta\alpha)^2 + \cdots,$$

where $\delta\alpha$ is the variable part added to α. The same method may be pursued, it is true, but if we follow it we take no account of the fact that the *form* of ϕ considered as a function of the x, y, α, t is unchanged, and it would seem as if this unchanged form should be made use of to simplify, if possible, the results. The new method of varying the constants which follows recognizes this fact and brings to light certain results which will be of value in the applications. As in art. 3, the α will be replaced by the u; the two conditions of art. 4 will also be imposed.

6. *Variation of arbitrary and given constants.* I suppose that the x, y retain the same *forms* when the u in ϕ are replaced by functions of the time different from those with which the original problem is solved, so that we replace not only the c, w *but also the* u by their new values in order to get the new values of x, y. The new values of c, w will, therefore, be different in this method from those found by the method of art. 5. In fact we now have, since t is present in the x, y only through its presence in the c, w, u,

$$\sum_j\left(\frac{dx_i}{dc_j}\dot{c}_j + \frac{dx_i}{dw_j}\dot{w}_j\right) + \sum_h\frac{dx_i}{du_h}\dot{u}_h = \frac{\partial\phi}{\partial y_i},$$

$$\sum_j\left(\frac{dy_i}{dc_j}\dot{c}_j + \frac{dy_i}{dw_j}\dot{w}_j\right) + \sum_h\frac{dy_i}{du_h}\dot{u}_h = -\frac{\partial\phi}{dx_i}.$$

On account of (9), (14), (15), these give

$$(17)\qquad \begin{aligned} \frac{dw_j}{dt} &= \frac{\partial\phi}{\partial c_j} - \sum_h[c_j, u_h]\dot{u}_h = \frac{\partial}{\partial c_j}(\phi + \sum_h U_h\dot{u}_h) = \frac{\partial Q}{\partial c_j}, \\ \frac{dc_j}{dt} &= -\frac{\partial\phi}{dw_j} + \sum_h[w_j, u_h]\dot{u}_h = -\frac{\partial}{\partial w_j}(\phi + \sum_h U_h\dot{u}_h) = -\frac{\partial Q}{\partial w_j}, \end{aligned}$$

if u_h be independent of c_j, w_j, and

$$Q = \phi + \Sigma_h U_h u_h$$

The equations are therefore still canonical.

Now when the u, c, w have their original values, denoted by the index 0,

$$\phi^0 + \Sigma_h U_h^0 \dot{u}_h^0 = -B^0 \tag{18}$$

This equation is an identity after the x, y, u, U have been replaced by their values in terms of the w, c, u, and t, and we can, therefore, give to the c, w, u any arbitrary variations, symbolically denoted by δ_1, so that

$$\phi^0 + \delta_1 \phi + \Sigma_h (U_h^0 + \delta_1 U_h)(\dot{u}_h^0 + \delta_1 u_h) = -B^0 - \delta_1 B. \tag{19}$$

The actual variations in Q are produced by adding portions, denoted symbolically by δ, to the c, w, u, so that

$$Q = \phi^0 + \delta\phi + \Sigma_h (U_h + \delta U_h) \frac{d}{dt}(u_h^0 + \delta u_h). \tag{20}$$

As ϕ, U, B, $\dot{u}$ are functions of the c, w, u only and not of their derivatives with respect to t, we can take $\delta_1 = \delta$. But we cannot in general put $\delta_1 \dot{u} = d\delta u/dt$. For the identity (19) demands that $\dot{u}$ be first expressed in terms of u by means of (10) and the variation then taken, while in Q a portion δu is to be added to u and the derivative with respect to t then formed. It was the failure to recognize this distinction that caused the error in my former paper.* The method there used implies the substitution of $-B$ for Q under the false assumption that the derivative of δu with respect to t would be the same as $\delta \dot{u}$. The error is immediately evident in the case of those u_h which were originally constants; for them, $\dot{u}_h = 0$ and, therefore, $\delta \dot{u}_h = 0$, while obviously $d\delta u/dt$ will only be zero if δu be constant, and this will not generally be the case.

Combining (19), (20), with $\delta_1 = \delta$, we obtain

$$Q = -B + \Sigma_h U_h \left(\frac{d}{dt} \delta u_h - \delta \dot{u}_h \right), \tag{21}$$

the final form of Q.

I have also obtained this result, when squares and higher powers of δ are neglected, by direct transformation from the ordinary canonical equations of art. 5 above, making use of the relations given in art. 3.

7. There is, of course, no special necessity to confine this method to those variations which are produced by varying the given constants in ϕ. Suppose

* Loc. cit., art. 17.

that the present method is retained for the consideration of a function R added to ϕ, while the given constants in ϕ do not vary. We must then add to Q the function

$$-\delta'\phi + R.$$

the first term of which is necessary to counterbalance the variations of the u in ϕ. But this proceeding appears to be of little or no value, as the results are more complicated than in the usual method. On the other hand the variations of the u are now entirely arbitrary and we might be able to so choose them that the form of Q is more simple or the canonical equations more easy to solve. What advantage may be gained will depend on the particular problem in hand and I shall not pursue this development further in the present paper.

8. *Case of the u_h satisfying canonical equations.* Suppose that the u are divided into two classes c'_h, $w'_h = b'_h t + \epsilon'_h$, where $b'_h = -\partial B'/\partial c'_h$. I suppose that in the initial problem, the c', ϵ' are constants and that B' is a function of the c' only. Instead of using the symbols U, put

$$C'_h = \frac{\partial S'}{\partial c'_h}, \qquad W'_h = \frac{\partial S'}{\partial w'_h} \qquad [S' = S(x_i, c_i, c'_i, w'_i)],$$

so that

$$-\phi - B = \frac{\partial S}{\partial t} = \Sigma_h b'_h W'_h,$$

$$Q = -B + \Sigma_h C'_h \frac{d}{dt}\delta c'_h + \Sigma_h W'_h \left(\frac{d}{dt}\delta w'_h - \delta b'_h\right). \tag{22}$$

Further, I suppose that the variations of the c', w' are given by

$$\frac{dc'_h}{dt} = \frac{\partial}{\partial w'_h}(R' + B'), \qquad \frac{dw'_h}{dt} = -\frac{\partial}{\partial c'_h}(R' + B'),$$

or

$$\frac{d}{dt}\delta c'_h = \frac{\partial R'}{\partial w'_h}, \qquad \frac{d}{dt}\delta w'_h = -\frac{\partial R'}{\partial c'_h} + \delta b'_h.$$

Substituting in (22), we obtain

$$Q = -B + \Sigma_h \left(C'_h \frac{\partial R'}{\partial w'} - W'_h \frac{\partial R'}{\partial c'_h}\right), \tag{23}$$

where it is supposed, in accordance with the previous work, that R' is independent of the c, w.

9. *Properties of the solution in the case of the lunar theory.* The application of these results to the lunar theory is immediate when the necessary limita-

tions which constitute this particular case are imposed. For the sake of brevity, I shall not prove these limitations here but merely state those of them that are necessary for my purpose. They are as follows:

1°. The function R' is the disturbing function due to a planet on the motion of the earth. It can be expanded as a sum of periodic terms and it has a small factor whose square can and will be neglected.

2°. A periodic term in R' of period α years will give rise to periodic terms in the $\delta c'$ of the same period with a factor α in the coefficients: in the $\delta w'$ the factor α^2 also occurs.

3°. It is supposed that the orbit of the earth lies in a fixed plane. There are then two angles w'_h, and three angles w_i.

4°. The W'_h consist only of sums of cosines of multiples of the four differences of the five angles w'_h, w_i, the terms independent of t being zero; the C'_h consist of sines with similar arguments.

5°. The c'_h are functions of the mean motion and of the eccentricity of the earth's orbit.

10. *The indirect planetary inequalities in the moon's motion.* There are very few sensible planetary inequalities in the moon's motion except when α is a number fairly large compared with unity; only the cases where α is large are treated here. It is further supposed that squares of the small factor which accompanies R' are neglected. Then from (17), (23),

$$\frac{d}{dt}\delta c_i = \frac{\partial Q_1}{\partial w_i}, \qquad \frac{d}{dt}\delta w_i = -\frac{\partial Q_1}{\partial c_i} + \Sigma_j \frac{db_i}{dc_j}\delta c_j + \Sigma_h \frac{db_i}{dc'_h}\delta c'_h,$$

where

$$Q_1 = \Sigma_h \left(C'_h \frac{\partial R'}{\partial w'_h} - W'_h \frac{\partial R'}{\partial c'_h} \right).$$

If R' contains a term of long period, α years, the factor α may be introduced in the δc, $\delta c'$ and therefore the factor α^2 in the δw, $\delta w'$, and no higher power of α can be present. Hence the general

THEOREM. *When squares and higher powers of the ratio of the mass of a planet to that of the sun are neglected, the large factor α due to a long period inequality can never occur in the corresponding term in the moon's motion to a power higher than α^2, even if its square is present in the corresponding inequality of the earth's motion.*

The importance of this result lies in the fact that it gives us a means of rejecting in advance many terms of long period, which under the old method would have to be examined in order to find out whether their coefficients were sensible in the moon's motion. The advantage of this method lies, to a great extent, in the theorem just stated and in the calculation of particular classes of inequalities; this will appear in the following article.

11. *Special classes of inequalities.* I divide the long-period inequalities into three classes for which certain results can be immediately deduced. The *order* of these refers to the power of α present. In the periodic inequalities it is supposed that at least one of the arguments belonging to the disturbing planet is present.

(*a*) *Inequalities containing the lunar arguments.* They are of two kinds. First, those produced by terms of moderately long period in the moon-earth-sun system combined with terms of nearly the same period in the earth-sun-planet system. The former terms have all quite small coefficients and there will be few of such inequalities with sensible coefficients. Second, those produced by combining terms of short period in the two systems. We shall generally be able to neglect the $\delta c'$, $\delta w'$, and the principal parts of the coefficients arise only in the δw_i obtained from limiting the equations to

$$\frac{d}{dt}\delta c_i = \frac{\partial Q_1}{\partial w_i}, \qquad \frac{d}{dt}\delta w_i = \Sigma_j \frac{db_i}{dc_j}\delta c_j .$$

(*b*) *Periodic inequalities not containing the lunar arguments.* Here

$$\delta c_i = 0$$

and the C', W' are limited to those terms which have the arguments w'_h only. The latter have all small coefficients and moreover the terms produced by Q_1 have only the factor α. Hence it will be possible to obtain the chief parts of the coefficients by solving the equations

$$\frac{d}{dt}\delta w_i = \Sigma_h \frac{db_i}{dc'_h}\delta c'_h, \qquad c_i = \text{const.}$$

(*c*) *Non-periodic inequalities.* These again are of two classes. First, the terms containing t to the first power only. These merely give constant additions to the mean motions of the lunar arguments. Second, the secular inequalities, which contain t^2. These latter are determined by

$$\delta c_i = 0, \qquad \frac{d}{dt}\delta w_i = \Sigma_h \frac{db_i}{dc'_h}\delta c_h,$$

where $\delta c'_h$ is determined from the non-periodic term of R'. This is NEWCOMB's theorem.* The theorem therefore only applies exactly, with the given limita-

* *Action of the Planets on the Moon,* Washington Astronomical Papers, vol. 5 (1894), p. 191.

tions, to the secular inequalities, but it is approximately true for the greater part of the inequalities independent of the lunar arguments, while it is not applicable to inequalities containing the lunar arguments. In arts. 17–19 of my former paper (loc. cit. ante), it was stated that the theorem applied to all ndirect inequalities.

I have contented myself here with a brief outline of the chief results which the method furnishes for the lunar problem. There are numerous details and properties connected with the functions C', W' which might find a place. But these are of more interest in connection with a numerical calculation of the inequalities than of theoretical interest in the variation of the constants or even in the problem of three bodies. It therefore appears advisable to give them later with the publication of the details of a method for such an investigation.

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