# THEOREMS CONVERSE TO RIEMANN'S ON LINEAR DIFFERENTIAL EQUATIONS*

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In the fragment *Zwei allgemeine Sätze über lineare Differentialgleichungen mit algebraischen Coefficienten* numbered XXI in his collected works, RIEMANN laid the foundation of the modern theory of linear differential equations by regarding the solutions as a linear family and studying the substitution group of such a family. The importance of his results will perhaps lend interest to an investigation of theorems of a converse character. The present paper is not, however, concerned with the two theorems from which RIEMANN's fragment takes its name, but with two others from which they are deduced.

Although the paper referred to discusses only differential equations all of whose singular points are regular, we add no new difficulties to our problem if we generalize the results there obtained to the case of $n+1$ linear families of the $n$th order, $y^{(1)}, y^{(2)}, \cdots, y^{(n+1)}$, all analytic in the same $m$-tuply connected region $T_m$, and having the same substitution group (Monodromiegruppe) in $T_m$. The term *basis* will be used here to designate any system of $n$ linearly independent members of a family. If bases chosen from different families have the same group of substitutions they will be said to *correspond*, and homologous members of such bases will be called *corresponding branches*. With this terminology the two theorems of RIEMANN which we here consider may be stated as follows:

*A. If in the matrix*

$$\left\| \begin{matrix} y_1^{(1)} & y_1^{(2)} & \cdots & y_1^{(n+1)} \\ y_2^{(1)} & y_2^{(2)} & \cdots & y_2^{(n+1)} \\ \cdot & \cdot & \cdot & \cdot \\ y_n^{(1)} & y_n^{(2)} & \cdots & y_n^{(n+1)} \end{matrix} \right\|$$

*the columns are corresponding bases of the families $y^{(1)}, y^{(2)}, \cdots, y^{(n+1)}$ respectively, the $n+1$ determinants formed by striking out in succession each column in the above matrix are functions which have the same substitutions. These substitutions are merely multiplicative, the multiplier for any circuit in $T_m$*

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*being the determinant of the substitution undergone by the columns of the matrix.*

*B. Every set of corresponding branches* $y_i^{(1)}, y_i^{(2)}, \cdots, y_i^{(n+1)}$ *satisfies a linear relation*

$$a_1 y_i^{(1)} + a_2 y_i^{(2)} + \cdots + a_{n+1} y_i^{(n+1)} = 0 \qquad (i=1,2,\cdots,n)$$

*whose coefficients are single-valued and analytic in* $T_m$.

In what follows we will suppose given a part, or all, of what appears in the conclusions above, deducing therefrom criteria that different families have the same group.

## § 1. *Theorems converse to A.*

We now proceed to invert the order of theorem $A$, obtaining thereby criteria in terms of certain determinants that two families of the $n$th order, $y$ and $z$, have the same group. With the family $y$ we associate $n-1$ other families $y^{(1)}, y^{(2)}, \cdots, y^{(n-1)}$ which are known to have the same group as $y$, and which are analytic in the same region $T_m$. Let $z$ also be analytic in this region. We now form the matrix

$$(1) \qquad \left\| \begin{matrix} z_1 & y_1 & y_1^{(1)} & \cdots & y_1^{(n-1)} \\ z_2 & y_2 & y_2^{(1)} & \cdots & y_2^{(n-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ z_n & y_n & y_n^{(1)} & \cdots & y_n^{(n-1)} \end{matrix} \right\|$$

in which each column is a basis of its respective family, the last $n$ being corresponding bases. By $Z, Y, Y^{(1)}, \cdots, Y^{(n-1)}$ we shall designate the determinants, taken with alternate positive and negative signs, formed by striking out in succession the columns of (1). Our first theorem, then, is:

*If there is no set of branches (not identically zero)* $\zeta_1, \zeta_2, \cdots, \zeta_n$ *of the family* $z$, *which, substituted for the first column of* $Y$, *causes the resulting determinant to vanish, then the families* $y$ *and* $z$ *will have the same group provided* $Y$ *has the same group as* $Z$. *In this case corresponding bases are furnished by the columns of* (1).

To prove this theorem, continue $Y$ and $Z$ analytically about any circuit $C$, denoting the final values of the functions concerned by dashes over the former symbols. We shall have

$$\left.\begin{aligned} &(2) \qquad \bar{z}_i = \sum_{k=1}^{k=n} \alpha_{ki} z_k \\ &(3) \qquad \bar{y}_i = \sum_{k=1}^{k=n} \beta_{ki} y_k \end{aligned}\right\} \qquad (i=1,2,\cdots,n);$$

with equations similar to (3) for the families $y^{(1)}, y^{(2)}, \cdots, y^{(n-1)}$. If we denote by $\lambda$ the determinant of substitution (3) we have

$$\bar{Z} = \lambda Z,$$

so that if $Y$ and $Z$ have the same group the equation

$$(4) \qquad \bar{Y} - \lambda Y = \begin{vmatrix} z_1' & \bar{y}_1^{(1)} & \cdots & \bar{y}_1^{(n-1)} \\ z_2' & \bar{y}_2^{(1)} & \cdots & \bar{y}_2^{(n-1)} \\ \cdot & \cdot & \cdot & \cdot \\ z_n' & \bar{y}_n^{(1)} & \cdots & \bar{y}_n^{(n-1)} \end{vmatrix} = 0$$

must be satisfied, where

$$z_i' = \sum_{k=1}^{k=n} (\beta_{ki} - \alpha_{ki}) z_k \qquad (i = 1, 2, \cdots, n).$$

If $\alpha_{ki}$ were not equal to $\beta_{ki}$ for all values of $k$ and $i$ analytic continuation backward over $C$ would change (4) into an equation

$$\begin{vmatrix} \zeta_1 & y_1^{(1)} & \cdots & y_1^{(n-1)} \\ \zeta_2 & y_2^{(1)} & \cdots & y_2^{(n-1)} \\ \cdot & \cdot & \cdot & \cdot \\ \zeta_n & y_n^{(1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = 0,$$

which is contrary to our first hypothesis. Hence for every circuit in $T_m$ substitutions (2) and (3) must be the same, as required by our theorem.

Let us now discard the first condition of the previous theorem, but add the hypothesis that $Z$ does not vanish identically. The reason for introducing this restriction will appear in what follows, but we may notice here that if $Z \equiv 0$, the $n$ families $y, y^{(1)}, \cdots, y^{(n-1)}$ satisfy among themselves a relation given by theorem $B$.* Our next theorem is:

*If, for analytic continuation over every circuit in $T_m$, $Y^{(1)}, Y^{(2)}, \cdots, Y^{(n-1)}$ undergo the same substitutions as $Z$, then either there exists a relation*

$$z_i'' = \rho y_i \qquad (i = 1, 2, \cdots, n),$$

*between bases of the families $z$ and $y$, where $d/dx \log \rho$ is single-valued in $T_m$; or else $z$ has the same group as $y$, the columns of* (1) *being composed of corresponding bases.*

To establish this, continue analytically over any circuit $C$ the $n$ relations

$$(5) \qquad z_i Z + y_i Y + y_i^{(1)} Y^{(1)} + \cdots + y_i^{(n-1)} Y^{(n-1)} = 0 \quad (i = 1, 2, \cdots, n).$$

Using the same notation as before, we obtain from linear combinations of the resulting equations and the original set (5) the system

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* For if we associate with $y, y^{(1)}, \cdots, y^{(n-1)}$ a family $y^{(n)}$ having the same group, and by a suitable change of notation adapt the formulæ of theorems $A$ and $B$ to this case, we shall have $\pm Y^{(n)} = Z$. But the coefficients of the equation of $B$ are shown by RIEMANN to be proportional to the determinants $Y, Y^{(1)}, \cdots, Y^{(n)}$, so that if $Z = 0$, then $a_n = 0$.

$$z_i'\lambda Z + \bar{y}_i(\bar{Y} - \lambda Y) = 0 \qquad (i = 1, 2, \cdots, n), \tag{6}$$

since by hypothesis $\bar{Y}^{(k)} = \lambda Y^{(k)}$ $(k = 1, 2, \cdots, n-1)$.

If the substitutions (2) and (3) are always the same, the second conclusion of our theorem holds good; it remains then to consider the case where (2) and (3) are not the same for at least one circuit. In this event $(z_1', z_2', \cdots, z_n')$ must be a basis, for otherwise a suitable linear combination of equations (6) would give the relation

$$\sum_{i=1}^{i=n} c_i \bar{y}_i [\bar{Y} - \lambda Y] = 0.$$

But the branches $\bar{y}_i$ are linearly independent, being analytic continuations of a basis, so that $\sum c_i \bar{y}_i$ cannot vanish. The same is true of the expression $\bar{Y} - \lambda Y$, for in equations (6) $Z$, by hypothesis, does not vanish, nor can the determinant $\lambda$, while the vanishing of all the branches $z_i'$ is impossible since they are linear combinations, in which the coefficients are not all zero, of the basis $(z_1, z_2, \cdots, z_n)$. The equation above is therefore inadmissible.

If we continue equations (6) backward over $C$ the result can be written in the form

$$z_i'' = \rho y_i \qquad (i = 1, 2, \cdots, n). \tag{7}$$

We can now show that $\rho$ undergoes only multiplicative substitutions in $T_m$, i. e., that $d/dx \log \rho$ is single-valued. To prove this, let the independent variable describe any circuit in $T_m$; $\rho$ will take on a value $\bar{\rho}$, and from (7) will result a system of equations

$$\sum_{k=1}^{k=n} \gamma_{ki} z_k'' = \bar{\rho} \sum_{k=1}^{k=n} \delta_{ki} y_k \qquad (i = 1, 2, \cdots, n), \tag{8}$$

where the determinants of the two substitutions $(\gamma)$ and $(\delta)$ cannot vanish. Combining (7) and (8) we have

$$\sum_{k=1}^{k=n} (\rho\gamma_{ki} - \bar{\rho}\delta_{ki}) y_k = 0 \qquad (i = 1, 2, \cdots, n).$$

The determinant of the coefficients of $y_1, y_2, \cdots, y_n$ in this system of equations must vanish, and since $\rho \not\equiv 0$ we have thus an equation of degree $n$ in $\bar{\rho}/\rho$ with constant coefficients. This equation cannot be illusory since two of its coefficients are the determinants of $(\gamma)$ and $(\delta)$, one of these being multiplied by $(-1)^n$. We therefore have $\bar{\rho} = \kappa\rho$, where $\kappa$ is some constant, so that the substitutions of $\rho$ are in fact all multiplicative.

With two families of the second order it becomes unnecessary to introduce a third. If the determinant

$$\begin{vmatrix} z_1 & y_1 \\ z_2 & y_2 \end{vmatrix}$$

is multiplicative, its multipliers being the determinants of the substitutions on $(y_1, y_2)$, then the consequences of our theorem follow.

## § 2. *Theorems converse to B.*

In deciding whether linear families which satisfy a relation of the type given in theorem $B$ have the same group, the question of reducibility assumes such importance that a brief reference to that subject may be advisable. A linear family of given order is said to be reducible when it includes all the members of a linear family of lower order. A family thus contained within another we shall refer to as a subfamily of the first. If a family of order $n$ contains a member $y_a$ having only $p < n - 1$ analytic continuations which, with itself, are linearly independent, these $p + 1$ branches form a basis of a subfamily (not necessarily irreducible) of order $p + 1$. The branch $y_a$ cannot belong to any other subfamily of order as small as $p + 1$.

In the case of two linear families $y$, $z$ of order $\lambda$, $\mu$ respectively, when $\lambda > \mu$, it may be possible to choose a set of branches $z_1, z_2, \cdots, z_\lambda$, none of whose members vanish, that has the same group of substitutions as a basis $(y_1, y_2, \cdots, y_\lambda)$*. It is easy to prove that the relation of theorem $B$ holds good even when some of the families concerned are of order $< n$ and the sets of branches $y_1^{(\nu)}, y_2^{(\nu)}, \cdots, y_n^{(\nu)}$ are not all bases, provided these sets have the same group. Returning to our notation for the families $y$ and $z$, we can now easily show that $y$ must be reducible, for, by theorem $B$ in its altered form, we have

$$a_1 z_i + a_2 y_i + a_3 \frac{dy_i}{dx} + \cdots + a_{\lambda+1} \frac{d^{(\lambda-1)} y_i}{dx^{(\lambda-1)}} = 0 \qquad (i = 1, 2, \cdots, \lambda),$$

where $a_1 \neq 0$, and consequently the remaining coefficients cannot all vanish. But the branches $z_1, z_2, \cdots, z_\lambda$ are linearly dependent, since $z$ is of order $\mu < \lambda$; hence we can form a linear combination of the above relations so as to eliminate $z$. We shall then have a linear differential equation of order $< \lambda$ satisfied by a member of the family $y$; in such a case, by a well-known theorem, $y$ must be reducible.

Let $\Gamma$ be a system of $s$ linear families $y^{(1)}, y^{(2)}, \cdots, y^{(s)}$ analytic in $T_m$. These may be of different orders, and any of them may be reducible. Between certain branches $y_1^{(1)}, y_1^{(2)}, \cdots, y_1^{(s)}$ of these families we shall suppose there exists a relation

$$(1) \qquad a_1 y_1^{(1)} + a_2 y_1^{(2)} + \cdots + a_s y_1^{(s)} = 0,$$

where $a_1, a_2, \cdots, a_s$ are single-valued and analytic in $T_m$. Further, in order to avoid results which apply only to subfamilies, we shall suppose that none of the branches in (1) belongs to a subfamily. As a matter of notation we take

* A simple illustration is furnished by the set of branches $x$, $x$ and the basis $(x + \sqrt{x}, x - \sqrt{x})$.

$y^{(1)}$ as the family whose order, $n$, is larger than, or at least equal to that of any other.

We can now obtain $n$ linearly independent branches $y_1^{(1)}, y_2^{(1)}, \cdots, y_n^{(1)}$ by analytic continuation of $y_1^{(1)}$ over $n-1$ suitably chosen paths $C_1, C_2, \cdots, C_{n-1}$. Continuing equation (1) over these paths in succession we obtain the set

$$a_1 y_i^{(1)} + a_2 y_i^{(2)} + \cdots + a_s y_i^{(s)} = 0 \qquad (i=1, 2, \cdots, n). \tag{2}$$

Designate by $\bar{y}_1^{(\nu)}$ the result of continuing analytically the branch $y_1^{(\nu)}$ about any circuit $C$ in $T_m$. Since $(y_1^{(1)}, y_2^{(1)}, \cdots, y_n^{(1)})$ is a basis we have

$$\bar{y}_1^{(1)} = c_1 y_1^{(1)} + c_2 y_2^{(1)} + \cdots + c_n y_n^{(1)}, \tag{3}$$

where the coefficients $c_1, c_2, \cdots, c_n$ are constants. Unless the sets of branches $y_1^{(\nu)}, y_2^{(\nu)}, \cdots, y_n^{(\nu)}$ (of which some may be bases and some not) have the same group for all the values $\nu = 1, 2, \cdots, s$, there must exist a circuit $C$ for which the equation

$$\bar{y}_1^{(\nu)} = c_1 y_1^{(\nu)} + c_2 y_2^{(\nu)} + \cdots + c_n y_n^{(\nu)}$$

holds only for certain values of $\nu$. Then if we multiply equations (2) by suitable constants and add to the equation

$$a_1 \bar{y}_1^{(1)} + a_2 \bar{y}_1^{(2)} + \cdots + a_s \bar{y}_1^{(s)} = 0 \tag{4}$$

we can obtain a relation

$$a_p y_{n+1}^{(p)} + a_q y_{n+1}^{(q)} + \cdots + a_r y_{n+1}^{(r)} = 0 \tag{5}$$

from which $y^{(1)}$ has been eliminated. Hence *if there exists no relation* (5) *linearly independent of relations* (2), *the families of* $\Gamma$ *have corresponding sets of branches* (i. e., sets having the same group)

$$y_1^{(\nu)}, y_2^{(\nu)}, \cdots, y_n^{(\nu)} \qquad (\nu=1, 2, \cdots, s),$$

*where* $y_i^{(\nu)}$, *for each value of* $i$ *and* $\nu$, *is an analytic continuation of* $y_1^{(\nu)}$.

The following theorem gives a condition that the families of $\Gamma$ have the same group: *If there exists no relation of the same type as* (1) *between less than* $s$ *branches chosen each from a different family of* $\Gamma$, *then the existence of relation* (1) *is a sufficient condition that the families of* $\Gamma$ *have the same group. In this case corresponding bases* $(y_1^{(\nu)}, y_2^{(\nu)}, \cdots, y_n^{(\nu)})$ *are obtained by simultaneously continuing the branches* $y_1^{(\nu)}$ *over* $n-1$ *suitably chosen circuits in* $T_m$. The existence of a relation linearly independent of relations (2) may be easily proved incompatible with our hypothesis. Hence to complete the proof of this theorem we need only show that the corresponding sets of branches which must exist in accordance with the previous theorem are bases. But if this were not true we could obtain a linear combination of equations (2) so as to eliminate at least one family from the resulting relation, in contradiction to our hypothesis.

Here we must have $s \leqq n+1$, since theorem $B$ always gives a linear relation between $n+1$ families of order $n$ that have the same group. It can easily be shown that if the conditions of this theorem are satisfied no family of $\Gamma$ can have a subfamily of order $\sigma < s-1$, for in such a case every family of $\Gamma$ must have a subfamily of order $\sigma$ and all these subfamilies must have the same group. But theorem $B$ gives a linear relation between branches of $\sigma+1$ of these, which contradicts our hypothesis that no such relation exists between less than $s$ branches. In particular, if $s = n+1$, all the families of $\Gamma$ must be irreducible.

If, therefore, $n+1$ irreducible linear families of order $n$ have no relation of type (1) between less than $n+1$ branches chosen one from each of the families, the preceding results combined with theorem $B$ show us that the existence of a relation (1) is a necessary and sufficient condition that these families have the same group. Under these conditions we can eliminate $y_1^{(1)}, y_1^{(2)}, \cdots, y_1^{(n+1)}$, between (1) and the differential equations of their respective families so as to obtain a system of differential equations in $a_1, a_2, \cdots, a_{n+1}$. The condition that the families have the same group is that these equations have a system of single-valued solutions. Such a system of equations is generally of degree $\geqq 2$, but in case $n-1$ of the families are the successive derivatives of a family $y^{(2)}$ the system is linear. Its study by HEUN has shed some light on the conditions which the parameters of two linear differential equations must satisfy in order that the equations may have the same group.*

If, now, we set aside the conditions of the second theorem of this section, though still assuming the existence of a relation (1), an especially interesting case presents itself when all the families of $\Gamma$ are irreducible. In this case the sets of branches $y_1^{(\nu)}, y_2^{(\nu)}, \cdots, y_n^{(\nu)}$ cannot correspond unless they are all bases, otherwise, by the results of page 103, $y^{(1)}$ would be reducible. Hence if the sets of branches $y_1^{(\nu)}, y_2^{(\nu)}, \cdots, y_n^{(\nu)}$ are not corresponding bases for $\nu = 1, 2, \cdots, s$, we can not only obtain a relation (5) from which $y^{(1)}$ has been eliminated, but we can also deduce a relation from which any given family appearing in (5) has been eliminated, but which involves every family of $\Gamma$ except $y^{(p)}, y^{(q)}, \cdots, y^{(r)}$ (and may involve some of these also). This can be done by analytically continuing (5) over $n-1$ suitably chosen circuits and combining the results with (1). We now apply the same reasoning to these two relations until we finally reach a set of relations from none of which further eliminations are possible. By the preceding theorem, the families involved in any one of these relations have the same group. Thus either all the families of $\Gamma$ have the same group, or else we can group them into systems $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ whose members have the same group. Every family of $\Gamma$ will belong to at least one of these systems.

*Acta Mathematica, vol. 11 (1887), p. 97; vol. 12 (1888), p. 103. Mathematische Annalen, vol. 33 (1889), p. 161.

If a family is common to two systems they must have the same group; by the use of this principle it is possible to separate $\Gamma$ into mutually exclusive systems in each of which all the families have the same group.

In conclusion we illustrate these results by considering three linear families of the second order, $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, between branches of which there holds a relation of form (1)

$$a_1 y_1^{(1)} + a_2 y_1^{(2)} + a_3 y_1^{(3)} = 0. \tag{6}$$

If the branches in this equation all belong to subfamilies it is easily seen that these subfamilies must have the same group. If $y_1^{(1)}$ does not belong to a subfamily of $y^{(1)}$ we can obtain, by analytic continuation of (6), a relation

$$a_1 y_2^{(1)} + a_2 y_2^{(2)} + a_3 y_2^{(3)} = 0. \tag{7}$$

In case the pairs of branches $y_1^{(\nu)}$, $y_2^{(\nu)}$ do not have the same group for $\nu = 1, 2, 3$, we can deduce from (6), (7), and suitable analytic continuations of (6), a set of equations

$$\begin{aligned} a_1 y_3^{(1)} + a_2 y_3^{(2)} &= 0, \\ a_2 y_4^{(2)} + a_3 y_4^{(3)} &= 0, \\ a_3 y_5^{(3)} + a_1 y_5^{(1)} &= 0. \end{aligned} \tag{8}$$

Hence if the families $y^{(1)}$, $y^{(2)}$ have no pairs of branches possessing the same group they are reducible and contain subfamilies which have the same group; similarly for $y^{(2)}$, $y^{(3)}$ and $y^{(3)}$, $y^{(1)}$.

If one of the three families is irreducible the same is true of the others. For let $y^{(1)}$ be irreducible, and $y^{(2)}$, if possible, be reducible. A suitable linear combination of equations (6) and (7) will then give an equation of form (6) in which $y_1^{(2)}$ is a member of a subfamily of $y^{(2)}$. If we use the notation of (6) for this relation we see, by the results of page 103, that $y_1^{(1)}$, $y_2^{(1)}$, being linearly independent members of an irreducible family, cannot have the same group as $y_1^{(2)}$, $y_2^{(2)}$. But the first equation of (8) would then hold good, though an evident impossibility if $y^{(1)}$ is irreducible and $y^{(2)}$ reducible.

From these results we deduce the conclusion that if one family represented in a relation (6) is irreducible the same must be true of the others, and the families must all have the same group. To obtain corresponding bases we continue analytically the branches in relations (8), if such exist; otherwise those in relation (6).

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