DETERMINATION OF THE ABSTRACT GROUPS OF ORDER p^2qr; p, q, r BEING DISTINCT PRIMES*

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Since the publication† in 1899 of Professor MILLER's "Report on recent progress in the theory of groups of finite order," WESTERN‡ has published his determination of the groups of order p^3q, and LE VASSEUR§ has discussed the order p^2q^2. This paper is devoted to the determination of all groups of the order p^2qr. It thus completes the discussion of the problem of groups whose orders are products of four primes.‖

With the exception of the group of order $2^2 \cdot 3 \cdot 5$, simply isomorphic with the icosahedron-group, all groups of order p^2qr are solvable. The maximal self-conjugate subgroups will therefore serve as the basis of classification. The twelve possible arrangements of the factors of composition are

$$(1)\ ppqr,\quad (2)\ pprq,\quad (3)\ pqpr,\quad (4)\ pqrp,\quad (5)\ prpq,\quad (6)\ prqp,$$

$$(7)\ qppr,\quad (8)\ qprp,\quad (9)\ qrpp,\quad (10)\ rqpp,\quad (11)\ rppq,\quad (12)\ rpqp.$$

If for a given type of group precisely the arrangements (i), (j), (k), $\cdots$, of the factors of composition are possible, then we symbolize¶ the group $(i, j, k, \cdots)$. Two groups having distinct symbols cannot be simply isomorphic.

The group G always contains a maximal invariant subgroup** of order p^2q, and may contain maximal subgroups†† of order p^2r and pqr. We shall discuss

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† Bulletin, American Mathematical Society, vol. 1 (1899), p. 227.

‡ Proceedings of the London Mathematical Society, vol. 30 (1899), p. 209.

§ Annales Toulouse, 1903, p. 63. Comptes Rendus, vol. 128 (1899), p. 1152, and lithographed book.

‖ HÖLDER, Mathematische Annalen, vol. 43 (1893), p. 335. BURNSIDE, *Finite Groups*, p. 81. HÖLDER, Göttinger Nachrichten (1895), p. 211.

¶ Additional abbreviations used throughout are the following: P, Q, $\cdots$, operations of order p, q, $\cdots$; $H_{h,i}$, a maximal invariant subgroup of G, order h and type i; $\rho_{\Omega,h}$, number of subgroups of G, order h, permutable with Ω; N_h, number of subgroups of G of order h.

** FROBENIOUS, Berliner Sitzungsberichte, vol. 1 (1895), p. 170.

†† HÖLDER, loc. cit., COLE and GLOVER, American Journal of Mathematics, vol. 15 (1893), p. 202 BURNSIDE, *Theory of Groups*, p. 63.

in detail in this paper only two classes of groups: those possessing invariant subgroups of both the types H_{p^2q} and H_{p^2r}, and those possessing maximal invariant subgroups of the type H_{p^2q} only. A detailed summary of the results obtained in the other classes is given at the end. We shall thus be concerned principally with the subgroups $H_{p^2\sigma}$ $(\sigma = q, r)$ all types of which are given in the following table, in which τ denotes the number of distinct types, while (p) signifies (modulo p):

$H_{p^2\sigma, i}$	$S_3^{-1}SS_3$	$S_3^{-1}S_1S_3$	$S_3^{-1}S_2S_3$	$S_2^{-1}S_1S_2$	Parameters	τ
$i = I$	S	.	.	.		1
II	.	S_1	S_2	S_1		1
III	.	S_1^a	S_2	S_1	$a^\sigma \equiv 1\,(p),\ p \equiv 1\,(\sigma)$	1
IV	S^a	.	.	.	$a^\sigma \equiv 1\,(p^2),\ p \equiv 1(\sigma)$	1
V	.	S_1^a	$S_2^{a^h}$	S_1	$a^\sigma \equiv 1\,(p),\ p \equiv 1\,(\sigma)$	1 or $\frac{1}{2}[\sigma+1]$
VI	.	S_2	$S_1^{-1}S_2^{\iota p+\iota}$	S_1	* $\iota^\sigma \equiv 1\,(p),\ p \equiv -1\,(\sigma)$	1

$$\sigma = q, r;\ S^{p^2} = 1,\ S_1^p = S_2^p = 1,\ S_3^\sigma = 1.$$

§ 1. *Determination of* $\rho_{\Omega, h}$.

By SYLOW's theorem,† $N_\sigma = qr/\sigma, p, p^2, pqr/\sigma, p^2qr/\sigma$ or 1. If $N_\sigma = 1$ then $\rho_{\Omega,\sigma} = 1$, Ω being any operator of prime order in G. When $N_\sigma > 1$, the result of transforming the single conjugate set of N_σ subgroups

$$g_1, g_2, g_3, \cdots, g_{N_\sigma}$$

by Ω is to permute them among themselves. Hence

$$\Omega^{-1}(g_1, g_2, \cdots, g_{N_\sigma})\Omega = \begin{pmatrix} g_1, & g_2, & \cdots, & g_{N_\sigma} \\ g_{i_1}, & g_{i_2}, & \cdots, & g_{i_{N_\sigma}} \end{pmatrix} = J_{\Omega,\sigma}.$$

It follows that $J_{\Omega,\sigma}^\omega = 1$ and

$$(1) \qquad N_\sigma - \rho_{\Omega,\sigma} \equiv 0 \pmod{\omega};\ \rho_{\Omega,\sigma} \geqq 1.$$

Next let $\omega = \sigma$. Then $N_p = (p^2 - 1)/(p-1) = p+1$, and

$$(2) \qquad p + 1 - \rho_{\Omega,p} \equiv 0 \pmod{\sigma}.$$

Hence either $\rho_{\Omega,p} = 0$ or else $\rho_{\Omega,p} \geqq 2$ $(\omega = q, r)$. Now if the subgroup I_{p^2} of $H_{p^2\sigma, i}$ is cyclical the order of its group of isomorphisms is

$$I = \phi(p^2) = p(p-1).$$

* Throughout the paper ι denotes a non-integral mark of the $GF\,[p^2]$. Thus $\iota^\sigma \equiv 1(p)$ is an abbreviation for $\iota^\sigma \equiv 1$ (modd p, P), P being any quadratic function irreducible modulo p.

† SYLOW, Mathematische Annalen, vol. 5 (1872).

If I_{p^2} is of type $[1, 1]$ its group of isomorphisms is simply isomorphic with the congruence group $\{S_1, S_2 \cdots\}$ of order $I = p(p-1)^2(p+1)$, where S_1 is

$$y_1 \equiv a_{11}x_1 + a_{12}x_2, \qquad y_2 \equiv a_{21}x_1 + a_{22}x_2 \pmod{p},$$

or say

$$S_1 = (a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2).$$

Since Ω corresponds to an isomorphism of G, $\{\Omega\}$ corresponds to a subgroup of the group of isomorphisms of G and ω divides I. Hence when I_{p^2} is cyclical, or when $I_{p^2} = [1, 1]$ and $p \equiv 1(\sigma)$, $\rho_{\Omega, p} \geqq 2$. But when $p \equiv -1(\sigma)$ and p is odd, $\rho_{\Omega, p} = 0$. Also since $\rho_{\Omega, \sigma} \geqq 1$, $J_{Q, \sigma}$ and $J_{R, \sigma}$ may be permutable. If

$$S_2 = (b_{11}x_1 + b_{12}x_2, b_{21}x_1 + b_{22}x_2)$$

the necessary and sufficient conditions that $S_1 S_2 = S_2 S_1$ are

$$(3)\quad \delta_{12} = \begin{vmatrix} a_{12} & b_{12} \\ a_{11} - a_{22} & b_{11} - b_{22} \end{vmatrix} \equiv 0,^{*} \qquad \delta'_{12} = \begin{vmatrix} a_{21} & b_{21} \\ a_{11} - a_{22} & b_{11} - b_{22} \end{vmatrix} \equiv 0,$$

$$d_{12} = \begin{vmatrix} a_{12} & a_{21} \\ b_{12} & b_{21} \end{vmatrix} \equiv 0.$$

§ 2. *Class* (9, 10), $p > q > r$.

We now consider the groups whose symbol is (9, 10), having the maximal subgroups $H_{p^2q, i}$ and $H_{p^2r, j}$ $(i, j = \text{IV, V, VI})$. Since I_{p^2} is invariant in G the existence of a subgroup of type IV excludes the possibility of a subgroup of type V or VI, and vice versa. There are thus five cases to consider.

[1] $i = j = \text{IV}$. Here $I_{p^2} = \{P\}$ is cyclical and P may be regarded as the generator of order p^2 in both H-sub-groups. Since $\rho_{Q, r} \geqq 1$, we may choose $\{R\}$ permutable with Q and, since $q > r$, $QR = RQ$, so that G is defined by

$$P^{p^2} = Q^q = R^r = 1, \qquad Q^{-1}PQ = P^{\alpha}, \qquad R^{-1}PR = P^{\beta}, \qquad QR = RQ;$$

or for brevity $G = (\alpha : \beta : 1)$, where

$$\alpha^q \equiv 1, \qquad \beta^r \equiv 1(p^2), \qquad p \equiv 1(qr), \qquad \tau = 1.$$

[2] $i = j = \text{V}$. Let $H_{p^2q, i} = \{P'_1, P'_2, Q\}$, $H_{p^2r, j} = \{P_1, P_2, R\}$, wherein $QR = RQ$. We may write

$$R^{-1}P_1R = P_1^{\alpha}, \qquad R^{-1}P_2R = P_2^{\beta}, \qquad \alpha^r \equiv 1(p), \qquad \beta \equiv \alpha^h.$$

$$Q^{-1}P_1Q = P_1^{a_{11}}P_2^{a_{21}}, \qquad Q^{-1}P_2Q = P_1^{a_{12}}P_2^{a_{22}},$$

and from the permutable isomorphisms of I_{p^2}

$$J_Q = \begin{pmatrix} P_1^{x_1} P_2^{x_2} \\ P_1^{a_{11}x_1 + a_{12}x_2} P_2^{a_{21}x_1 + a_{22}x_2} \end{pmatrix}, \qquad J_R = \begin{pmatrix} P_1^{x_1} P_2^{x_2} \\ P_1^{\alpha x_1} P_2^{\beta x_2} \end{pmatrix},$$

* All congruences are taken modulo p unless otherwise indicated

$$(4) \qquad \delta_{12} = a_{12}(\alpha - \beta) \equiv 0, \qquad \delta'_{12} = a_{21}(\alpha - \beta) \equiv 0.$$

Reserving for later treatment the ambiguous case $h = 1$, we deduce $a_{12} \equiv a_{21} \equiv 0$. Suppose next that

$$R^{-1}P'_i R = P_1^{b_{1i}} P_2^{b_{2i}} \qquad (i = 1, 2).$$

Then

$$(RQ)^{-1}P'_1(RQ) = P_1^{a_{11}b_{11}} P_2^{a_{22}b_{21}} = (QR)^{-1}P'_1(QR) = P_1^{\gamma b_{11}} P_2^{\gamma b_{21}},$$

$$(5) \qquad \begin{aligned} &b_{11}(a_{11} - \gamma) \equiv 0, \qquad b_{21}(a_{22} - \gamma) \equiv 0, \qquad \gamma^q \equiv 1, \\ &b_{12}(a_{11} - \delta) \equiv 0, \qquad b_{22}(a_{22} - \delta) \equiv 0, \qquad \delta \equiv \gamma^k. \end{aligned}$$

Thus when $h \neq 1$, $k \neq 1$ we have one of the two equivalent results

$$a_{11} \equiv \gamma, \ a_{22} \equiv \delta \qquad \text{or} \qquad a_{11} \equiv \delta, \ a_{22} \equiv \gamma.$$

In case $h \neq 1$, $k = 1$, the set (5) becomes

$$\begin{aligned} &b_{11}(a_{11} - \gamma) \equiv 0, \qquad b_{21}(a_{22} - \gamma) \equiv 0, \\ &b_{12}(a_{11} - \gamma) \equiv 0, \qquad b_{22}(a_{22} - \gamma) \equiv 0, \end{aligned}$$

and there are three possibilities to consider, viz.,

(i) $\quad a_{11} \not\equiv \gamma, \quad b_{11} \equiv 0, \quad b_{12} \equiv 0, \quad b_{21} \not\equiv 0, \quad b_{22} \not\equiv 0, \quad a_{22} \equiv \gamma;$

(ii) $\quad a_{11} \equiv \gamma, \quad a_{22} \not\equiv \gamma, \quad b_{21} \equiv b_{22} \equiv 0, \quad b_{11} \not\equiv 0, \quad b_{12} \not\equiv 0;$

(iii) $\quad a_{11} \equiv \gamma, \quad a_{22} \equiv \gamma.$

Case (i) implies

$$R^{-1}P'_1 R = P_2^{b_{21}}, \qquad R^{-1}P'_2 R = P_2^{b_{22}},$$

$$R^{-1}P_1'^{b_{22}} R = R^{-1}P_2'^{b_{21}} R \qquad \text{or} \qquad P_1'^{b_{22}} = P_2'^{b_{21}},$$

contrary to the independence of P'_1 and P'_2. Likewise, case (ii) is excluded. Hence $a_{11} \equiv a_{22} \equiv \gamma$.

In a similar manner, when $h = 1$, $k \neq 1$, we get $a_{11} \equiv a_{22} \equiv \alpha$.

Next let $h = 1$, $k = 1$, so that

$$R^{-1}P_i R = P_i^{\alpha}, \qquad Q^{-1}P_i Q = P_i^{\gamma} \qquad (i = 1, 2).$$

One of the operations P'_1, P'_2 must be independent of P_1. As $\gamma^q \equiv 1 \pmod{p}$, we may assume that P_1 and P'_2 are independent. These will generate I_{p^2}, so that

$$Q^{-1}P_1 Q = P_1^{a_{11}} P_2'^{a_{21}}, \qquad R^{-1}P'_2 R = P_1^{b_{12}} P_2'^{b_{22}}.$$

The abelian conditions from J_Q and J_R are [Eq. (3)]

$$\delta_{12} = b_{12}(a_{11} - \delta) \equiv 0, \qquad \delta'_{12} = a_{21}(b_{22} - \alpha) \equiv 0, \qquad d_{12} = a_{21} b_{12} \equiv 0.$$

Thus three possibilities arise, viz.,

(i) $a_{21} \equiv 0, \quad b_{12} \equiv 0 \quad a_{11} \equiv \delta;$

(ii) $a_{21} \not\equiv 0, \quad b_{12} \equiv 0, \quad b_{22} \equiv \alpha;$

(iii) $a_{21} \equiv 0, \quad b_{12} \equiv 0.$

For (i), let $P_1' = P_1^x P_2'^y$, $P_2 = P_1^z P_2'^w$, whence

$$Q^{-1}P_1'Q = P_1^{x\gamma}P_2'^{y\gamma} = P_1^{\delta x}P_2'^{\delta y},$$

$$R^{-1}P_2R = P_1^{z\beta}P_2^{w\beta} = P_1^{\alpha z+b_{12}w}P_2'^{b_{22}w},$$

$$(\gamma-\delta)x \equiv 0, \qquad (\gamma-\delta)y \equiv 0,$$

$$w(b_{22}-\beta) \equiv 0, \qquad z(\alpha-\beta)+b_{12}w \equiv 0.$$

Hence $\gamma \equiv \delta$ and $k = 1$; but as P_1, P_2 are independent, $w \not\equiv 0$, $b_{22} \equiv \beta$, $\alpha \not\equiv \beta$ and $h \neq 1$, contrary to hypothesis. Since (ii) is likewise excluded, we have $a_{21} \equiv b_{12} \equiv 0$,

$$Q^{-1}P_1Q = P_1^{a_{11}}, \qquad R^{-1}P_2'R = P_2'^{b_{22}},$$

$$x(a_{11}-\gamma) \equiv 0, \qquad y(\delta-\gamma) \equiv 0,$$

$$z(\beta-\alpha) \equiv 0, \qquad w(b_{22}-\beta) \equiv 0,$$

where $x \not\equiv 0$, $w \not\equiv 0$. Hence when $\alpha \equiv \beta$, $\delta \equiv \gamma$ there results $a_{11} \equiv \gamma$, $b_{22} \equiv \alpha$. We are thus led to a single set of defining relations:

$$P_1^p = P_2^p = Q^q = R^r = 1, \qquad P_1P_2 = P_2P_1, \qquad Q^{-1}P_1Q = P_1^\gamma,$$

$$Q^{-1}P_2Q = P_2^{\gamma^k}, \qquad R^{-1}P_1R = P_1^\alpha, \qquad R^{-1}P_2R = P_2^{\alpha^h}, \qquad RQ = QR,$$

$$\alpha^r \equiv 1(p), \qquad \gamma^q \equiv 1(p) \quad (h = 1, 2, \cdots, r-1;\ k = 1, 2, \cdots, q-1),$$

or, briefly, say $G = (1:\gamma 0:0\gamma^k:\alpha 0:0\alpha^h:1)$. Proceeding to the determination of τ we observe that there are, by hypothesis, two subgroups, $\{P_1\}$, $\{P_2\}$, both permutable with Q and R. In any isomorphism of G with itself either $\{P_1\} \sim \{P_2\}$, $\{P_2\} \sim \{P_1\}$ or else $\{P_1\} \sim \{P_1\}$, $\{P_2\} \sim \{P_2\}$. Hence there are two choices of generators of order p. Every element of G is of the form $\Omega = R^xQ^yP_1^uP_2^v$. Hence $\Omega^s = R^{sx}Q^{sy}P_1^{u_s}P_2^{v_s}$, so that Ω is of order r only when $y \equiv 0 \pmod q$ and of order q when $x \equiv 0 \pmod r$. Thus the most general operator of order q is $Q_0' = Q^yP_1^uP_2^v$, which transforms G in the same manner as $Q_0 = Q^y$. Similarly $R_0 = R^x$. Employing the new generators R_0, Q_0, $P_{1_0} = P_1$, $P_{2_0} = P_2$, we get

$$(1:\gamma 0:0\gamma^k:\alpha 0:0\alpha^h:1) \sim (1:\gamma^y 0:0\gamma^{ky}:\alpha^x 0:0\alpha^{hx}:1).$$

Hence any set of relations involving arbitrary primitive roots (α^a, γ^b) can be transformed into the original set. Next let $P_{1_0} = P_2$, $P_{2_0} = P_1$. Then

$$(1:\gamma 0:0\gamma^k:\alpha 0:0\alpha^h:1) \sim (1:\gamma^{ky} 0:0\gamma^y:\alpha^{hx} 0:0\alpha^x:1)$$

if

(6) $$ky \equiv 1 \pmod{q}, \qquad hx \equiv 1 \pmod{r}.$$

The group characterized by $[h, k]$ is thus isomorphic with $[x, y]$ when (6) is satisfied. Further τ equals the number of distinct solutions of (6), e. g., when $r = 2$, $\tau = \frac{1}{2}(q+1)$, and when r is odd, $\tau = \frac{1}{4}(qr + q + r + 1)$.

[3] $i = \mathrm{VI}$, $j = \mathrm{V}$. When $h \neq 1$ we have $Q^{-1}P_jQ = P_j^{a_{jj}}$ $(j = 1, 2)$. Assuming that

$$R^{-1}P'_1R = P_1^xP_2^y, \qquad R^{-1}P'_2R = P_1^zP_2^w,$$

we derive

$$a_{11}x - z \equiv 0, \qquad x - (\iota^p + \iota - a_{11})z \equiv 0,$$

$$a_{22}y - w \equiv 0, \qquad y - (\iota^p + \iota - a_{22})w \equiv 0.$$

The elimination of x, y, z, w gives

$$a_{jj}^2 - (\iota^p + \iota)a_{jj} + 1 \equiv 0 \qquad (j = 1, 2),$$

whence $a_{jj} = \iota^p$ or ι. Hence a_{11}, a_{22} are galoisian imaginaries* and G, for $i = \mathrm{VI}$, $j = \mathrm{V}$, does not exist.

Before considering the ambiguous case $h = 1$ a few general results must be established.

Let S and T be any set of generators of I_{p^2}, so that $G = \{S, T, Q, R\}$. We may write

$$P'_1 = S^xT^y, \qquad P'_2 = S^zT^w,$$

$$Q^{-1}SQ = S^{a_{11}}T^{a_{21}}, \qquad Q^{-1}TQ = S^{a_{12}}T^{a_{22}}.$$

Hence

$$Q^{-1}P'_1Q = P'_2 = S^zT^w = S^{a_{11}x+a_{12}y}T^{a_{21}x+a_{22}y},$$

$$Q^{-1}P'_2Q = P_1'^{-1}P_2'^{\iota p+\iota} = S^{-x+(\iota p+\iota)z}T^{-y+(\iota p+\iota)w} = S^{a_{11}z+a_{12}w}T^{a_{21}z+a_{22}w},$$

whence results the eliminant

$$D = \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \begin{vmatrix} a_{11} & a_{12} & -1 & 0 \\ a_{21} & a_{22} & 0 & -1 \\ 1 & 0 & a_{11} - t & a_{12} \\ 0 & 1 & a_{21} & a_{22} - t \end{vmatrix} \end{array} \equiv 0 \pmod{p},$$

where $t = \iota^p + \iota$. Its expansion gives

$$D_{12}^2 - t(a_{11} + a_{22} - t)D_{12} + a_{22}^2 - a_{11}^2 + t(a_{11} - a_{22}) + 2a_{12}a_{21} + 1 \equiv 0.$$

Now assume $S = P_1$. Then, since $p \equiv -1 \pmod{q}$, $\rho_{Q,p} = 0$ and we may take $Q^{-1}P_1Q \equiv U$ as T. Then

*Serret, *Cours d'Algebre Superieur*, cinq. ed. (1885), tome 2, sec. 3, chap. 3. See also Dickson, *Linear Groups*, pp. 14–19.

$$J_Q = \begin{pmatrix} P_1^{x_1} U^{x_2} \\ P_1^{a_{12}x_2} U^{x_1 + a_{22}x_2} \end{pmatrix}, \quad J_Q^q = 1,$$

$$D_{12}^q = \begin{vmatrix} 0 & a_{12} \\ 1 & a_{22} \end{vmatrix}^q \equiv (-a_{12})^q \equiv 1 \pmod p.$$

Now $-a_{12}$ cannot be a primitive root of this congruence; for, if so $p \equiv 1 \pmod q$, whereas $p \equiv -1 \pmod q$ and $q > r$. It follows that $a_{12} \equiv -1 \pmod p$ and

$$D \equiv (a_{22} - t)^2 \equiv 0, \quad a_{22} \equiv t \equiv \iota^p + \iota.$$

This gives $I_{p^2} = \{P_1 U\}$ and

(7)
$$Q^{-1} P_1 Q = U, \qquad Q^{-1} U Q = P_1^{-1} U^{\iota^p + \iota},$$
$$R^{-1} P_1 R = P_1^{\alpha}, \qquad R^{-1} U R = P_1^{\xi} U^{\eta},$$

$$\delta_{12} = \begin{vmatrix} -1 & \xi \\ -\iota^p - \iota & \alpha - \eta \end{vmatrix} \equiv 0, \qquad \delta'_{12} = \begin{vmatrix} 1 & 0 \\ -\iota^p - \iota & \alpha - \eta \end{vmatrix} \equiv 0,$$

and thus, when $h = 1$, $\eta \equiv \alpha$, $\xi \equiv 0 \pmod p$.

Inversely let $P_2 = P_1^{\xi'} U^{\eta'}$. Then

$$R^{-1} P_2 R = P_1^{\xi' \alpha^h} U^{\eta' \alpha^h} = P_1^{\xi' \alpha} U^{\eta' \alpha}$$

and hence $h = 1$. Thus when $h = 1$ there exists a group

$$G = \{P_1, U, Q, R\} = (1:01:-1\iota^p + \iota : \alpha 0 : 0\alpha : 1),$$

where $\alpha^r \equiv 1(p)$, $p \equiv 1(r)$, $\tau = 1$. Also $p \equiv -1 \pmod q$ and, in the $GF[p^2]$, $\iota^q \equiv 1 \pmod p$.

[4] $i = V$, $j = VI$. Since r is necessarily an odd prime, the argument of [3] again gives for G a single type, $G = (1:\gamma 0:0\gamma:01:-1\iota^p + \iota:1)$, with $\gamma^q \equiv 1(p)$, $p \equiv 1(q)$, $\tau = 1$. Likewise $p \equiv -1 \pmod r$; and $\iota^r \equiv 1 \pmod p$ in the $GF[p^2]$.

[5] $i = \mathrm{VI}$, $j = \mathrm{VI}$. Employing as in [3] the theory of the determinant D we are led to the same equations (7), viz.,

$$Q^{-1} P_1 Q = U, \qquad Q^{-1} U Q = P_1^{-1} U^{\iota_1^p + \iota_1}, \qquad \iota_1^q \equiv 1(p).$$

Let us assume that

$$R^{-1} P_1 R = P_2 = P_1^x U^y, \qquad R^{-1} U R = P_1^z U^w.$$

Then

$$\delta_{12} = \begin{vmatrix} -1 & z \\ -\iota_1^p - \iota_1 & x - w \end{vmatrix} \equiv 0, \qquad \delta'_{12} = \begin{vmatrix} 1 & y \\ -\iota_1^p - \iota_1 & x - w \end{vmatrix} \equiv 0,$$

$$d_{12} = \begin{vmatrix} -1 & 1 \\ z & y \end{vmatrix} \equiv 0, \qquad D_{12} = \begin{vmatrix} x & z \\ y & w \end{vmatrix} \not\equiv 0.$$

Thus

$$z \equiv -y, \qquad w \equiv x + (\iota_1^p + \iota_1)y, \qquad D_{12} \equiv x^2 + (\iota_1^p + \iota_1)xy + y^2.$$

Since

$$R^{-1}P_2R = P_1^{-1}P_2^{\iota_2^p + \iota_2}, \qquad \iota_2^r \equiv 1\,(p),$$

so that

$$R^{-1}U^yR = P_1^{-y^2}U^{xy+(\iota_1^p+\iota)y^2} = P_1^{-(x^2+1)+(\iota_2^p+\iota_2)x}U^{-xy+(\iota_2^p+\iota_2)y}.$$

Since P_1 and P_2 are independent, $y \not\equiv 0$; hence

$$(8) \qquad 2x + (\iota_1^p + \iota_1)y - (\iota_2^p + \iota_2) \equiv 0,$$

$$(9) \qquad y^2 - x^2 + (\iota_2^p + \iota_2)x - 1 \equiv 0.$$

From the latter we at once derive

$$D_{12} = x^2 + (\iota_1^p + \iota_1)xy + y^2 \equiv 1,$$

$$(10) \qquad (\iota_2 - \iota_1^2\iota_2)^2x^2 - (1 - \iota_1^2)(\iota_2 - \iota_2^3)x + (1 - \iota_1^2\iota_2^2)(\iota_2^2 - \iota_1^2) \equiv 0,$$

$$(11) \qquad (\iota_1 - \iota_1^p)^2y^2 - (\iota_2 - \iota_2^p)^2 \equiv 0.$$

There always exist integral solutions of (10) and (11), $x = \epsilon_j,\, y = \sigma_j\ (j = 1, 2)$. Thus

$$R^{-1}P_1R = P_1^{\epsilon_j+(\iota_1^p+\iota_1)\sigma_j}U^{-\sigma_j}, \qquad R^{-1}UR = P_1^{\sigma_j}U^{\epsilon_j}.$$

THEOREM. *The two general types of G characterized by the two distinct sets of solutions of* (10) *and* (11), *viz.* $[\epsilon_1, \sigma_1]$ *and* $[\epsilon_2, \sigma_2]$ *are simply isomorphic.*

In proof, $\sigma_2 \equiv -\sigma_1$, and congruence (8) gives

$$2\epsilon_2 - (\iota_1^p + \iota_1)\sigma_1 - (\iota_2^p + \iota_2) \equiv 0, \qquad \epsilon_2 \equiv \epsilon_1 + (\iota_1^p + \iota_1)\sigma_1.$$

Hence the two types of G are characterized by

$$R^{-1}P_1R = P_1^{\epsilon_1+(\iota_1^p+\iota_1)\sigma_1}U^{-\sigma_1}, \qquad R^{-1}UR = P_1^{\sigma_1}U^{\epsilon_1},$$

and

$$R^{-1}P_1R = P_1^{\epsilon_1}U^{\sigma_1}, \qquad R^{-1}UR = P_1^{-\sigma_1}U^{\epsilon_1+(\iota_1^p+\iota_1)\sigma_1}.$$

Let us select a new operation of order q from $\{Q\}$, e. g. $Q' = Q^{-1}$. Then $Q'R = RQ'$, $Q'^{-1}UQ' = P_1$,

$$Q'^{-1}P_1Q' = U^{r_1}P_1^{r_2} = U^{-1}P_1^{\iota_1^p+\iota_1}, \qquad r_j = \frac{\iota_1^{(q-j)p} - \iota_1^{q-j}}{\iota_1^p - \iota_1}.$$

The result of selecting Q' and (ϵ_2, σ_2) is thus to interchange P_1 and U and to reproduce the relations given by Q and (ϵ_1, σ_1). Hence $[\epsilon_2, \sigma_2] \sim [\epsilon_1, \sigma_1]$.

The quantities ι_1 and ι_2 are marks of the $GF[p^2]$ and in that field appertain respectively to the exponents q and r. Let ρ be any primitive root in the $GF[p^2]$. It is easy to show that $\tau = 1$ and hence we may select *

$$\iota_1 \equiv \rho^{(p^2-1)/q}, \qquad \iota_2 \equiv \rho^{(p^2-1)/r},$$

thus

$$G = (1:01:-1,\ \iota_1^p + \iota_1 : \epsilon + (\iota_1^p + \iota_1)\sigma,\ -\sigma : \sigma\epsilon : 1),$$

where

$$\iota_1 \equiv \rho^{(p^2-1)/q},\ \iota_2 \equiv \rho^{(p^2-1)/r},\ \rho^{p^2-1} \equiv 1; \qquad p \equiv -1 \pmod{qr},\ \tau = 1,$$

$$(\iota_1 - \iota_1^p)^2\sigma^2 - (\iota_2 - \iota_2^p)^2 \equiv 0, \qquad 2\epsilon + (\iota_1^p + \iota_1)\sigma - (\iota_2^p + \iota_2) \equiv 0.$$

§ 3. *The generating function* $[k]$.

Consider the relation $R^{-z}P_1R^z = P_1^{u_z}U^{v_z}$. From it

$$u_{z+1} - (2x + t_1 y)u_z + (x^2 + t_1xy + y^2)u_{z-1} \equiv 0,$$

$$u_{z+1} - t_2 u_z + u_{z-1} \equiv 0 \qquad (t_j = \iota_j^p + \iota_j;\ j = 1, 2),$$

These recurring formulæ give

$$u_k \equiv [k]_2 x - [k-1]_2, \qquad v_k = [k]_2 y,$$

where

$$[k]_j \equiv \frac{\iota_j^{kp} - \iota_j^k}{\iota_j^p - \iota_j}.$$

Following are some of the properties of the generating function $[k]_j$.

(12) $$\frac{[k+1]_j}{[k]_j} = \frac{1}{t_j +}\,\frac{1}{t_j +}\,\frac{1}{t_j +} \cdots k \text{ terms},$$

(13) $$[k]_j^2 - [k+1]_j[k-1]_j - 1 \equiv 0,$$

(14) $$[0]_j \equiv 0, \qquad [1]_j \equiv 1, \qquad [-k]_j \equiv -[k]_j,$$

(15) $$[k+1]_j \equiv [2]_j[k]_j - [k-1]_j,$$

(16) $$\{[k+1]_j - [k-1]_j - [2]_j\}\iota_j^k \equiv (\iota_j^{k+1} - 1)(\iota_j^{k-1} - 1).$$

§ 4. *Class* (10), $p > q > r$.

We shall consider next groups possessing a single maximal self-conjugate subgroup $H_{p^2q,\,i}$ of non-abelian type ($i = \mathrm{III}, \mathrm{IV}, \mathrm{V}, \mathrm{VI}$). It is readily shown that class (10, 12), with $i = \mathrm{III}$, must contain an invariant subgroup H_{p^2r}. Class (10) remains to be considered.

[1] $i = \mathrm{IV}$. Here $H_{p^2q,\,\mathrm{IV}} = \{P\,.\,Q\}$ and since $\{P\}$ is self-conjugate in G, $R^{-1}PR = P^\beta$. Since $\rho_{R,\,q} \geqq 1$ [Eq. (1)], $R^{-1}QR = Q^\gamma$. Hence

$$(QR)^{-1}P(QR) = P^{\alpha\beta} = (RQ^\gamma)^{-1}P(RQ^\gamma) = P^{\beta\alpha^\gamma}, \qquad \alpha^q \equiv 1\,(p^2),$$

$$\alpha\beta(\alpha^{\gamma-1} - 1) \equiv 0 \pmod{p^2}, \qquad \gamma \equiv 1 \pmod{q}.$$

* DICKSON, *Linear Groups*, p. 13.

Hence $\{P_1, P_2, R\}$ is self-conjugate in $\{P_1, P_2, Q, R\} = G$, contrary to hypothesis.

[2] $i = \text{V}$. Let $H_{p^2q, \text{V}} = \{P'_1, P_2, Q\}$. Assuming that

$$R^{-1}P'_1R = P_1'^{a_{11}}P_2^{a_{21}}, \qquad R^{-1}P_2R = P_1'^{a_{12}}P_2^{a_{22}},$$

we deduce

$$a_{11}\alpha(\alpha^{\gamma-1} - 1) \equiv 0, \qquad a_{21}(\beta^{\gamma} - \alpha) \equiv 0,$$

$$a_{22}\beta(\beta^{\gamma-1} - 1) \equiv 0, \qquad a_{12}(\alpha^{\gamma} - \beta) \equiv 0,$$

where $\alpha^q \equiv 1(p)$, $\beta \equiv \alpha^h$. Now $\gamma \not\equiv 1 \pmod q$. Hence

$$a_{11} \equiv 0, \qquad a_{22} \equiv 0, \qquad \alpha^{\gamma h} \equiv \alpha, \qquad \alpha^{\gamma} \equiv \alpha^h \pmod p,$$

$$\gamma \equiv h \pmod q, \qquad \alpha^{\gamma^2} \equiv \alpha \pmod p, \qquad \gamma^2 \equiv 1 \pmod q.$$

But γ appertains to the exponent r modulo q, and therefore $r = 2$ and $\gamma \equiv -1 \pmod q$. Thus

$$R^{-1}P'_1R = P_2^{a_{21}}, \qquad R^{-1}P_2R = P_1'^{a_{12}}, \qquad a_{12}a_{21} \equiv 1 \pmod p.$$

Then $P_1 = P_1'^{a_{12}}$, P_2, Q, R, generate a group of order $2p^2q$, viz., $G = (1:\alpha 0:0\alpha^{q-1}:01:10:-1)$. Also $p \equiv 1(q)$, $\tau = 1$.

[3] $i = \text{VI}$. It has been shown [§ 1], that $p \equiv \pm 1 \pmod r$.

(a) First let $p \equiv 1(r)$. Then $\rho_{R,p} \geqq 2$ and two subgroups $\{P_1\}$, $\{P_3\}$ may be selected which are permutable with R. If

$$Q^{-1}P_1Q = P_2, \qquad Q^{-1}P_2Q = P_1^{-1}P_2^{\iota p+\iota},$$

then

$$R^{-1}P_1R = P_1^{\beta}, \qquad R^{-1}QR = Q^{\gamma}, \qquad \gamma \not\equiv 1 \pmod q.$$

Since I_{p^2} is invariant in G we may assume that

$$P_3 = P_1^zP_2^w, \qquad R^{-1}P_2R = P_1^xP_2^y,$$

Hence

$$(QR)^{-1}P_1(QR) = P_1^xP_2^y = (RQ^{\gamma})^{-1}R_1(RQ^{\gamma}) = P_1^{-\beta[\gamma-1]}P_2^{\beta[\gamma]},$$

$$(QR)^{-1}P_2(QR) = P_1^{-\beta+[2]x}P_2^{[2]y} = (RQ^{\gamma})^{-1}P_2(RQ^{\gamma}) = P_1^{-[\gamma-1]x-[\gamma]y}P_2^{[\gamma]x+[\gamma+1]y},$$

$$x \equiv -[\gamma-1]\beta, \qquad y \equiv [\gamma]\beta,$$

$$[\gamma]^2 \equiv [\gamma-1]^2 + [2][\gamma-1] + 1,$$

$$[\gamma]\{[\gamma+1] - [\gamma-1] - [2]\} \equiv 0.$$

Now $[\gamma] \not\equiv 0 \pmod q$. Since $[-k] \equiv -[k]$ and

$$[\gamma+1] - [\gamma-1] - [2] \equiv (\iota^{\gamma+1} - 1)(\iota^{\gamma-1} - 1) \equiv 0 \text{ [Eq. (16)]},$$

there results $\gamma \equiv -1 \pmod q$, $\gamma^r \equiv (-1)^r \equiv +1 \pmod q$, whence $r = 2$. If $R^{-1}P_3R = P_3^{\alpha}$, then $\alpha \equiv \pm 1 \pmod p$,

$$w(y \mp 1) \equiv 0, \qquad xw + z(\beta \mp 1) \equiv 0,$$
$$w(-\beta \mp 1) \equiv 0, \qquad [2]\beta w + z(\beta \mp 1) \equiv 0.$$

First let the upper sign hold. If $\beta \equiv 1$, then $w \equiv 0$ which is impossible, since P_1, P_3 are independent. Hence $\beta \equiv -1$, $x \equiv -[2]$, $y \equiv +[1] \equiv +1$. Likewise if we use the lower sign, $\beta \equiv +1$, $x \equiv +[2]$, $y \equiv -[1] \equiv -1$. We thus obtain the two sets of defining relations:

$$(1:01:-1\iota^p + \iota : \mp 10 : \iota^{\mp p} + \iota^{\mp 1}, \pm 1 : -1).$$

To determine τ, let $Q_0 = Q^x$, $R_0 = R$, $P_{1_0} = P_1$, $P_{2_0} = P_1^{-[x-1]}P_2^{[x]}$; there results

$$\{P_{1_0}, P_{2_0}, Q_0, R_0\} = (1:01:-1\iota^{xp} + \iota^x : \mp 10 : \pm[x-1] \mp [2][x], \pm[x] : -1).$$

But

$$\pm[x-1] \mp [2][x] \equiv \mp \lfloor x+1] \equiv \mp(\iota^{xp} + \iota^x) \mp [x-1],$$

[Eq. (15)]. Hence

$$\{P_{1_0}, P_{2_0}, Q_0, R_0\} = (1:01:-1\iota^{xp} + \iota^x : \mp 10 : \mp(\iota^{xp} + \iota^x), \pm 1 : -1) \sim G.$$

Thus the same defining relations are reproduced with ι replaced by ι^x, and so $\tau = 1$.

It will now be proved that these two types are simply isomorphic. Select new operators as follows:

$$q_1 = Q, \; r_1 = R, \; p_1 = P_1^a P_2^b, \; p_2 = P_1^{-b} P_2^{a+[2]b} = q_1^{-1} p_1 q_1.$$

Then using the first set of defining relations we will have

$$q_1^{-1}p_2q_1 = p_1^{-1}p_2^{\iota^p+\iota}, \; r_1^{-1}p_1r_1 = p_1, \; r_1^{-1}p_2r_1 = p_1^{\iota^p+\iota}p_2^{-1}, \; r_1^{-1}q_1r_1 = q_1^{-1}$$

if

$$2a + [2]b \equiv 0 \pmod p.$$

Hence when a new operator $p_1 = P_1^a P_2^b$ is selected, where a and b are solutions of $2a + (\iota^p + \iota)b \equiv 0 \pmod p$, the first type is transformed into the second. They are therefore isomorphic.

(*b*) When $p \equiv -1(r)$, r odd, $\rho_{R,p} = 0$. As before, we deduce

$$Q^{-1}P_1Q = P_2, \qquad Q^{-1}P_2Q = P_1^{-1}P_2^{\iota_1^p+\iota_1}, \qquad \iota_1^q \equiv 1(p),$$
$$R^{-1}P_1R = P_3, \qquad R^{-1}P_3R = P_1^{-1}P_3^{\iota_2^p+\iota_2}, \qquad \iota_2^r \equiv 1(p),$$

Let $P_3 = P_1^x P_2^y$ and $R^{-1}P_2R = P_4 = P_1^z P_2^w$. Then

$$(17) \quad R^{-1}P_2^yR = P_1^{-(x^2+1)+[2]_2 x} P_2^{y[2]_2 - yx} = P_1^{-[\gamma-1]_1 xy - [\gamma]_1 y^2} P^{[\gamma]_1 xy + [\gamma+1]_1 y^2}.$$

In addition to the latter, but not independent of them, we have the congruences derived from

$$(QR)^{-1}P_2^y(QR) = (RQ^\gamma)^{-1}P_2^y(RQ^\gamma). \tag{18}$$

The equations (17) and (18) give us the dialytic eliminant

$$\Delta_{12} = \{\iota_2^p + \iota_2\}\{[\gamma]_1^2 - (\iota_2^{2p} + \iota_2^2)[\gamma]_1 + 1\}\{(\iota_1^{\gamma+1} - 1)(\iota_1^{\gamma-1} - 1)\}^2 \equiv 0.$$

Now $[\gamma]_1$ is an integer, and since $r \neq 2$, and $\gamma \not\equiv -1$, it follows that $\gamma \equiv 1 \pmod q$, contrary to hypothesis. Hence when $p \equiv -1 \pmod r$ and r is odd, no corresponding group G exists.

The results of this section may be summarized in the following

THEOREM. *A group $G_{p^2qr}(p > q > r)$ always contains a maximal self-conjugate subgroup H of order p^2q. If H is the only maximal invariant subgroup of G and if r is odd, then $N_q = 1$ and H is necessarily abelian. If r is even ($r = 2$) and $p \equiv 1 \pmod q$ there exists one type whose subgroup H_{p^2q} is non-abelian, and if r is even and $p \equiv -1 \pmod q$ there exists a second type possessing a non-abelian H_{p^2q}. These two types of G contain respectively q and pq operators (and subgroups) of order 2, and in each type $N_q = p^2$. Moreover, with exception of the two types just described, every group of order $p^2qr(p > q > r)$, in which $N_r \equiv 0 \pmod q$, possesses an abelian maximal self-conjugate subgroup H_{p^2q}.*

A general summary of all the existent types of G follows. Except for ι and ρ, every parameter occurring in the tables is an integer; while ι and ρ are marks of the $GF[p^2]$. See footnote on the second page of the paper.

TABLE 1. $p > q > r$.

I_{p^2} non-cyclical; $P_1^p = P_2^p = Q^q = R^r = 1$, $P_1P_2 = P_2P_1$. I_{p^2} cyclical; $P_1^{p^2} = Q^q = R^r = 1$.

Case (*a*), $QR = RQ$.

Class	$Q^{-1}P_1Q$	$Q^{-1}P_2Q$	$R^{-1}P_1R$	$R^{-1}P_2R$	Parameters.	Arith. Rel.	τ
[12 ⋯ 12]	P_1	.	P_1	.	.	.	1
"	P_1	P_2	P_1	P_2	.	.	1
[3467891012]	P_1^α	P_2	P_1	P_2	$\alpha^q \equiv 1\,(p)$	$p \equiv 1\,(q)$	1
[3489101112]	P_1	P_2	P_1^α	P_2	$\alpha^r \equiv 1\,(p)$	$p \equiv 1\,(r)$	1
[46891012]	P_1^α	P_2	P_1^β	P_2	$\alpha^q \equiv \beta^r \equiv 1\,(p)$	$p \equiv 1\,(qr)$	1
[78910]	P_1^α	.	P_1	.	$\alpha^q \equiv 1\,(p^2)$	$p \equiv 1\,(q)$	1
"	P_1^α	$P_2^{\alpha^k}$	P_1	P_2	$\alpha^q \equiv 1\,(p)$	$p \equiv 1\,(q)$	$\frac{1}{2}(q+1)$
"	P_2	$P_1^{-1}P_2^{\iota p+\iota}$	P_1	P_2	$\iota^q \equiv 1\,(p)$	$p \equiv -1\,(q)$	1
[891012]	P_1^α	P_2	P_1	P_2^β	$\alpha^q \equiv \beta^r \equiv 1\,(p)$	$p \equiv 1\,(qr)$	1
[9101112]	P_1	.	P_1^α	.	$\alpha^r \equiv 1\,(p^2)$	$p \equiv 1\,(r)$	1
"	P_1	P_2	P_1^α	$P_2^{\alpha^k}$	$\alpha^r \equiv 1\,(p)$	$p \equiv 1\,(r)$	1 or $\frac{1}{2}(r+1)$
"	P_1	P_2	P_2	$P_1^{-1}P_2^{\iota p+\iota}$	$\iota^r \equiv 1\,(p)$	$p \equiv -1\,(r)$	1
[8910]	P_1^β	$P_2^{\beta^k}$	P_1^α	P_2	$\alpha^r \equiv \beta^q \equiv 1\,(p)$	$p \equiv 1\,(qr)$	$q-1$
[91012]	P_1^α	P_2	P_1^β	$P_2^{\beta^k}$	$\alpha^q \equiv \beta^r \equiv 1\,(p)$	$p \equiv 1\,(qr)$	$r-1$
[910]	P_1^α	.	P_1^β	.	$\alpha^q \equiv \beta^r \equiv 1\,(p^2)$	$p \equiv 1\,(qr)$	1
"	P_1^γ	$P_2^{\gamma^k}$	P_1^α	$P_2^{\alpha^k}$	$\gamma^q \equiv \alpha^r \equiv 1\,(p)$	$p \equiv 1\,(qr)$	$\frac{1}{2}(q+1)$ or $\frac{1}{4}(r+1)(q+1)$
"	P_2	$P_1^{-1}P_2^{\iota p+\iota}$	P_1^α	P_2^α	$\iota^q \equiv 1(p)$ $\alpha^r \equiv 1\,(p)$	$p \equiv -1\,(q)$ $p \equiv 1\,(r)$	1
"	P_2	$P_1^{-1}P_2^{\rho^{\frac{p^3-p}{q}}+\rho^{\frac{p^2-1}{q}}}$	$P_1^{\epsilon+\left(\rho^{\frac{p^3-p}{q}}+\rho^{\frac{p^2-1}{q}}\right)\sigma}P_2^{-\sigma}$	$P_1^\sigma P_2^\epsilon$	$\rho =$ prim. root in $GF[p^2]$; $\iota_1, \iota_2 = \rho^{(p^2-1)/q,\, r}$ $2\epsilon + [2]_1\sigma - [2]_2 \equiv 0$ $(\iota_1 - \iota_1^p)^2\sigma^2 - (\iota_2 - \iota_2^p)^2 \equiv 0$	$p \equiv -1\,(qr)$	1
"	P_1^γ	P_2^γ	P_2	$P_1^{-1}P_2^{\iota p+\iota}$	$\iota^r \equiv 1\,(p)$ $\gamma^q \equiv 1\,(p)$	$p \equiv -1(r)$ $p \equiv 1\,(q)$	1

Case (b). $R^{-1}QR = Q^{\gamma^h}$; $\gamma^r \equiv 1(q)$.

Class.	$Q^{-1}P_1Q$	$Q^{-1}P_2Q$	$R^{-1}P_1R$	$R^{-1}P_2R$	Parameters.	Arith. rel.	τ
[25610 11 12]	P_1	P_2	P_1	P_2	$h = 1$	$q \equiv 1(r)$	1
"	P_1	.	P_1	.	$h = 1$	$q \equiv 1(r)$	1
[56101112]	P_1	P_2	P_1^{α}	P_2	$h = 1, 2 \cdots r-1$ $\alpha^r \equiv 1(p)$	$p \equiv q \equiv 1(r)$	$r-1$
[101112]	P_1	.	P_1^{α}	.	$h = 1, 2 \cdots r-1$ $\alpha^r \equiv 1(p^2)$	$p \equiv q \equiv 1(r)$	$r-1$
"	P_1	P_2	P_1^{α}	$P_2^{\alpha^k}$	$h, k = 1, 2 \cdots r-1$ $\alpha^r \equiv 1(p)$	$p \equiv q \equiv 1(r)$	1 or $\frac{1}{2}(r^2-1)$
"	P_1	P_2	P_2	$P_1^{-1}P_2^{\iota p+\iota}$	$h = 1, 2 \cdots r-1$ $\iota^r \equiv 1(p)$	$p \equiv -q \equiv -1(r)$	$r-1$
[10]	P_1^{α}	$P_2^{\alpha^{q-1}}$	P_2	P_1	$h = 1, \gamma \equiv -1$ $\alpha^q \equiv 1(p)$	$r = 2$ $p \equiv 1(q)$	1
"	P_2	$P_1^{-1}P_2^{\iota p+\iota}$	P_1^{-1}	$P_1^{\iota-p+\iota^{-1}}P_2$	$h = 1, \gamma \equiv -1$ $\iota^q \equiv 1(p)$	$r = 2$ $p \equiv -1(q)$	1

TABLE 2. $q > p > r$.

I_{p^2} non-cyclical; $P_i^p = Q^q = R^r = 1$ $(i = 1, 2)$, $P_1P_2 = P_2P_1$, $RP_2 = P_2R$,
I_{p^2} cyclical; $P_1^{p^2} = Q^q = R^r = 1$, $RP_1 = P_1R$.

Class.	$P_1^{-1}QP_1$	$P_2^{-1}QP_2$	$R^{-1}QR$	$R^{-1}P_1R$	Parameters.	Arith. Rel.	τ
[1234561112]	Q^{α}	.	Q	P_1	$\alpha^p \equiv 1(q)$	$q \equiv 1(p)$	1
[12511]	Q^{α}	.	Q	P_1	$\alpha^{p^2} \equiv 1(q)$	$q \equiv 1(p^2)$	1
[1112]	Q^{α}	.	Q^{γ}	P_1	$\alpha^p \equiv \gamma^r \equiv 1(q)$	$q \equiv 1(pr)$	1
[11]	Q^{α}	.	Q^{γ}	P_1	$\alpha^{p^2} \equiv \gamma^r \equiv 1(q)$	$q \equiv 1(p^2r)$	1
[1251112]	Q	Q^{α}	Q	P_1	$\alpha^p \equiv 1(q)$	$q \equiv 1(p)$	1
[251112]	Q	Q^{γ}	Q^{α}	P_1	$\gamma^p \equiv \alpha^r \equiv 1(q)$	$q \equiv 1(pr)$	1
[4561112]	Q	Q^{α}	Q	P_1^{δ}	$\alpha^p \equiv 1(q)$ $\delta^r \equiv 1(p)$	$q \equiv 1(p)$ $p \equiv 1(r)$	1
[561112]	Q	Q^{α}	Q^{γ^h}	P_1^{δ}	$\alpha^p \equiv \gamma^r \equiv 1(q)$ $\delta^r \equiv 1(p)$	$q \equiv 1(pr)$ $p \equiv 1(r)$	$r-1$

TABLE 3. $q > r > p$.

Case (a).

I_{p^2} non-cyclical; $P_i^p = Q^q = R^r = 1 \, (i = 1, 2)$, $P_1P_2 = P_2P_1$, $RQ = QR$,
I_{p^2} cyclical; $P_1^{p^2} = Q^q = R^r = 1$, $QR = RQ$.

Class.	$P_1^{-1}QP_1$	$P_2^{-1}QP_2$	$P_1^{-1}RP_1$	$P_2^{-1}RP_2$	Parameters.	Arith. Rel.	τ
[12345678]	Q	.	R^{α}	.	$\alpha^p \equiv 1(r)$	$r \equiv 1(p)$	1
[1237]	Q	.	R^{α}	.	$\alpha^{p^2} \equiv 1(r)$	$r \equiv 1(p^2)$	1
[123456]	Q^{α}	.	$R^{\beta^{\lambda}}$	.	$\alpha^p \equiv 1(q)$ $\beta^p \equiv 1(r)$	$q \equiv r \equiv 1(p)$	$p-1$
[125]	Q^{α}	.	$R^{\beta^{\lambda}}$	.	$\alpha^{p^2} \equiv 1(q)$ $\beta^p \equiv 1(r)$	$q \equiv 1(p^2)$ $r \equiv 1(p)$	$p-1$
[234]	$Q^{\alpha^{\lambda}}$	.	R^{β}	.	$\alpha^p \equiv 1(q)$ $\beta^{p^2} \equiv 1(r)$	$r \equiv 1(p^2)$ $q \equiv 1(p)$	$p-1$
[12]	Q^{α}	.	$R^{\beta^{\lambda}}$	.	$\alpha^{p^2} \equiv 1(q)$ $\beta^{p^2} \equiv 1(r)$	$q \equiv r \equiv 1(p^2)$	p^2-1
[12345678]	Q	Q	R	R^{α}	$\alpha^p \equiv 1(r)$	$r \equiv 1(p)$	1
[123]	Q	Q^{α}	R	$R^{\beta^{\lambda}}$	$\alpha^p \equiv 1(q)$ $\beta^p \equiv 1(r)$	$q \equiv r \equiv 1(p)$	$p-1$
[1235]	Q	Q^{β}	R^{α}	R	$\alpha^p \equiv 1(r)$ $\beta^p \equiv 1(q)$	$q \equiv r \equiv 1(p)$	1

Case (b). *The simple group* $G_{\frac{1}{2}5!}$, $p = 2$, $q = 5$, $r = 3$.

$$Q^5 = 1, \quad P^2 = 1, \quad (QP)^3 = 1, \quad [R = QP].$$