GENERAL MEAN VALUE AND REMAINDER THEOREMS WITH APPLICATIONS TO MECHANICAL DIFFERENTIATION AND QUADRATURE*†

BY

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Introduction.

We consider a real function $f(x)$ where x is a real variable. It is assumed that $f(x), f'(x), \cdots, f^{n-1}(x)$ are continuous functions of x on the interval (α, β), viz., for

$$\alpha \leqq x \leqq \beta,$$

and that $f^{n}(x)$ exists on (α, β). Let $x_0, x_1, \cdots, x_n$ be points of (α, β), which points need not all be distinct. Also let $k_0, k_1, \cdots, k_n$ be integers such that

$$0 \leqq k_i \leqq n-1 \qquad (i = 0, 1, \cdots, n).$$

The $n+1$ pairs of numbers

(1) $$(k_i, x_i) \qquad (i = 0, 1, \cdots, n)$$

are predicated distinct. Further, if $\phi(x)$ be a given continuous function on (α, β) and if $\phi'(x)$ exists except at a finite number of points, we write

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* Presented to the Society February 27, 1904, under the title *A general remainder theorem.* Received for publication April 15, 1905, and November 2, 1905.

† This paper develops and applies an important relation (6). I read a paper containing this relation and the principal theorems at the New York meeting of the Society in February, 1904. In a later version I added certain applications.

The theorems initially involved the hypothesis of the continuity of the nth derivative. This unnecessary restriction was removed in the final revision of October, 1905, and the theory stated completely in terms of the differential calculus, the symbol S denoting the antiderivative D^{-1} between limits.

I am very grateful to Professor E. H. Moore for his valuable suggestions in connection with this paper.

$$\phi(x) = S[\phi'(x)].^* \tag{2}$$

This paper is concerned with the connection of the $n+1$ numbers

$$f^{k_i}(x_i) \qquad (i=0, 1, \cdots, n) \tag{3}$$

and $f^n(x)$. It is clear that such a connection exists in certain simple cases, e. g., if

$$(k_i, x_i) = (0, a), (1, a), \cdots, (n-1, a), (0, b), \tag{4}$$

it is

$$\begin{aligned} f(b) = f(a) + f'(a)\cdot(b-a) + \cdots + f^{n-1}(a)\cdot\frac{(b-a)^{n-1}}{(n-1)!} \\ + \int_a^b f^n(x)\cdot\frac{(b-x)^{n-1}}{(n-1)!}\,dx, \end{aligned} \tag{5}$$

an important known formula.† Here $f^n(x)$ is taken continuous.

In the notation of this paper the general relation is

$$\sum_{i=0}^{n} f^{k_i}(x_i)\cdot\Delta_i = S_A^B[f^n(x)\cdot\Delta(x)]. \tag{6}$$

Here A, B are the end points of the set of points x_i. Further the Δ_i's are numbers and the $\Delta(x)$ a function of x depending on the system of number pairs (1). If (6) is not to be trivial, a certain restriction must be imposed on the integers k_i.

As far as I am aware, the principal case of the relation (6) heretofore published is that given by (4)‡. The general formula (6) is important in that it gives rise to a new mean value theorem, and to a new remainder theorem, having applications in the fields of mechanical differentiation and mechanical quadrature.

In § 1 is developed the fundamental formula (6) and in § 2 are given certain important properties of $\Delta(x)$.

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* If $\phi'(x)$ is continuous we have

$$S[\phi'(x)] = \int\phi'(x)\,dx.$$

It is to be noted that

$$S[\phi_1'(x)\cdot\phi_2'(x)]$$

exists if $S[\phi_1'(x)]$, $S[\phi_2'(x)]$ exist and $\phi_2''(x)$ is continuous ; indeed

$$S[\phi_1'(x)\cdot\phi_2'(x)] = \phi_1(x)\cdot\phi_2'(x) - \int\phi_1(x)\cdot\phi_2''(x)\,dx.$$

We write S_γ^δ for $S\big|_{x=\delta} - S\big|_{x=\gamma}$. Then $S_\gamma^\delta[\phi'(x)]$ is a definite number. In this paper S is used always in connection with a ϕ explicitly given.

† Cf. C. JORDAN, *Cours d'analyse*, vol. 1, p. 245.

‡ HERMITE gives a more general case than (4), where, however, instead of a single real integral, he has a multiple real integral : *Sur la formule d' interpolation de Lagrange*, Crelle's Journal, vol. 84 (1877).

In § 3 is considered the mean value theorem. Here we write

$$\text{(7)}\qquad \begin{aligned} &\Delta_i = 0 \qquad && (i=0, 1, \cdots, r-1), \\ &\Delta_i \neq 0 \qquad && (i=r, r+1, \cdots, n), \end{aligned}$$

properly arranging the Δ_i's, and assume

$$\text{(8)}\qquad f^{k_i}(x_i) = 0 \qquad (i=r, r+1, \cdots, n).$$

The mean value theorem then states that there exist points ξ_1, ξ_2 within (A, B) such that

$$\text{(9)}\qquad f^n(\xi_1) - E\cdot f^n(\xi_2) = 0 \qquad (0 \leqq E \leqq 1),$$

the E being a certain number, the *characteristic* number, of the system of pairs (1). The ordinary mean value theorem is the simplest example. Of especial interest are the extreme cases $E = 0$ and $E = 1$.

If $E = 0$ we have

$$\text{(10)}\qquad f^n(\xi_1) = 0.$$

For certain systems (k_i, x_i) of this sort [as for instance (4)], (9) can be proved by an application of the ordinary mean value theorem to $f(x), f'(x), \cdots, f^{n-1}(x)$ in succession. In general this method of proof fails. A table of the various new cases $E = 0$ for $n = 3$ is given.

If $E = 1$ the equation (9) is trivial for $\xi_1 = \xi_2$, but in this event it is shown that we have

$$\text{(11)}\qquad f^{n+\lambda}(\xi_1) - E_\lambda \cdot f^{n+\lambda}(\xi_2) = 0 \qquad (0 \leqq E_\lambda < 1),$$

in case $f^n(x), f^{n+1}(x), \cdots, f^{n+\lambda-1}(x)$ are continuous and $f^{n+\lambda}(x)$ exists on (A, B). Here λ, E_λ are dependent on the system of pairs (k_i, x_i) and ξ_1, ξ_2 are again points within (A, B). Furthermore in this section a generalization of BONNET's extended mean value theorem is given.

In § 4 is given the remainder theorem

$$\text{(12)}\qquad f^{k_0}(x_0) = F^{k_0}(x_0) + \frac{f^n(\xi_1) - E\cdot f^n(\xi_2)}{1-E}\cdot\frac{Z^{k_0}(x_0)}{n!}$$

in the principal form; $F(x)$ is the polynomial in x of degree $n-1$ at most, such that

$$\text{(13)}\qquad F^{k_i}(x_i) = f^{k_i}(x_i) \qquad (i=1, 2, \cdots, n),$$

and $Z(x)$ is a polynomial of degree n with leading coefficient 1 such that

$$\text{(14)}\qquad Z^{k_i}(x_i) = 0 \qquad (i=1, 2, \cdots, n).$$

The theorem (12) is a restatement of (6) after $S_A^B[f^n(x)\cdot\Delta(x)]$ has been replaced by a mean value. The result MARKOFF* employs is a particular case of (12).

Further a more general form for the remainder term is derived which contains, for the system of type (4), SCHLÖMILCH's† general form of the remainder in TAYLOR's development.

For the case $k_0 = 0$, $F(x_0)$ can be considered an interpolation formula for $f(x_0)$: the theorem gives a method of determining limits for the *error* $f(x_0) - F(x_0)$. If $k_0 = 1, 2, \cdots, n-1$, $F^{k_0}(x_0)$ is an interpolation formula for $f^{k_0}(x_0)$, and (12) furnishes a method of computing the error as before.

In § 5 the character of the remainder is considered in some detail for the case of mechanical differentiation with equal intervals, and the formulæ of ENCKE‡ are supplemented in this way.

There is derived in § 6 as a direct case of (12) the remainder theorem for mechanical quadrature

$$(15) \qquad \int_{\rho_2}^{\rho_1} f(x)\,dx = \int_{\rho_2}^{\rho_1} F(x)\,dx + \frac{f^n(\xi_1) - E\cdot f^n(\xi_2)}{1-E}\cdot\frac{\int_{\rho_2}^{\rho_1} Z(x)\,dx}{n!}$$

where $F(x)$ and $Z(x)$ are defined as before and E is the characteristic number of the system

$$(0, \rho_1),\ (0, \rho_2);\ (k_i + 1, x_i) \qquad (i = 1, 2, \cdots, n).$$

An obvious generalization obtains for successive quadrature. For practical purposes the writer determined E for the case of equal intervals $n = 2, 3, \cdots, 11$, in which cases COTES computed certain numbers of importance.‖ MARKOFF's remainder theorem § for the system of GAUSS is one other case of (15) that has been treated.¶

§ 1. *Derivation of the fundamental formula.*

The function

$$(1) \qquad \phi_i(x) = \sum_{p=0}^{n-1} f^{n-p-1}(x)\cdot\frac{(x_i - x)^{l_i - p}}{(l_i - p)!},$$

where

$$(2) \qquad l_i = n - k_i - 1,$$

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* MARKOFF: *Differenzenrechnung*, chap. 1 (German translation). This remainder theorem contains as special cases the remainder theorem of LAGRANGE for TAYLOR's development and the remainder theorem of CAUCHY for LAGRANGE's interpolation formula.

† SCHLÖMILCH: Liouville's Journal, ser. 2, vol. 3 (1858), p. 384.

‡ ENCKE: *Über mechanische Quadratur*, *Gesammelte Abhandlungen*, vol. 1.

§ MARKOFF, loc. cit., p. 61.

‖ MARKOFF, loc. cit., chap. 5.

¶ For general related references cf. *Encyklopædie der mathematischen Wissenschaften*, IE, ID 3; IIA 2, §§ 7, 11, 13, 51, 52, 54.

is continuous * and

$$\phi_i'(x) = f^n(x) \cdot \frac{(x_i - x)^{l_i}}{l_i!} \tag{3}$$

on (A, B). Moreover,

$$\phi_i(x_i) = f^{k_i}(x_i). \tag{4}$$

By definition, therefore,

$$\phi_i(x) = f^{k_i}(x_i) - S_x^{x_i}\left(f^n(x) \cdot \frac{(x_i - x)^{l_i}}{l_i!}\right). \tag{5}$$

Expanding and writing $x = A$ we find

$$\sum_{p=0}^{n-1} f^{n-p-1}(A) \cdot \frac{(x_i - A)^{l_i - p}}{(l_i - p)!} = f^{k_i}(x_i) - S_A^{x_i}\left(f^n(x) \cdot \frac{(x_i - x)^{l_i}}{l_i!}\right) \tag{6}$$

where

$$i = 0, 1, \cdots, n.$$

The $n + 1$ equations (6) are linear in the n quantities

$$f^{n-1}(A), f^{n-2}(A), \cdots, f(A).$$

Therefore the determinant †

$$\left| f^{k_0}(x_0) - S_A^{x_0}\left(f^n(x) \cdot \frac{(x_0 - x)^{l_0}}{l_0!}\right), \frac{(x_1 - A)^{l_1}}{l_1!}, \frac{(x_2 - A)^{l_2 - 1}}{(l_2 - 1)!}, \cdots, \frac{(x_n - A)^{l_n - n + 1}}{(l_n - n + 1)!} \right| = 0. \tag{7}$$

We denote the cofactor of the element

$$f^{k_i}(x_i) - S_A^{x_i}\left(f^n(x) \cdot \frac{(x_i - x)^{l_i}}{l_i!}\right)$$

of this determinant by Δ_i. The equation (7) is then written

$$\sum_{i=0}^{n} \Delta_i \cdot \left\{ f^{k_i}(x_i) - S_A^{x_i}\left(f^n(x) \cdot \frac{(x_i - x)^{l_i}}{l_i!}\right) \right\} = 0. \tag{8}$$

If the discontinuous function $\epsilon_i(x)$ be defined as follows:

$$\epsilon_i(x) = 1; \ x < x_i: \quad \epsilon_i(x) = 0; \ x > x_i, \tag{9}$$

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* If $l_i - p$ is negative we write $1/(l_i - p)! = 0$, e. g.,

$$\sum_{p=0}^{3} f^{3-p}(x) \cdot \frac{(a - x)^{1-p}}{(1-p)!} = f'''(x) \cdot (a - x) + f''(x) + 0 + 0.$$

In general the last k_i terms of (1), i. e., those containing $f(x), f'(x), \cdots, f^{k_i - 1}(x)$ are zero.

† The notation is

$$|a_0, b_1, \cdots, e_m| = \begin{vmatrix} a_0 & b_0 & \cdots & e_0 \\ a_1 & b_1 & \cdots & e_1 \\ \cdot & \cdot & \cdot & \cdot \\ a_m & b_m & \cdots & e_m \end{vmatrix}.$$

it is clear that

$$S_A^{x_i}\left(f^n(x)\cdot\frac{(x_i-x)^{l_i}}{l_i!}\right)=S_A^B\left(f^n(x)\cdot\epsilon_i(x)\cdot\frac{(x_i-x)^{l_i}}{l_i!}\right);$$

this new S function is the continuous function $\bar{\phi}_i(x)$ where

$$\bar{\phi}_i(x)=\phi_i(x);\ x\leqq x_i:\quad \bar{\phi}_i(x)=\phi_i(x_i);\ x\geqq x_i.$$

From (8), therefore,

$$\sum_{i=0}^{n}\Delta_i\ \left\{f^{k_i}(x_i)-S_A^B\left(f^n(x)\cdot\epsilon_i(x)\cdot\frac{(x_i-x)^{l_i}}{l_i!}\right)\right\}=0,$$

and more simply

$$\sum_{i=0}^{n}f^{k_i}(x_i)\cdot\Delta_i=S_A^B\left[f^n(x)\cdot\Delta(x)\right], \tag{10}$$

where the new S function is $\sum_{i=0}^{n}\bar{\phi}_i(x)$, so that

$$\Delta(x)=\sum_{i=0}^{n}\epsilon_i(x)\cdot\frac{(x_i-x)^{l_i}}{l_i!}\cdot\Delta_i. \tag{11}$$

Here $\Delta(x)$ and its derivatives are regarded as not defined at $x_0, x_1, \cdots, x_n$. It is evident that $\Delta(x)$ is made up of polynomial parts of degree at most $n-1$.

From (11) we have

$$\Delta(x)=\left|\epsilon_0(x)\cdot\frac{(x_0-x)^{l_0}}{l_0!},\ \frac{(x_1-A)^{l_1}}{l_1!},\ \frac{(x_2-A)^{l_2-1}}{(l_2-1)!},\ \cdots,\ \frac{(x_n-A)^{l_n+1}}{(l_n-n+1)!}\right|,$$

But the function of y,

$$\left|\epsilon_0(x)\cdot\frac{(x_0-x)^{l_0}}{l_0!},\ \frac{(x_1-y)^{l_1}}{l_1!},\ \frac{(x_2-y)^{l_2-1}}{(l_2-1)!},\ \cdots,\ \frac{(x_n-y)^{l_n-n+1}}{(l_n-n+1)!}\right|,$$

is independent of y, the derivative in y of the first and last column being zero, and of each other column the succeeding one. Hence

$$\Delta(x)=\left|\epsilon_0(x)\cdot\frac{(x_0-x)^{l_0}}{l_0!},\ \frac{x_1^{l_1}}{l_1!},\ \frac{x_2^{l_2-1}}{(l_2-1)!},\ \cdots,\ \frac{x_n^{l_n-n+1}}{(l_n-n+1)!}\right|. \tag{12}$$

Let D_g denote the number of pairs of the system (k_i, x_i) for which

$$k_i=g.$$

For example, in the system

$$(0,0),\ (0,3),\ (0,4),\ (1,1),\ (1,3),\ (2,3)$$

we have

$$D_0=3;\ D_1=2;\ D_2=1;\ D_3=0;\ D_4=0.$$

We assume that

$$\sum_{g=0}^{m} D_g \geqq m+2 \qquad (m=0, 1, \cdots, n-1). \tag{13}$$

Then no one of the quantities Δ_i vanishes identically in $x_0, x_1, \cdots, x_n$, as we prove for Δ_0.

In Δ_0 appear the n pairs

$$(k_i, x_i) \qquad (i=1, 2, \cdots, n). \tag{14}$$

We assume these pairs (14) so ordered that

$$k_1 \leqq k_2 \leqq \cdots k_n.$$

But for this system from (13)

$$\sum_{g=0}^{m} D_g \geqq m+1,$$

and therefore

$$k_1 \leqq 0,\ k_2 \leqq 1,\ \cdots,\ k_n \leqq n-1,$$

or

$$l_i = n-i-1+\mu_i;\ \mu_i \geqq 0,$$

and we find

$$\frac{\partial^{\mu_1}}{\partial x_1}\frac{\partial^{\mu_2}}{\partial x_2}\cdots\frac{\partial^{\mu_n}}{\partial x_n}\Delta_0 = \pm 1.*$$

If

$$\Delta_i = 0 \qquad (i=0, 1, \cdots, n),$$

the relation (10) is a trivial identity, although in all cases there exists at least one relation of the type of (10). We assume that *not all the Δ_i's are zero.* If this condition and (13) obtain, the system (k_i, x_i) is a *normal* system.

Existence theorem. *If the derivative function $\phi(x)$, defined on (A, B), and the constants $g_0, g_1, \cdots, g_n$ satisfy the relation*

$$\sum_{i=0}^{n} g_i \cdot \Delta_i = S_A^B[\phi(x)\cdot\Delta(x)] \tag{15}$$

with respect to a given normal system (k_i, x_i), then there exists precisely one function $f(x)$ such that

$$f^{k_i}(x_i) = g_i \qquad (i=0, 1, \cdots, n),$$

$$f^{n}(x) = \phi(x).$$

* Indeed we find on obtaining Δ_0 from (12) that

$$\frac{\partial^{\mu_1}}{\partial x_1^{\mu_1}}\frac{\partial^{\mu_2}}{\partial x_2^{\mu_2}}\cdots\frac{\partial^{\mu_n}}{\partial x_n^{\mu_n}}\Delta_0 = \begin{vmatrix} \frac{x_1^{n-1}}{(n-1)!} & \frac{x_1^{n-2}}{(n-2)!} & \cdots & 1 \\ \frac{x_2^{n-2}}{(n-2)!} & \frac{x_2^{n-3}}{(n-3)!} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \cdots & 0 \end{vmatrix}.$$

COROLLARY. *If $\Delta_0 \neq 0$ for a given normal system, then there exists precisely one function $f(x)$ such that*

$$f^{k_i}(x_i) = g_i \qquad (i = 1, 2, \cdots, n),$$

$$f^n(x) = \phi(x),$$

$\phi(x)$ being a given derivative function on (A, B) and $g_1, g_2, \cdots, g_n$ being n arbitrary numbers.

Proof. The argument is simple. First we note that the numbers

$$S_A^{x_i}\left(\phi(x) \cdot \frac{(x_i - x)^{l_i}}{l_i!}\right) \qquad (i = 0, 1, \cdots, n)$$

are well defined. Write in the equations (6) g_i for $f^{k_i}(x_i)$ and $\phi(x)$ for $f^n(x)$. Then the numbers $f(A), f'(A), \cdots, f^{n-1}(A)$ are given uniquely, the determinant of the system vanishing because of (15), while $\Delta_0 \neq 0$. If we determine $f(x)$ so that $f(x)$ and its derivatives have these values at A while

$$f^n(x) = \phi(x),$$

$f(x)$ is the required function.

For the corollary we choose g_0 so that (15) obtains, noting that $\Delta_0 \neq 0$.

§ 2. *The function* $\Delta(x)$.

As of use in the determination of the constants E of the mean value and remainder theorems, we give here a theorem concerning the number of changes of sign of $\Delta(x)$ on (A, B). The proof is given in § 7. We first make some definitions.

A set of pairs of the system (k_i, x_i)

$$(1) \qquad (\kappa, \xi), (\kappa + 1, \xi), \cdots, (\kappa + \mu, \xi)$$

are said to form a *sequence;* if this is a *separate* sequence of the system, that is, if $(\kappa - 1, \xi)$, $(\kappa + \mu + 1, \xi)$ are not pairs of the system, then (κ, ξ) is the *first* member of the sequence and $(\kappa + \mu, \xi)$ is the *last.* It is clear that the system (k_i, x_i) can be grouped into separate sequences, some perhaps containing but one pair. For the same sequence the x_i's are the same. In the example of the first section, the sequences are

$$(0, 0);\ (1, 2);\ (0, 3), (1, 3), (2, 3);\ (0, 4).$$

In the adjoined *scheme* of the system an entry $\mu + 1$ in the κth row and the ξ column signifies that in the system there is a sequence (1).

$\xi =$	0	1	3	4
0	1		3	1
1		1		
2				

The interval (A_{1k}, B_{1k}) is defined as the interval whose end points are the end points of the set of points

$$x_i \qquad (k_i = 0, 1, \cdots, k).$$

All these intervals are well defined since $D_0 \geqq 2$. In the above example

$$(A_0, B_0) \equiv (0, 3);\ (A_1, B_1) \equiv (A_2, B_2) \equiv (A_3, B_3) \equiv (A_4, B_4) \equiv (0, 4).$$

It is evident that (A_{n-1}, B_{n-1}) is the same as (A, B). A sequence of pairs with first member (κ, ξ) is said to be *conservative* if the point ξ is at $A_{\kappa-1}$, $B_{\kappa-1}$ or *without* $(A_{\kappa-1}, B_{\kappa-1})$; also if $\kappa = 0$. In our example $(1, 2)$ is the single non-conservative sequence, $\xi = 2$ being within $(A_0, B_0) \equiv (0, 3)$.

A sequence is *odd* or *even* according as it contains an odd or even number of pairs (k_i, x_i). If there are no non-conservative odd sequences of pairs the system (k_i, x_i) is a *conservative system.*

THEOREM. *The number of changes of sign of* $\Delta(x)$ *cannot exceed the number of non-conservative odd sequences of pairs.*

COROLLARY 1. *For a conservative system* $\Delta(x)$ *does not change sign.*

COROLLARY 2. *The number of changes of sign of* $\Delta(x)$ *in no case exceeds* $n-1$.

§ 3. *Mean value theorem.*

Let us arrange the Δ_i's of the normal system (k_i, x_i) so that

$$(1) \qquad \begin{aligned} \Delta_i &= 0 \qquad (i = 0, 1, \cdots, r-1), \\ \Delta_i &\neq 0 \qquad (i = r, r+1, \cdots, n); \end{aligned}$$

in general $r = 0$, i. e., none of the Δ_i's vanish. For the mean value theorem we assume that

$$(2) \qquad f^{k_i}(x_i) = 0 \qquad (i = r, r+1, \cdots, n).$$

It is clear that then

$$(3) \qquad \sum_{i=0}^{n} f^{k_i}(x_i) \cdot \Delta_i = 0.$$

We first obtain a mean value for $S_A^B[f^n(x) \cdot \Delta(x)]$ that obtains whether or not (2) is assumed, and that is important for the remainder theorem. To this end we need the lemma: *If* $\phi(B) - \phi(A) = 0$ *and further*

$$\phi'(x) = h(x) \cdot s(x),$$

where $s(x)$ *is a function of one sign on* (A, B) *or zero defined except at a finite set of points, and* $h(x)$ *is defined within* (A, B), *and* $S[h(x)]$ *exists, then* $h(\xi) = 0$ *where* ξ *is some point within* (A, B). Proof: It is clear that the product $h(x) \cdot s(x)$ changes sign within (A, B), and since in general $s(x)$ is defined and nowhere changes sign, $h(x)$ must change sign within

(A, B). Hence $S[h(x)]$ has a maximum within (A, B) say at ξ. But $h(x)$ exists at ξ. Hence $h(\xi)=0$.

Let $G(x)$ denote some function defined and of one sign (not zero) *within* (A, B). Also we assume that $S[G(x)\cdot\Delta(x)]$ exists on (A, B).* Determine H_1 and H_2 so that

$$(4) \qquad S_A^B\left\{[f^n(x)-H_1\cdot G(x)]\cdot\frac{\Delta(x)+|\Delta(x)|}{2}\right\}=0,$$

$$(5) \qquad S_A^B\left\{[f^n(x)-H_2\cdot G(x)]\cdot\frac{\Delta(x)-|\Delta(x)|}{2}\right\}=0.$$

By the lemma then, since $\frac{1}{2}\{\Delta(x)\pm|\Delta(x)|\}$ does not change sign,

$$f^n(\xi_1)-H_1\cdot G(\xi_1)=0,$$

$$f^n(\xi_2)-H_2\cdot G(\xi_2)=0,$$

ξ_1, ξ_2, being points within (A, B). Adding (4) and (5) after substituting in the values of H_1, H_2 thus obtained we find

$$(6) \qquad S_A^B[f^n(x)\cdot\Delta(x)]=\frac{f^n(\xi_1)}{G(\xi_1)}\cdot\bar{P}+\frac{f^n(\xi_2)}{G(\xi_2)}\cdot\bar{N},$$

where

$$(7) \quad \bar{P}=S_A^B\left(G(x)\cdot\frac{\Delta(x)+|\Delta(x)|}{2}\right);\quad \bar{N}=S_A^B\left(G(x)\cdot\frac{\Delta(x)-|\Delta(x)|}{2}\right).$$

The relation (6) is obvious if $f^n(x)$ and $G(x)$ are continuous. We define $\bar{E}$, the *characteristic number* of the system with respect to $G(x)$, $-\bar{P}/\bar{N}$ or 1 or $-\bar{N}/\bar{P}$, according as $|\bar{P}|<|\bar{N}|$ or $|\bar{P}|=|\bar{N}|$ or $|\bar{P}|>|\bar{N}|$, so that

$$(8) \qquad 0\leqq\bar{E}\leqq 1.$$

It is clear that $\bar{P}$ and $\bar{N}$ are not both zero because $\Delta(x)$ is not altogether zero.

For this section $G(x)=1$ and we write

$$(9) \qquad P=\int_A^B\frac{\Delta(x)+|\Delta(x)|}{2}\,dx;\quad N=\int_A^B\frac{\Delta(x)-|\Delta(x)|}{2}\,dx;$$

here we call the characteristic number E. From (3) using our fundamental formula we find

$$S_A^B[f^n(x)\cdot\Delta(x)]=0,$$

*If $S[G(x)\cdot\Delta(x)]$ exists, so does $S[G(x)\cdot|\Delta(x)|]$, since the product function $G(x)\cdot\Delta(x)$ changes sign but a finite number of times.

and then, using (6) for $G(x)=1$,

$$f^n(\xi_1)\cdot P + f(\xi_2)\cdot N = 0.$$

Hence

$$f^n(\xi_1) - E\cdot f^n(\xi_2) = 0,$$

the ξ_1, ξ_2 being the same as before or interchanged.

MEAN VALUE THEOREM. *If* (k_i, x_i) *is a normal system and if*

$$f^{k_i}(x_i) = 0 \qquad (i = r, r+1, \cdots, n),$$

then is

(10) $$f^n(\xi_1) - E\cdot f^n(\xi_2) = 0 \qquad (A < \xi_1, \xi_2 < B),$$

where E is the characteristic number of the given system with respect to 1.

A simple example is given here:

$$n = 2;\ (k_i, x_i) \equiv (0, x_0),\ (1, x_1),\ (0, x_2) \qquad (x_0 < x_1 < x_2),$$

so that $x_1 = A$, $x_2 = B$ and

$$\Delta(x) = \begin{vmatrix} \epsilon_0(x)\cdot(x_0 - x), & x_0, & 1 \\ \epsilon_1(x)\cdot 1, & 1, & 0 \\ \epsilon_2(x)\cdot(x_2 - x), & x_2, & 1 \end{vmatrix},$$

whence

$$\Delta(x) = +(x - x_0);\ x_0 < x < x_1 : \Delta(x) = -(x_2 - x);\ x_1 < x < x_2.$$

$$P = \frac{(x_1 - x_0)^2}{\cdot 2};\quad N = -\frac{(x_2 - x_1)^2}{2};\quad E = \left(\frac{x_2 - x_1}{x_1 - x_0}\right)^2 \text{ or } \left(\frac{x_1 - x_0}{x_2 - x_1}\right)^2.$$

The theorem states then that, in the case $x_2 - x_1 \leqq x_1 - x_0$,

$$f^2(\xi_1) - \left(\frac{x_2 - x_1}{x_1 - x_0}\right)^2 \cdot f^2(\xi_2) = 0.$$

If $x_2 - x_1 = x_1 - x_0$, so that the statement is trivial for $\xi_1 = \xi_2$, we can infer by a method given later that

$$f^3(\xi) = 0.$$

Of especial interest are the extreme cases $E = 0$ and $E = 1$.

In the case $E = 0$ the system (k_i, x_i) is a *simple* system. The condition that a given normal system be simple is that $\Delta(x)$ does not change sign. If a system is simple and if furthermore

$$f^{k_i}(x_i) = 0 \qquad (i = r, r+1, \cdots, n),$$

then

(11) $$f^n(\xi) = 0$$

for some point ξ within (A, B). The theorem of § 2 makes it clear that conservative systems are a special class of simple systems. The ordinary mean value theorem furnishes a method of proving (11) for conservative systems.

Example: $(k_i, x_i) \equiv (0, x_1), (0, x_2), (0, x_3) \qquad (x_1 < x_2 < x_3);$

$$f(x_1) = f(x_2) = f(x_3) = 0.$$

First we infer

$$f'(\xi_1) = 0, \qquad f''(\xi_2) = 0 \qquad (x_1 < \xi_1 < \xi_2 < x_3),$$

and then

$$f'''(\xi_3) = 0.$$

No attempt at a general consideration of simple systems is here made. The theorem of § 2 is important for this problem. A table of the various simple non-conservative systems for $n = 3$ is given herewith. In this table we choose $A = -1$, $B = +1$ so that the four pairs are

$$(k_0, -1), (k_1, x_1), (k_2, x_2), (k_3, +1),$$

where

$$-1 \leqq x_1, x_2 \leqq +1.$$

Moreover we choose $k_0 \leqq k_3$; I–VII are the various non-conservative systems.

	k_0	k_1	k_2	k_3	$E = 0$
I	0	0	1	0	$\lvert 2x_2 - x_1 \rvert \leqq 1$
II	0	0	2	0	
III	0	0	2	1	
IV	0	1	0	1	$x_1 \geqq \frac{x_2 - 1}{2}$; cons. if $x_1 \geqq x_2$
V	0	1	0	2	$x_1 \geqq \frac{x_2 - 1}{2}$; cons. if $x_1 > x_2$
VI	0	1	1	0	$\lvert x_1 - x_2 \rvert \geqq 1$
VII	0	1	2	0	$\lvert x_1 \rvert \leqq \frac{(1 - x_2)^2}{4}$

An important case of the mean value theorem arises if r has been made a maximum. An example of this is the system on which MARKOFF* makes depend his remainder theorem for GAUSS' scheme of quadrature.† This system is

$$(0, x_1), (1, \alpha_1), (2, \alpha_1), (1, \alpha_2), \cdots, (1, \alpha_m), (2, \alpha_m), (0, x_2),$$

which is conservative. Here $n = 2m + 1$. The quantities $\alpha_1, \alpha_2, \cdots, \alpha_m$ are the m roots of the equation

$$\frac{d^m}{dx^m}\{(x - x_1)^m (x - x_2)^m\} = 0.$$

	x_1	α_1	α_2	$\cdots$	α_m	x_2
0	1					1
1		2	2	$\cdots$	2	
2						

It can be proved then that

$$\Delta_i = 0; \qquad k_i = 2,$$

* MARKOFF, loc. cit., chap. 5.

† GAUSS, *Methodus nova integralium valores per approximationem inveniendi*, *Werke*, vol. 3.

m of the $2m+2$ Δ_i's thus vanishing. Hence in this case, if

$$f(x_1) = f'(\alpha_1) = f'(\alpha_2) = \cdots f'(\alpha_m) = f(x_2) = 0,$$

of necessity

$$f^{2m+1}(\xi) = 0$$

for some point ξ within (x_1, x_2).

If $E = 1$, the mean value theorem is

$$f^n(\xi_1) - f^n(\xi_2) = 0,$$

which is trivial for $\xi_1 = \xi_2$. In this case the fact is that $f^n(x)$ need not vary at all, although the conditions (2) obtain: that is, there exists a polynomial $P(x)$ of degree n such that

$$P^{k_i}(x_i) = 0 \qquad (i = 0, 1, \cdots, n),$$

for then

$$\sum_{i=0}^{n} P^{k_i}(x_i) = S_A^B[P^n(x)\cdot\Delta(x)]$$

is satisfied $(P + N = 0)$, and accordingly there is a $P(x)$ by the existence theorem. The condition $E = 0$ or

$$\int_A^B \Delta(x)\,dx = 0$$

can be written

$$(12) \qquad \left| \frac{x_0^{l_0+1}}{(l_0+1)!},\ \frac{x_1^{l_1}}{l_1!},\ \ldots,\ \frac{x_n^{l_n-n+1}}{(l_n-n+1)!} \right| = 0.$$

Let us assume that $f^n(x)$, $f^{n+1}(x)$, etc., are continuous. Integrating $S_A^B[f^n(x)\cdot\Delta(x)]$ by parts we obtain

$$S_A^B[f^n(x)\cdot\Delta(x)] = [f^n(x)\cdot\Delta^{-1}(x)]_A^B - S_A^B[f^{n+1}(x)\cdot\Delta^{-1}(x)],$$

which holds since

$$\Delta^{-1}(x) = \int_A^x \Delta(x)\,dx$$

is continuous. But $\Delta^{-1}(A) = 0$ and $\Delta^{-1}(B) = 0$. Hence

$$S_A^B[f^n(x)\cdot\Delta(x)] = -S_A^B[f^{n+1}(x)\cdot\Delta^{-1}(x)].$$

If

$$\int_A^B \Delta^{-1}(x)\,dx = 0,$$

it is proved in a similar way that

$$S_A^B[f^n(x)\cdot\Delta(x)] = -S_A^B[f^{n+1}(x)\cdot\Delta^{-1}(x)] = S_A^B[f^{n+2}(x)\cdot\Delta^{-2}(x)],$$

where

$$\Delta^{-2}(x) = \int_A^x \Delta^{-1}(x)\,dx.$$

If it is possible to proceed λ steps in this way we obtain

$$S_A^B[f^n(x)\cdot\Delta(x)] = (-1)^\lambda S_A^B[f^{n+\lambda}(x)\cdot\Delta^{-\lambda}(x)], \tag{13}$$

where

$$\Delta^{-\rho}(x) = \int_A^x \Delta^{-\rho+1}(x)\,dx \qquad (\rho = 1, 2, \cdots, \lambda);$$

here $f^n(x)$, $f^{n+1}(x)$, $\cdots$, $f^{n+\lambda-1}(x)$ are continuous and $f^{n+\lambda}(x)$ exists on (A, B). Assuming (2) to hold we see that

$$S_A^B[f^{n+\lambda}(x)\cdot\Delta^{-\lambda}(x)] = 0,$$

whence

$$f^{n+\lambda}(\xi_1) - E_\lambda\cdot f^{n+\lambda}(\xi_2) = 0 \qquad (A < \xi_1, \xi_2 < B), \tag{14}$$

where E_λ depends on $\Delta^{-\lambda}(x)$ just as does E on $\Delta(x)$.

Of course, here

$$0 \leqq E_\lambda < 1.$$

The $\lambda + 1$ conditions

$$\int_A^B \Delta(x)\,dx = \int_A^B \Delta^{-1}(x)\,dx = \cdots \int_A^B \Delta^{-\lambda}(x)\,dx = 0 \tag{15}$$

can at once be written as the vanishing of $\lambda + 1$ determinants like (12).

We proceed to formulate the extended form of the BONNET's theorem. Consider the determinant function

$$\Phi(x) = |\phi_1(x), \phi_2^{k_0}(x_0), \phi_3^{k_1}(x_1), \cdots, \phi_{n+2}^{k_n}(x_n)|$$

dependent on $n + 2$ functions ϕ whose nth derivatives exist on (A, B). This function $\Phi(x)$ satisfies the conditions

$$\Phi^{k_i}(x_i) = 0 \qquad (i = 0, 1, \cdots, n),$$

and therefore, since $\Phi^n(x)$ exists on (A, B),

$$\Phi^n(\xi_1) - E\cdot\Phi^n(\xi_2) = 0,$$

where E is the characteristic number of the given normal system, or, in determinant form,

$$|\phi_1^n(\xi_1) - E\cdot\phi_1^n(\xi_2), \phi_2^{k_0}(x_0), \phi_3^{k_1}(x_1), \cdots, \phi_{n+2}^{k_n}(x_n)| = 0, \tag{16}$$

the extended form of the BONNET's theorem. This theorem states that

$$|\phi_1^n(\xi_1), \phi_1(x_1), \phi_2(x_2), \cdots, \phi_{n+2}(x_n)| = 0.*$$

* PEANO: *Lezioni di analisi infinitesimale*, vol. 1, p. 107.

§ 4. *The remainder theorem.*

In this section it is assumed that

(1) $$f^{k_i}(x_i) = g_i \qquad (i = 1, 2, \cdots, n),$$

where $g_1, g_2, \cdots, g_n$ are known and also that

(2) $$\Delta_0 \neq 0.$$

The fundamental formula (10) of § 1 is written then

(3) $$f^{k_0}(x_0) = -\sum_{i=1}^{n} g_i \cdot \frac{\Delta_i}{\Delta_0} + \frac{S_A^B[f^n(x)\cdot\Delta(x)]}{\Delta_0}.$$

Let $F(x)$ be the polynomial in x of degree $n-1$ or less such that

(4) $$F^{k_i}(x_i) = g_i \qquad (i = 1, 2, \cdots, n),$$

there being a polynomial of this sort by the corollary to the existence theorem. If (3) be applied to this polynomial we find

(5) $$F^{k_0}(x_0) = -\sum_{i=0}^{n} g_i \cdot \frac{\Delta_i}{\Delta_0};$$

whence from (3)

(6) $$f^{k_0}(x_0) = F^{k_0}(x_0) + \frac{S_A^B[f^n(x)\cdot\Delta(x)]}{\Delta_0},$$

the remainder theorem with exact remainder term.

As an example we consider the system

$$\begin{cases} (k_i', \alpha) & (i = 0, 1, \cdots, \mu), \\ (k_i'', \beta) & (i = 0, 1, \cdots, \nu), \end{cases}$$

where $\mu + \nu + 1 = n$. This system is conservative. Furthermore $\Delta(x)$ and its derivatives are continuous within (α, β) and in fact on this interval

$$\Delta(x) = c \cdot P(x)$$

where $P(x)$ is a polynomial in x of degree $n-1$ with leading coefficient 1 such that

$$\begin{cases} P^k(\alpha) = 0 & (k \neq n - k_i' - 1), \\ P_k(\beta) = 0 & (k \neq n - k_i'' - 1). \end{cases}$$

In this system both $(0, \alpha), (0, \beta)$ must occur. We write the theorem for $(k_0, x_0) = (0, \beta)$,

$$f(\beta) = F(\beta) + c \cdot \frac{\int_\alpha^\beta f^n(x) P(x)\, dx}{\Delta_0}.$$

If $k_i' = i$, $k_i'' = i$ we have more explicitly

$$f(\beta) = F(\beta) + (-1)^n \frac{\int_\alpha^\beta f^n(x)(x-\alpha)^\nu(x-\beta)^\mu}{n!}.*$$

We proceed to the derivation of the general remainder theorem.

If we write $G(x) = g^n(x)$ in the mean value relation (6) of last section so that $g^n(x)$ exists and is of one sign *within* (A, B), while $S[g^n(x)\cdot\Delta(x)]$ exists on (A, B), we find

$$(7) \qquad S_A^B[f^n(x)\cdot\Delta(x)] = \frac{f^n(\xi_1)}{g^n(\xi_1)}\cdot\bar{P} + \frac{f^n(\xi_2)}{g^n(\xi_2)}\cdot\bar{N}.$$

If the pairs with greatest k_i for $x_i = A$ and $x_i = B$ are respectively

$$(K_1, A),\ (K_2, B),$$

then $\Delta(x)$ is of the form

$$(x-A)^{n-K_1-1}\cdot P_1(x), \qquad (x-B)^{n-K_2-1}\cdot P_2(x)$$

near these points, where $P_1(x)$ and $P_2(x)$ are polynomials; † K_1, K_2 are said to be the *exponents* of the system (k_i, x_i). Hence $g^n(x)$ is defined *within* (A, B), of one sign, and $S[g^n(x)\Delta(x)]$ exists on (A, B), if

$$(8) \qquad g^n(x) = \frac{\lambda(x)}{(x-A)^{n-K_1-1}(x-B)^{n-K_2-1}},$$

where $\lambda(x)$ and $S[\lambda(x)]$ exist on (A, B), and $\lambda(x)$ is of one sign but not zero within (A, B). We assume then that $g^n(x)$ is of this character (8).

If $g^n(x)$ be defined at A and B and thus is finite, then by the corollary to the existence theorem, $g(x)$ exists such that

$$(9) \qquad g^{k_i}(x_i) = 0 \qquad (i = 1, 2, \cdots, n)$$

and further by (6)

$$(10) \qquad g^{k_0}(x_0) = \frac{S[g(x)\cdot\Delta(x)]}{\Delta_0}.$$

If, however, $g^n(x)$ has the more general form (8) there still exists a unique $g(x)$ which satisfies (9); for this $g(x)$, (10) is also satisfied.

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* Cf. HERMITE's article: *Sur la formule d'interpolation de Lagrange*, Crelle's Journal, vol. 84 (1877).

† Near B, $\Delta(x)$ is identical with

$$\sum_{x_i=B} \frac{(x_i - x)^{l_i}}{l_i!}\Delta_i$$

which is obviously of the stated form [see (11) § 1]. A similar remark applies near $x = A$ when one notes that

$$\sum_{i=0}^{n} \frac{(x_i - x)^{l_i}}{l_i!}\Delta_i \equiv 0.$$

In order to prove these two statements we write

$$g(x)=\frac{1}{\Delta_0}\cdot\left|\bar{g}(x),\ \frac{x_1^{l_1}}{l_1!},\ \frac{x_2^{l_2-1}}{(l_2-1)!},\ \cdots,\ \frac{x_n^{l_n-n+1}}{(l_n-n+1)}\right|,$$

where $\bar{g}(x)$ satisfies the conditions

$$\bar{g}^n(x)=g^n(x):\bar{g}(\bar{x})=\bar{g}'(\bar{x})=\cdots=\bar{g}^{n-1}_{\iota}(\bar{x})=0,$$

$\bar{x}$ being within (A, B). Here

$$\bar{g}^{k_i}(x) \qquad (i=0, 1, \cdots, n)$$

are finite at x_i; (*a*) within (A, B) of course; (*b*) at A because $\bar{g}^{K_1'}(x)$ is continuous at $x=A$; (*c*) at B for a similar reason. Then it is clear that $g(x)$ satisfies the required conditions (9). Moreover if $e(x_i)$ also satisfies these conditions then $d(x)=g(x)-e(x)$ has the properties

$$d^n(x)=0:d^{k_i}(x_i)=0 \qquad (i=1, 2, \cdots, n).$$

But if $d(x)=0$, these conditions are satisfied. Hence by the existence theorem $d(x)=0$ so that $g(x)$ is indeed given above.

Consider further the function $g_1(x)$ such that

$$g_1^n(x)=g^n(x) \qquad (A+\epsilon\leqq x\leqq B-\epsilon)$$

and that $g_1^n(x)$ is zero otherwise, and such that

$$g_1(x)=\frac{1}{\Delta_0}\cdot\left|\bar{g}_1(x),\ \frac{x_1^{l_1}}{l_1!},\ \frac{x^{l_2}}{l_2!},\ \cdots,\ \frac{x_n^{l_n-n+1}}{(l_n-n+1)!}\right|,$$

where $\bar{g}_1$ is defined with respect to g_1 as $\bar{g}$ was with respect to g. Applying (6) we find

$$\frac{1}{\Delta_0}\cdot\left|\bar{g}_1^{k_0}(x_0),\ \frac{x_1^{l_1}}{l_1!},\frac{x_2^{l_2-1}}{(l_2-1)!},\cdots,\frac{x_n^{l_n-n+1}}{(l_n-n+1)!}\right|=\frac{S_A^B[g_1^n(x)\cdot\Delta(x)]}{\Delta_0}.$$

As ϵ approaches zero $\bar{g}_1^{k_i}(x_i)$ approach $\bar{g}^{k_i}(x_i)$ and $S_A^B[g_1^n(x)\cdot\Delta(x)]$ approaches $S_A^B[g^n(x)\cdot\Delta(x)]$. In the limit then

$$g^{k_0}(x_0)=\frac{S_A^B[g^n(x)\cdot\Delta(x)]}{\Delta_0}$$

which we wished to prove.

From (10) then, if $\bar{P}+\bar{N}\neq 0$,

$$\frac{1}{\Delta_0}=\frac{g^{k_0}(x_0)}{\bar{P}+\bar{N}}. \tag{11}$$

On substituting from (7) and (11) in (6), (6) becomes

$$f^{k_0}(x_0)=F^{k_0}(x_0)+\frac{\frac{f^n(\xi_1)}{g^n(\xi_1)}\cdot P+\frac{f^n(\xi_1)}{g^n(\xi_1)}\cdot N}{P+N}\cdot g^{k_0}(x_0)$$

or

$$f^{k_0}(x_0)=F^{k_0}(x_0)+\frac{\frac{f^n(\xi_1)}{g^n(\xi_1)}-E\cdot\frac{f^n(\xi_2)}{g^n(\xi_2)}}{1-E}\cdot g^{k_0}(x_0)$$

where ξ_1, ξ_2 are the same as before or interchanged.

General remainder theorem. *Let (k_i, x_i) be a normal system for which $\Delta_0 \neq 0$ and let $f(x)$ be a function such that $f(x), f'(x), \cdots, f^{n-1}(x)$ are continuous and $f^n(x)$ exists on (A, B). Further let $g(x)$ be a function such that $g(x), g'(x), \cdots, g^{n-1}(x)$ are continuous on (A, B),*

$$g^{k_i}(x_i)=0 \qquad (i=1, 2, \cdots, n),$$

and

$$g^n(x)=\frac{\lambda(x)}{(x-A)^{n-K_1-1}(x-B)^{n-K_2-1}}$$

K_1, K_2 being the exponents of the system, $\lambda(x)$ and $S[\lambda(x)]$ existing on (A, B), $\lambda(x)$ being of one sign (not zero) within (A, B). Then

$$f^{k_0}(x_0)=F^{k_0}(x_0)+\frac{\frac{f^n(\xi_1)}{g^n(\xi_1)}-E\cdot\frac{f^n(\xi_2)}{g^n(\xi_2)}}{1-E}\cdot g^{k_0}(x_0) \tag{12}$$

where ξ_1, ξ_2 are points within (A, B), and $F(x)$ is the polynomial of degree $n-1$ at most such that

$$F^{k_i}(x_i)=f^{k_i}(x_i) \qquad (i=1, 2, \cdots, n),$$

*and E is the characteristic number of the system with respect to $g^n(x)$.**

A convenient form for computation is

$$f^{k_0}(x_0)=F^{k_0}(x_0)+\theta\cdot\frac{1+E}{1-E}\cdot\frac{f^n(\xi_1)}{g^n(\xi_1)}\cdot g^{k_0}(x_0) \tag{13}$$

where $|\theta|\leqq 1$.

In the special case $E=0$, when $\Delta(x)$ does not change sign, (12) is

$$f^{k_0}(x_0)=F^{k_0}(x_0)+\frac{f^n(\xi_1)}{g^n(\xi_1)}\cdot g^{k_0}(x_0). \tag{14}$$

For the conservative Taylor system

$$(0, x_0), (0, x_1), (1, x_1), \cdots, (n-1, x_1)$$

* For definition of *normal system*, cf. p. 113, of *exponents*, cf. p. 132, *characteristic number*, cf. p. 116.

the exponents are 0 and $n-1$ so that

$$g^n(x)=\frac{\lambda(x)}{(x-x_0)^{n-1}}.$$

The theorem is then

$$f(x_0)=\left(f(x_1)+f'(x_1)\cdot(x_0-x_1)+\cdots+f^{n-1}(x_1)\frac{(x_0-x_1)^{n-1}}{(n-1)!}\right)+\frac{f^n(\xi)}{g^n(\xi)}\cdot g(x_0)$$

and in this form can be at once identified with the SCHLÖMILCH general remainder theorem * in TAYLOR's development

$$f(x_0)=\left(f(x_1)+f'(x_1)\cdot(x_0-x_1)+\cdots+f^{n-1}(x_1)\frac{(x_0-x_1)^{n-1}}{(n-1)!}\right)$$
$$+\frac{f^n(\xi)(x_0-\xi)^{n-1}}{F'(\xi)\cdot(n-1)!}[F(x_0)-F(x_1)].$$

In fact we have

$$F'(x)=\lambda(x)$$

and, therefore,

$$g(x_0)=S_{x_0}^{x_1}[g^n(x)\cdot(x_0-x)^{n-1}]=S_{x_0}^{x_1}[F'(x)]=F(x_1)-F(x_0)$$

as the required transformation. The conditions imposed are the same as SCHLÖMILCH's: $F(x)=S[\lambda(x)]$ is continuous, $F'(x)=\lambda(x)$ exists on (A, B) and is of one sign (not zero) within (x_1, x_0).

The most important case of (12) is obtained by setting $g^n(x)=1$ so that $\bar{E}=E$. If we denote by $Z(x)$ the polynomial in x of degree n with leading coefficient 1 such that

$$Z^{k_i}(x_i)=0 \qquad (i=1, 2, \cdots, n),$$

then

$$g(x)=\frac{Z(x)}{n!}.$$

Therefore,

$$f^{k_0}(x_0)=F^{k_0}(x_0)+\frac{f^n(\xi_1)-E\cdot f^n(\xi_2)}{1-E}\cdot\frac{Z^{k_0}(x_0)}{n!}, \tag{15}$$

which is *the principal form of the remainder theorem.*

Example. HERMITE's † remainder theorem.

$$n=\alpha_1+\alpha_2+\cdots+\alpha_m,$$

$$f(x)=F(x)+f^n(\xi)\cdot\frac{(x-x_1)^{\alpha_1}\cdot(x-x_2)^{\alpha_2}\cdots(x-x_m)^{\alpha_m}}{n!}. \qquad \begin{array}{c|c} & x_1, x_2, \cdots, x_m, x \\ \hline 0 & \alpha_1, \alpha_2, \cdots, \alpha_m, 1 \end{array}$$

The system is conservative for all values of x.

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* SCHLÖMILCH, Liouville's Journal, ser. 2, vol. 3 (1858), p. 384.

† *Sur la formule d'interpolation de Lagrange*, Crelle's Journal, vol. 84 (1878).

Case 1. TAYLOR's development with LAGRANGE's remainder term,

$$f(x_0)=f(x_1)+f'(x_1)\cdot(x-x_1)+\cdots+f^n(\xi)\cdot\frac{(x-x_1)^n}{n!}. \qquad \begin{array}{c|cc} & x_1 & x \\ \hline 0 & n & 1 \end{array}$$

TAYLOR's series is obtained by increasing n in this system.

Case 2. LAGRANGE's interpolation formula with remainder term of CAUCHY.

$$f(x_0)=f(x_1)\cdot\frac{(x-x_2)\cdot(x-x_3)\cdots(x-x_n)}{(x_1-x_2)\cdot(x_1-x_3)\cdots(x_1-x_n)} + \cdots + f^n(\xi)\frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{n!}. \qquad \begin{array}{c|c} & x_1, x_2, \cdots, x_m, x \\ \hline 0 & 1,\ 1,\ \cdots,\ 1,\ 1 \end{array}$$

Example. $(0, x), (0, x_0), (1, x_1), (2, x_2), (3, x_3), \cdots, (n-1, x_{n-1})$.

Here the system is conservative if

$$x, x_0, x_1, \cdots, x_n \qquad \text{or} \qquad x, x_0, x_1, \cdots, x_n$$

are in order. We assume

$$x_i = x_0 + \frac{i}{n}(x_n - x_0) \qquad (i=1, 2, \cdots, n).$$

Then we have

$$f(x)=f(x_0)+f'(x_1)(x-x_0)+f''(x_2)\cdot\frac{(x-x_0)(x-x_2)}{2!}+\cdots + f^n(\xi)\cdot\frac{(x-x_0)\cdot(x-x_n)^{n-2}}{(n-1)!},$$

from which a well known development of ABEL's* arises.

Example. $(0, x), (0, x_0), (1, x_1), (2, x_0)\cdots(n-1, x_0)$ or $(n-1, x_1)$.

This system is conservative if x is on (x_0, x_1). Here choosing $x_0=1, x_1=0$ we have

$$F(x)=f(0)+f'(1)\cdot x+f''(0)\cdot\frac{x^2-2x}{2!}+f'''(0)\cdot\frac{x^3-3x}{3!}+\cdots \text{etc.}$$

If

$$E=E_1=\cdots=E_{\lambda-1}=1,$$

when our theorem does not hold, we apply the simplification of (8), § 3,

$$S_A^B[f^n(x)\cdot\Delta(x)]=(-1)^\lambda S_A^B[f^{n+\lambda}(x)\cdot\Delta^{-\lambda}(x)]$$

under the assumptions there stated. From the exact remainder theorem (6) then,

$$f^{k_0}(x_0)=F^{k_0}(x_0)+\frac{S_A^B[f^{n+\lambda}(x)\cdot\Delta^{-\lambda}(x)]}{\Delta_0}. \tag{16}$$

* ABEL; *Sur les fonctions génératrices et leur déterminantes, Oeuvres,* vol. 2, p. 67.

Let $Z_\lambda(x)$ be a polynomial of degree $n+\lambda$ with leading coefficient 1 such that

$$Z^{k_i}(x_i)=0 \qquad (i=1,2,\cdots,n),$$

$$Z^n(x)=\frac{(n+\lambda)!}{\lambda!}\cdot P_\lambda(x),$$

where

$$P_\lambda(x)=x^\lambda+a_1x^{\lambda-1}+a_2x^{\lambda-2}+\cdots+a_\lambda,$$

$a_1, a_2, \cdots, a_\lambda$ being arbitrary. Applying (16) to this polynomial we find

$$Z^{k_0}(x_0)=(-1)^\lambda\frac{\int_A^B(n+\lambda)!\,\Delta^{-\lambda}(x)\,dx}{\Delta_0},$$

so that

$$\text{(17)}\qquad \frac{1}{\Delta_0}=\frac{(-1)^\lambda}{P_\lambda+N_\lambda}\cdot\frac{Z^{k_0}(x_0)}{n+\lambda!}.$$

Also as in (6) § 2,

$$\text{(18)}\quad (-1)^\lambda S_A^B[f^{n+\lambda}(x)\cdot\Delta^{-\lambda}(x)]=(-1)^\lambda\cdot[f^{n+\lambda}(\xi_1)\cdot P_\lambda+f^{n+\lambda}(\xi_2)\cdot N_\lambda].$$

Substituting these values in (16) we get

$$\text{(19)}\qquad f^{k_0}(x_0)=F^{k_0}(x_0)+\frac{f^{n+\lambda}(\xi_1)-E_\lambda\cdot f^{n+\lambda}(\xi_1)}{1-E_\lambda}\cdot\frac{Z^{k_0}(x_0)}{n+\lambda!}.$$

By the aid of this reduction we obtain a remainder theorem in the exceptional cases.

It remains to consider the case

$$\Delta_0=0.$$

If $\Delta_0=0$ then $f^{k_0}(x_0)$ is arbitrary although $f(x)$ be subjected to the conditions

$$\text{(20)}\qquad \begin{aligned} &f^n(x)=\phi(x),\\ &f^{k_i}(x_i)=g_i \qquad (i=1,2,\cdots,n).\end{aligned}$$

For, if $\Delta_0=0$, by the existence theorem there exists a polynomial $M(x)$ such that

$$M^{k_0}(x_0)=1 : M^{k_i}(x_i)=0 \qquad (i=1,2,\cdots,n),$$

of degree less than n, i. e., $M^n(x)=0$.

Then

$$f_1(x)=f(x)+t\,.\,M(x)$$

satisfies the required conditions (20) while $f_1^{k_0}(x_0)$ has the arbitrary value $f^{k_0}(x_0)+t$.

§ 5. *Mechanical differentiation.*

The preceding section admits of direct application to mechanical differentiation as stated in the introduction. For this case $k_0 > 0$.

We give here the application to the case of mechanical differentiation of the first order when the values of the function are given at n points. The system is

$$(1, x_0),\ (0, x_1),\ (0, x_2),\ \cdots,\ (0, x_n)$$

where

$$x_1 < x_2 < \cdots x_n.$$

	$x_0, x_1, x_2, \cdots, x_n$
0	$1, 1, \cdots, 1$
1	1

This includes the important case of equidistant points. In the notation of § 4

$$F(x) = \sum_{i=1}^{n} f(x_i) \cdot \frac{\Pi(x)}{(x - x_i) \cdot \Pi'(x_i)}$$

where

$$\Pi(x) = (x - x_1) \cdot (x - x_2) \cdots (x - x_n),$$

whence $F'(x)$ can be at once computed. The formula (15) of § 4 is here

$$f'(x_0) = F'(x_0) + \frac{f^n(\xi_1) - E \cdot f^n(\xi_2)}{1 - E} \cdot \frac{\Pi'(x_0)}{n!}. \tag{1}$$

It is clear that $E = 0$ if x_0 lies without the interval (x_1, x_n) and also if

$$x_0 = x_i \qquad (i = 1, 2, \cdots, n),$$

since in these cases the system is conservative. Moreover from the theorem concerning the changes of sign of $\Delta(x)$ we infer that $\Delta(x)$ changes sign at most once. If x_0 is within (x_1, x_n), the necessary and sufficient condition that $\Delta(x)$ changes sign is

$$(-1)^n \Delta_1 \cdot \Delta_n < 0. \tag{2}$$

For, near x_1 and x_n from (11) § 1, $\Delta(x)$ is

$$(-1)^n \cdot \frac{(x - x_1)^{n-1}}{(n-1)!} \cdot \Delta_1$$

and

$$\frac{(x_n - x)^{n-1}}{(n-1)!} \cdot \Delta_n,$$

x_0 within (x_1, x_n).* If then $(-1)^n \Delta_1$ and Δ_n are of opposite sign $\Delta(x)$

* This is clear near x_n. Near x_1

$$\Delta(x) = \sum_{i=0}^{n} \epsilon_2(x) \cdot \frac{(x_2 - x)^{l_i}}{l_i!} \cdot \Delta_i = \sum_{i=0}^{n} \frac{(x_i - x)^{l_i}}{l_i!} \Delta - \frac{(x_1 - x)^{n-1}}{(n-1)!} \Delta_1,$$

whence the statement since

$$\sum_{i=0}^{n} \frac{(x_i - x)^{l_i}}{l_i} = 0.$$

changes sign; if $(-1)^n\Delta_1$ and Δ_n are of the same sign $\Delta(x)$ does not change sign. For if it did it would either change sign an even number of times or be discontinuous. Applying the fundamental formula (10) of § 1 to the pair of functions

$$F_1(x) = (x - x_2)(x - x_3)\cdots(x - x_n),$$

$$F_n(x) = (x - x_1)(x - x_2)\cdots(x - x_{n-1}),$$

we have

$$F_1(x_1)\cdot\Delta_1 + F_1'(x_0).\Delta_0 = 0,$$

$$F_n(x_n)\cdot\Delta_n + F_n'(x_0).\Delta_0 = 0,$$

whence

$$\Delta_1.\Delta_n = \frac{F_1'(x_0)\cdot F_n'(x_0)}{F_1(x_1)\cdot F_n(x_n)}\cdot\Delta_0^2.$$

But

$$(-1)^{n-1}F_1(x_1) > 0;\ F_n(x_n) > 0,$$

so that the condition (2) becomes

$$F_1'(x_0)\cdot F_n'(x_0) > 0.$$

The equation

$$F_1'(x_0)\cdot F_n'(x_0) = 0$$

has precisely $2n - 4$ roots, one in each of the intervals (x_1, x_2), (x_{n-1}, x_n) and two in each of the remaining intervals (x_i, x_{i+1}). Moreover these $2n - 4$ roots are distinct, for if $F_1'(x) = 0$ and $F_n'(x) = 0$ had a common root at X, then

$$T(x) = F_n(X)\cdot F_1(x) - F_1(X)\cdot F_n(x)$$

would be a polynomial not identically zero of degree less than n with n roots since

$$T(x_2) = T(x_3) = \cdots = T(x_{n-2}) = 0;\ T(X) = T'(X) = 0.$$

If the roots of $F_1'(x_0)\cdot F_n'(x_0) = 0$, in order, are

$$\theta_1, \theta_{21}, \theta_{22}, \theta_{31}, \theta_{32}, \cdots, \theta_{n-1}$$

we have

$$x_1 < \theta_1 < x_2 < \theta_{21} < \theta_{22} < x_3 < \cdots \theta_{n-1} < x_n.$$

Since $E = 0$ for

$$x_0 = x_i \qquad (i = 1, 2, \cdots, n).$$

$\Delta(x)$ does not change sign on the intervals

$$(\theta_1, \theta_{21}),\ (\theta_{22}, \theta_{31}),\ \cdots,\ (\theta_{n-2\,2}, \theta_{n-1}),$$

where then $E = 0$. Moreover for the $n - 1$ roots of $\Pi'(x) = 0$, we have $E = 1$ from the condition (12) of § 3. At these points we can use the remainder theorem of (19) § 4. At points x_0 where $E \neq 0$ or 1, to compute E one must determine the position of the root of $\Delta(x) = 0$. These same methods can be extended. ENCKE* has given difference formulæ for obtaining $F'(x)$ in

* ENCKE: *Über mechanische Quadratur, Gesammelte Abhandlungen*, vol. 1, p. 21.

the case of equal intervals. The method here indicated gives rigorously the character of the remainder.

§ 6. *Mechanical quadrature.*

Consider a normal system $(\bar{k}_i, \bar{x}_i)$ of the type

$$(0, \rho_1),\ (0, \rho_2);\ (k_i + 1, x_i) \qquad (i = 1, 2, \cdots, n), \tag{1}$$

where the $\bar{\Delta}_0$ corresponding to $(0, \rho_1)$ does not vanish. Then the Δ_0 of the system

$$(k_0, x_0),\ (k_i, x_i) \qquad (i = 1, 2, \cdots, n)$$

does not vanish for here

$$\bar{\Delta}_0 = (-1)^n \cdot \Delta_0 .$$

Write now

$$\bar{f}(x) = \int_{\rho_2}^{x} f(x)\,dx,$$

then $\bar{F}(x)$ satisfies the conditions

$$\bar{F}(\rho_2) = \bar{f}(\rho_2) = 0 : \bar{F}^{k_i+1}(x_i) = f^{k_i}(x_i) \qquad (i = 1, 2, \cdots, n),$$

whence it appears that

$$\bar{F}(x) = \int_{\rho_2}^{x} F(x)\,dx,$$

where $F(x)$, as before, is the polynomial of degree $n-1$ at most such that

$$F^{k_i}(x_i) = f^{k_i}(x_i) \qquad (i = 1, 2, \cdots, n).$$

Similarly

$$\bar{Z}(x) = (n+1) \cdot \int_{\rho_2}^{x} Z(x)\,dx,$$

where $Z(x)$ is defined as in § 4. Thus we obtain, by substituting these values in

$$\bar{f}(\rho_1) = \bar{F}(\rho_1) + \frac{\bar{f}^{n+1}(\xi_1) - \bar{E}\cdot \bar{f}^{n+1}(\xi_2)}{1 - \bar{E}} \cdot \frac{\bar{Z}(\rho_1)}{(n+1)!},$$

that

$$\int_{\rho_2}^{\rho_1} f(x)\,dx = \int_{\rho_2}^{\rho_1} F(x)\,dx + \frac{f^n(\xi_1) - \bar{E}\cdot f^n(\xi_2)}{1 - \bar{E}} \cdot \frac{\int_{\rho_2}^{\rho_1} Z(x)\,dx}{n!} \tag{2}$$

the remainder theorem for mechanical quadrature. Here $\bar{E}$ is the characteristic number of the system (1) with respect to 1. If $\bar{E} = 1$ we can obtain another expression as in (19) § 4.

Of especial practical importance in single quadrature is the system

$$(0, x_1),\ (1, x_1),\ (1, x_2),\ \cdots,\ (1, x_n),\ (0, x_n),$$

where

$$x_2 - x_1 = x_3 - x_2 = \cdots x_n - x_{n-1} = h .$$

COTES * determined the members H_n^i for $n = 2, 3, \cdots, 11$ where

$$\int_{x_1}^{x_n} F(x)\,dx = (x_n - x_1)\cdot \sum_{i=1}^{n} H_n^i \cdot f\left(x_1 + i\,\frac{x_n - x_1}{n-1}\right)$$

for practical use in mechanical quadrature. On actually constructing $\Delta(x)$ with the aid of these numbers for $n = 2, 3, \cdots, 11$ I found that for $n = 2, 4, 6, 8, 10, E = 0$ so that the remainder is

$$f^n(\xi)\cdot \int_{x_1}^{x_n} \frac{(x - x_1)\cdot(x - x_2)\cdots(x - x_n)}{n!}\,dx, \tag{3}$$

and that if $n = 3, 5, 7, 9, 11,\ E = 1,\ E_1 = 0$ so that using (19) § 4 we could write the remainder

$$f^{n+1}(\xi)\cdot \int_{x_1}^{x_n} \frac{(x - x_1)\cdot(x - x_2)\cdots(x - x_n)}{n!}\,dx. \tag{3$'$}$$

Another system of interest is GAUSS's system. For this system

$$(0, c),\ (1, \alpha_1),\ (2, \alpha_1),\ (1, \alpha_2),\ (2, \alpha_2),\ \cdots,\ (1, \alpha_m),\ (2, \alpha_m),\ (0, d),$$

where $\alpha_1, \alpha_2, \cdots, \alpha_m$ are roots of

$$\frac{d^m}{dx^m}\{(x - c)^m (x - d)^m\} = 0;$$

the formula is

$$\int_c^d f(x)\,dx = c_1 \cdot f(\alpha_1) + c_2 f(\alpha_2) + \cdots + c_m \cdot f(\alpha_m)$$
$$+ \frac{(d - c)^{2m+1}}{2m + 1}\cdot\left\{\frac{1\cdot 2\cdot 3\cdots m}{m + 1\cdot m + 2\cdots 2m}\right\}\cdot\frac{f^{2m}(\xi)}{2m!}.\ \dagger$$

§ 7. *Proof of the theorem of* § 2.

We first consider the discontinuities of $\Delta(x)$ and of its derivatives. The kth derivative of any term of $\Delta(x)$ in the expression (11) of § 1 is

$$\epsilon_i(x)\cdot(-1)^k\cdot\frac{(x_i - x)^{l_i - k}}{(l_i - k)!}\,\Delta_i \qquad (l_i > k),$$

$$\epsilon_i(x)\cdot(-1)^k\cdot\Delta_i \qquad (l_i = k),$$

$$0 \qquad (l_i < k).$$

In the first and last case, this derivative has nowhere a discontinuity; in the second case there is a discontinuity at x_i providing $\Delta_i \neq 0$.

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*COTES, *Harmonia mensurarum* ; MARKOFF, loc. cit., pp. 60, 61.

† MARKOFF, loc. cit., p. 68.

Accordingly $\Delta^k(x), \Delta^{k+1}(x), \cdots, \Delta^{n-1}(x)$ have their discontinuities at the points

$$x_i;\ k_i = 0, 1, \cdots, n - k_i - 1 \qquad (\Delta_i \neq 0);$$

these points lie on the interval $(A_{\bar{k}}, B_{\bar{k}})$* where we write in general

$$\bar{k} = n - k - 1.$$

It is to be noted that $\Delta(x)$ vanishes everywhere without (A, B). Hence $\Delta^k(x)$ vanishes everywhere without $(A_{\bar{k}}, B_{\bar{k}})$, $\Delta^k(x)$ being of polynomial form and neither itself or its derivatives having discontinuities without $(A_{\bar{k}}, B_{\bar{k}})$. Also $\Delta^k(x)$ does not vanish just within $(A_{\bar{k}}, B_{\bar{k}})$, if none of the Δ_i's are zero.

We first make our proof for a special case assuming

(α) none of the Δ_i's are zero,

$(\beta)\, \Delta^k(x)$ has no zero stretch within $(A_{\bar{k}}, B_{\bar{k}})$ for $k = 0, 1, \cdots, n-1$.

It is to be noted that $\Delta(x)$ and its derivatives are not defined at $x_0, x_1, \cdots, x_n$. Further, $\Delta^k(x)$ is said to be continuous at a point if the same value is approached on opposite sides of the point. On account of (β), $\Delta^k(x)$ changes sign at points but not along stretches. Let Z_k denote the number of changes of sign of $\Delta^k(x)$ continuous or discontinuous according as $\Delta^k(x)$ is continuous or discontinuous at the point.

The following table gives a classification of the discontinuities of $\Delta^k(x)$ at a point according simultaneously to the position of the discontinuities and to the change or permanence of sign at a discontinuity, with a notation for the number of discontinuities of each class.

Number of points of discontinuity of $\Delta^k(x)$	Position: $x = A_{\bar{k}}$ or $x = B_{\bar{k}}$	$A_{\bar{k}} < x \leqq A_{\overline{k+1}}$† or $B_{\overline{k+1}} \leqq x < B_{\bar{k}}$	$A_{\overline{k+1}} < x < B_{\overline{k+1}}$†
Permanence of $\Delta^k(x)$	E_k	C_k	P_k
Change of $\Delta^k(x)$ with			
(a) permanence of $\Delta^{k+1}(x)$	0‡	C'_k	P'_k
(b) change of $\Delta^{k+1}(x)$	0‡	0‡	P''_k

* For definition of (A_k, B_k), see § 2.

† Here $(A_{\bar{n}}, B_n)$ is not defined and we write

$$P_{n-1} = P'_{n-1} = P''_{n-1} = 0,$$

and because $\Delta^{n-1}(x)$ is constant except at discontinuities

$$Z_{n-1} = C'_{n-1}.$$

Also $E_{n-1} = 2$, there being a discontinuity at either end.

‡ The zero entries are due to the fact $\Delta^k(x)$ is zero without $(A_{\bar{k}}, B_{\bar{k}})$ and $\Delta^{k-1}(x)$ zero without $(A_{\overline{k+1}}, B_{\overline{k+1}})$.

We can now proceed. First the number of continuous changes of sign of $\Delta^k(x)$ is

$$Z_k - (C'_k + P'_k + P''_k),$$

the sum $C'_k + P'_k + P''_k$ being the number of discontinuous changes of sign. Also the function $\Delta^k(x)$ tends to zero at $2 - E_k$ points $A_{\bar{k}}$, $B_{\bar{k}}$.

Between any *adjacent* two of these

$$\{Z_k - (C'_k + P'_k + P''_k)\} + \{2 - E_k\} \tag{1}$$

points there is either

I) a change of sign of Δ^{k+1}, $\Delta^k(x)$ not changing sign, or

II) a change of sign of $\Delta^k(x)$, $\Delta^{k+1}(x)$ not changing sign.

That one or the other of these changes does occur is evident from the fact that the product function $\Delta^k(x) : \Delta^{k+1}(x)$ changes sign between two such zero values. The total number of changes of sign of $\Delta^{k+1}(x)$ is Z_{k+1}. The number of these changes at which $\Delta^k(x)$ also changes sign is P'', for if $\Delta^k(x)$ changes sign at such a point it must do so discontinuously. Hence $Z_{k+1} - P''_k$ is the number of changes I.

Moreover since $\Delta^k(x)$ does not change sign continuously at the intermediate points, P'_k is the number of changes II.

There are

$$\{Z_k - (C'_k + P'_k + P''_k)\} + \{2 - E_k\} - 1 \tag{2}$$

pairs of adjacent points;* hence

$$\{Z_{k+1} - P''_k\} + P'_k \geqq \{Z_k - (C'_k + P'_k + P''_k)\} + \{Z - E_k\} - 1$$

or more simply

$$Z_{k+1} + 2P'_k + C'_k + E_k \quad - 1 \geqq Z_k \qquad (k = 0, 1, \cdots, n-2). \tag{3}$$

Therefore since also

$$C'_{n-1} = Z_{n-1}, \tag{4}$$

$$2P'_{n-1} = 0, \tag{5}$$

$$E_{n-1} - 2 = 0, \tag{6}$$

we have adding (3), (4), (5) and (6)

$$2\sum_{k=0}^{n-1} P'_k + \sum_{k=0}^{n-1} C'_k + \sum_{k=0}^{n-1} E_k - (n+1) \geqq Z_0.$$

Or if we write in general

$$\sum_{k=0}^{n-1} T_k = T$$

we have

$$Z_0 \leqq 2P' + C' + E - (n+1).$$

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* If there are no *pairs* of points this sum is 0 or -1.

But since the enumeration of discontinuities is complete and no Δ_i's are zero

$$E + C + C' + P + P' + P'' = n + 1.$$

Hence

(7) $$Z_0 \leqq P' - (C + P + P''),$$

an inequality affording a method for determining a superior limit to the number of changes of sign of $\Delta(x)$.

Now consider the discontinuities of $\Delta(x)$, $\Delta'(x)$, $\cdots$, $\Delta^{n-1}(x)$ all of which can be grouped into separate sequences of discontinuities as of

$$\Delta^k(x), \Delta^{k+1}(x), \cdots, \Delta^\nu(x)$$

at ξ, corresponding to which we have the inverted separate sequence of pairs

$$(\bar{k}, \xi), (\bar{k} - 1, \xi), \cdots, (\bar{\nu}, \xi).$$

Here $\Delta^k(x)$ at ξ is the *first* discontinuity of the sequence and $\Delta^\nu(x)$ at ξ is the *last* discontinuity. In such a sequence a discontinuity of type P'_k cannot be directly succeeded by one of type P'_{k+1}, C'_{k+1} or E_{k+1}. For if $\Delta^k(x)$ has a P'_k discontinuity at ξ, $\Delta^{k+1}(x)$ does not change sign at ξ, excluding types P'_{k+1} and C'_{k+1}, and

$$A_{\overline{k+1}} < \xi < B_{\overline{k+1}},$$

excluding type E_{k+1}. Hence to every discontinuity P' not the last member of its sequence we have a succeeding discontinuity of type C, P or P''.

Let us consider the discontinuities as grouped in separate sequences t. Unless then in such a sequence a P' discontinuity terminates the sequence

$$P'_t - (C_t + P_t + P''_t) \leqq 0.$$

Moreover when t is terminated by a discontinuity P'

$$P'_t - (C_t + P_t + P''_t) \leqq 1.$$

Let t' be a sequence of β terms in which

(8) $$P_{t'} - (C_{t'} + P_{t'} + P''_{t'}) = 1.$$

Let the final discontinuity be of $\Delta^\nu(x)$ at ξ corresponding to the initial pair $(\bar{\nu}, \xi)$ of the sequence of pairs. Then

(9) $$A_{\bar{\nu}-1} = A_{\overline{\nu+1}} < \xi < B_{\overline{\nu+1}} = B_{\bar{\nu}-1},$$

this discontinuity being of type P. Hence the sequence of pairs is *non-conservative*. Also no preceding discontinuity can be of types C, C' or E. Therefore,

$$P_{t'} + P'_{t'} + P''_{t''} = \beta,$$

and from (8)

$$P'_{t'} - (P_{t'} + P''_{t'}) = 1.$$

Adding these last equations we find

$$P'_{\iota'} = \frac{\beta + 1}{2}$$

so that the sequence, besides being non-conservative, is *odd*. If W be the number of non-conservative odd sequences, then from (7) and (8)

$$Z_0 \leqq W,$$

which is the theorem.

We proceed to the proof for the case when not both (α) and (β) hold, by considering the effect of a slight variation of the x_i's.

Let

(10) $$(K_1, X_1)\ (K_2, X_2), \cdots, (K_s, X_s)$$

be the s leading pairs of the sequences of pairs written so that

(11) $$X_1 \leqq X_2 \leqq X_3 \cdots \leqq X_s$$

and so that if

$$X_\sigma = X_{\sigma+1},$$

then

$$K_\sigma < K_{\sigma+1}$$

if $X_\sigma < A_0$, and

$$K_\sigma > K_{\sigma+1}$$

if $X_\sigma > B_0$. By the definition of conservative and non-conservative sequences, it is clear that if we vary $X_1, X_2, \cdots, X_s$ in the system of pairs, preserving the order relation (11), that conservative sequences remain conservative.

Choose now ϵ so that for

(12) $$|\bar{X}_\sigma - X_\sigma| < \epsilon \qquad (\sigma = 1, 2, \cdots, s)$$

the transformed $\bar{\Delta}(x)$ changes sign at least as often as $\Delta(x)$. Let us then make any slight change in the $X_1, X_2, \cdots, X_s$ of the system in accordance with (11) and (12). This change does not increase the number of non-conservative odd sequences of pairs W. If then (α) and (β) hold for some such sequence we have for $\bar{\Delta}(x)$

$$\bar{Z}_0 \leqq W,$$

and hence for $\Delta(x)$

$$Z_0 \leqq \bar{Z}_0 \leqq W,$$

which we desired to prove.

But the conditions (α) and (β) do hold for some such variation system unless one of the conditions

(13) $$\Delta_i \equiv 0 \qquad (i = 0, 1, \cdots, n),$$

or

(14) $$\Delta^k(x) \equiv 0 \qquad (A_k \leqq X_\sigma < x < X_{\sigma+1} \leqq \bar{B}_k)$$

holds identically in $X_1, X_2, \cdots, X_s$. That none of these do hold can be proved thus. Rearrange the leading pairs (10) in

$$(\kappa_1, \xi_1), (\kappa_2, \xi_2), \cdots, (\kappa_s, \xi_s)$$

where

$$\kappa_\sigma \leqq \kappa_{\sigma+1} \qquad (\sigma = 0, 1, \cdots, s),$$

and consider the highest term of (13) and (14) in

$$x, \xi_1, \cdots, \xi_s \tag{15}$$

as it appears *formally*. The coefficient of this term is not zero.

For in the expression $\Delta^k(x)$, the x coefficient of this highest formal term is some Δ_i. It suffices then to prove that none of the conditions (12) can obtain, say that

$$\Delta_0 \not\equiv 0.$$

The argument of page 113, § 1, shows that Δ_0 does not vanish formally, i. e., that no rows or columns are made up of zeros. First differentiate Δ_0 in ξ_1 to the highest formal power. We obtain a number of determinants, each the same, and we have

$$\frac{\partial^{\nu_1}}{\partial \xi_1^{\nu_1}} \Delta_0 = d_1 \cdot u_1.$$

We then proceed in the same way for $\xi_2, \xi_3, \cdots, \xi_n$ in order and find

$$\frac{\partial^{\nu_1}}{\partial \xi_1^{\nu_1}} \frac{\partial^{\nu_2}}{\partial \xi_2^{\nu_2}} \cdots \frac{\partial^{\nu_s}}{\partial \xi_s^{\nu_s}} \Delta_0 = d_1 \cdot d_2 \cdots d_n u_n.$$

Precisely as on page 113, § 1,

$$u_n = \pm 1.$$

Hence $\Delta_0 \not\equiv 0$ in $\xi_1, \xi_2, \cdots, \xi_s$.

THE UNIVERSITY OF CHICAGO,
October 31, 1905.