ON QUADRATIC, HERMITIAN AND BILINEAR FORMS*

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Introduction.

Part I is concerned with the reduction of quadratic forms in an arbitrary field to canonical types, a problem hitherto treated only for finite fields and for the field of all real or all complex numbers.

Part II treats of the reduction of hermitian forms in a field Q obtained by the adjunction to an arbitrary field F of a root of a quadratic equation belonging to and irreducible in F . The problem is completely solved when F is any finite field, the field of all real numbers, or the field of all rational numbers.

Part III deals with the bilinear forms in an arbitrary field F which are invariant under a given substitution S with coefficients in F . The necessary and sufficient conditions on S for the existence of such invariants are obtained, and the reduction of the invariants to a single normal form is effected by a transformation commutative with S . Here and in Part IV use is made of the writer's determination of the canonical form of a linear transformation in an arbitrary field. †

Part IV treats the analogous questions on quadratic forms and gives the generalization to an arbitrary field F of JORDAN's recent results for the case of a field of order p , a prime. ‡ The explicit form of the general invariant is determined with the same ease, but the difficult problem of its reduction to canonical forms by substitutions commutative with S becomes much more troublesome for an arbitrary field than for a finite field. When F does not have modulus 2, the general character of the result may be indicated as follows. As semi-normal forms of the invariant we obtain

$$\sum B_i + \sum a_i H_i + \sum a'_i H'_i + \cdots + \sum A_i + \sum b_i Q_i + \sum b'_i Q'_i + \cdots,$$

where the invariants B_i , H_i , A_i and Q_i have respectively the character of bilinear, hermitian, alternate bilinear and quadratic forms, each with completely

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† American Journal of Mathematics, vol. 24 (1902), pp. 101-108.

‡ *Mémoire sur les formes quadratiques, suivant un module premier p , invariables par une substitution linéaire donnée*, Journal de Mathématiques, ser. 6, vol. 1 (1905), pp. 217-284.

fixed coefficients, while $b_i, b'_i, \dots$ are any non-vanishing elements of F and $a_i, a'_i, \dots$, any non-vanishing elements of certain fields $F(\rho)$. The question of ultimate normal forms is the question of the extent to which these parameters can be specialized by the application of a substitution commutative with S . It is shown that normalization must take place in each sum separately, that the normalization of the quadratic function $\sum b_a Q_a$ of certain variables x_j^a must be made by a substitution T on the x_0^a , the same substitution on the x_1^a , the same on x_2^a , etc., with analogous remarks on the sums $\sum a_i H_i, \dots$. But the effect on the b_i is the same as if we had applied to $\sum b_a (x_0^a)^2$ the substitution T on the variables x_0^a alone. Moreover, this partial substitution on the x_0^a may be chosen arbitrarily. Hence the problem of normalizing $\sum_{a=1}^l b_a Q_a$ by a substitution commutative with S and cogredient in the various sets of variables is replaced by the problem (treated in Part I) of normalizing an l -ary quadratic form in F by means of unrestricted l -ary substitutions in F . Similarly, the problem on $\sum a_i H_i$ reduces to that on hermitian forms (Part II).

I. *Reduction of quadratic forms* in a general field F .*

1. Within any field F , not having modulus 2, a quadratic form of non-vanishing determinant is linearly reducible to †

$$(1) \quad q \equiv \sum_{i=1}^n a_i x_i^2 \quad (\text{each } a_i \text{ an element } \neq 0 \text{ of } F).$$

Hence for the field of real numbers the canonical types are

$$f_s \equiv \sum_{i=1}^s x_i^2 - \sum_{i=s+1}^n x_i^2.$$

For $s \neq \sigma$, f_s cannot be transformed into f_σ by a real n -ary linear substitution; this invariance of s is the JACOBI-SYLVESTER law of inertia of real quadratic forms. ‡

2. Under the transformation

$$x_i = \sum_{j=1}^n b_{ij} y_j \quad (i = 1, \dots, n),$$

q becomes

$$\sum_{j=1}^n A_j y_j^2 + 2 \sum_{j,k,j < k}^{1, \dots, n} B_{jk} y_j y_k, \quad A_j \equiv \sum_{i=1}^n a_i b_{ij}^2, \quad B_{jk} \equiv \sum_{i=1}^n a_i b_{ij} b_{ik}.$$

We discuss the question: Given $b_{11}, b_{21}, \dots, b_{n1}$, in the general field F , such that $A_1 \neq 0$, can we determine elements $b_{ij} (j > 1)$ of F , such that every

* The same treatment applies to the reduction of symmetric bilinear forms by cogredient transformations of the two sets of variables.

† The usual proof for the field of all real numbers is valid for F . Or we may proceed as in § 6, identifying every element with its conjugate.

‡ References in BALTZER, *Determinanten*, 5th ed., p. 176.

$B_{jk} = 0$ and $\Delta \equiv |b_{ij}| \neq 0$? Since the b_{ii} enter the question symmetrically, we may assume that $b_{11} \neq 0$. The conditions $B_{1k} = 0$ are satisfied if we take

$$b_{1k} = -a_1^{-1} b_{11}^{-1} \sum_{i=2}^n a_i b_{i1} b_{ik} \quad (k=2, \dots, n).$$

If, with these values inserted in Δ , we remove the factor $a_1^{-1} b_{11}^{-1}$ from the first row of Δ , then multiply the i th row by $a_i b_{i1}$ and add the sum to the first row, for $i=2, \dots, n$, we find that

$$\Delta = a_1^{-1} b_{11}^{-1} A_1 \Delta_{11}, \quad \Delta_{11} \equiv |b_{is}| \quad (i, s=2, \dots, n).$$

If $n=2$, we take $b_{22} \neq 0$ and obtain an affirmative answer to our question. Let next $n > 2$. In $B_{jk} = 0$ ($2 \leq j < k$), we replace b_{ij} and b_{ik} by their values and get

$$B'_{jk} \equiv \sum_{i=2}^n R_i b_{ij} b_{ik} + \sum_{i, t, i \neq t}^{2, \dots, n} P_{it} b_{ij} b_{ik} = 0, \quad R_i \equiv a_i^2 b_{i1}^2 + a_1 a_i b_{i1}^2, \quad P_{it} \equiv a_i a_t b_{i1} b_{t1}.$$

(i) Suppose first that every $R_i = 0$ ($i=2, \dots, n$). Then every $b_{ii} \neq 0$, $P_{it} \neq 0$. With the exception of b_{32} , we take every $b_{is} = 0$ ($i, s=2, \dots, n; i > s$). We assign any value not zero to b_{ii} ($i > 1$); and any values to b_{23}, b_{32} such that $b_{23} b_{32} = -b_{22} b_{33}$. Then $\Delta_{11} = 2b_{22} b_{33} \dots b_{nn} \neq 0$. Finally, for $l=4, \dots, n$, we set

$$C_{il} \equiv \sum_{t=2}^{i-2, \dots, l} P_{it} b_{tl} = 0 \quad (i=2, \dots, l-1).$$

Then conditions $B'_{jk} = 0$ ($j, k=2, \dots, n; j < k$) are all satisfied, since the coefficient of b_{ij} equals C_{ik} if $k \geq 4$, while for $k=3, j=2$, the condition reduces to $P_{23}(b_{22} b_{33} + b_{32} b_{23}) = 0$. But for a fixed $l, l > 3$, equations $C_{il} = 0$ serve to express the b_{it} ($t=2, \dots, l-1$) in terms of b_{il} , since the determinant of the coefficients of the former is*

$$\begin{vmatrix} 0 & P_{23} & P_{24} & \dots & P_{2, l-1} \\ P_{32} & 0 & P_{34} & \dots & P_{3, l-1} \\ P_{42} & P_{43} & 0 & \dots & P_{4, l-1} \\ \dots & \dots & \dots & \dots & \dots \\ P_{l-1, 2} & P_{l-1, 3} & P_{l-1, 4} & \dots & 0 \end{vmatrix} = r \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 \end{vmatrix} = (-1)^{l-3} (l-3)!,$$

* To reach the second determinant, replace each P_{it} by $a_i a_t b_{i1} b_{t1}$, remove the factor $a_i b_{i1}$ from the $(i-1)$ th row, and the factor $a_t b_{t1}$ from the $(t-1)$ th column. To evaluate the second determinant, subtract the first row from each of the remaining rows, then add the $2^d, \dots, (l-2)$ th columns to the first. The resulting determinant has the first row $l-3, 1, 1, \dots, 1$, and zeros elsewhere outside the main diagonal.

where $r \equiv a_2^2 a_3^2 \cdots a_{l-1}^2 b_{21}^2 b_{31}^2 \cdots b_{l-11}^2 \neq 0$. Hence the equations $C_{ii} = 0$ can all be satisfied if * F does not have a modulus $\leq n-3$.

(ii) If not every R_i vanishes, we may set $R_2 \neq 0$ in view of the symmetry. We determine the b_{2k} to make the coefficient of b_{2j} in B'_{jk} vanish:

$$b_{2k} = -R_2^{-1} \sum_{t=3}^n P_{2t} b_{tk} \quad (k=3, \dots, n).$$

We give to b_{22} any value $\neq 0$ and set $b_{i2} = 0$ ($i=3, \dots, n$). Then $B'_{2k} \equiv 0$ and

$$\Delta_{11} = b_{22} \Delta_{22}, \quad \Delta_{22} \equiv |b_{is}| \quad (i, s=3, \dots, n).$$

If $n=3$, we take $b_{33} \neq 0$ and obtain an affirmative answer to our question. Let next $n > 3$. In $B'_{jk} = 0$ ($3 \leq j < k$), we replace b_{tk} for $t=2$ by its value and obtain

$$B''_{jk} \equiv \sum_{i=3}^n b_{ij} \left\{ R'_i b_{ik} + \sum_{t=3, \dots, n}^{t \neq i} P'_{it} b_{tk} \right\} = 0, \quad R'_i \equiv R_2 R_i - P_{2i}^2, \quad P'_{it} \equiv a_1 a_2 b_{11}^2 P_{it},$$

the value of P'_{it} being initially $R_2 P_{it} - P_{2i} P_{2t}$. The present problem—to determine the b_{is} ($i \geq 3, s \geq 3$) so that their determinant $\Delta_{22} \neq 0$ and each $B''_{jk} = 0$ ($3 \leq j < k$)—is the exact analogue of the former problem—to determine the b_{is} ($i \geq 2, s \geq 2$) so that $\Delta_{11} \neq 0$ and each $B'_{jk} = 0$ ($2 \leq j < k$). After suitable repetitions of the argument, the final problem is to determine the b_{is} ($i, s = n-1, n$) so that $|b_{is}| \neq 0$ and

$$B^*_{n-1n} \equiv \sum_{t=n-1}^n b_{it} \left\{ R^*_i b_{in} + \sum_{t=3}^{t=n-1, n}^{t \neq i} P^*_{it} b_{tn} \right\} = 0, \quad P^*_{n-1n} = P^*_{nn-1} \equiv P.$$

If R^*_{n-1} , R^*_n and P all vanish, we may choose the b 's to be any elements of determinant not zero. In the contrary case, we take

$$b_{n-1n-1} = R^*_n b_{nn} + P_{n-1n}, \quad b_{nn-1} = -R^*_{n-1} b_{n-1n} - P b_{nn}.$$

Then $B^*_{n-1n} \equiv 0$, $|b| = R^*_n b_{nn}^2 + 2P b_{nn} b_{n-1n} + R^*_{n-1} b_{n-1n}^2 \neq 0$.

THEOREM. *If F is any field not † having a modulus $\leq n-3$, there exists an n -ary linear transformation (b_{ij}) in F , with preassigned values of $b_{11}, b_{21}, \dots, b_{n1}$ making $\sum_{i=1}^n a_i b_{i1}^2 \neq 0$, which replaces a given quadratic form $\sum_{i=1}^n a_i x_i^2$ by one of the type $\sum_{i=1}^n A_i x_i^2$ with $A_1 = \sum_{i=1}^n a_i b_{i1}^2$.*

* This condition is necessary for the solvability of the $C_u = 0$. If F has a modulus which divides $l-3$, the above determinant vanishes. Call $M_2, M_3, \dots, M_{l-1}$ the minors of the elements $0, P_{32}, \dots, P_{l-12}$. Then must $b_u(P_{2u} M_2 - P_{3u} M_3 + \dots) = 0$. The determinant whose expansion is the second factor is seen as above to have the value 1.

† It is unnecessary for our applications to inquire whether or not this restriction on F is necessary for the validity of the theorem.

3. If F is a finite field the equation $a_1 b_{11}^2 + a_2 b_{21}^2 = 1$ has solutions b_{11}, b_{21} in F . Hence, applying the theorem of § 2 for $n = 2$, we see that any form $\sum_{i=1}^m a_i x_i^2$ in the $GF[p^k]$, $p > 2$, can be reduced by a succession of binary transformations to $\sum_{i=1}^{m-1} x_i^2 + ax_m^2$. Since one-half of the marks, not zero, of the $GF[p^k]$, $p > 2$, are squares, while the ratio of any two not-squares is a square, we can make $a = 1$ or a particular not-square ν .

THEOREM.* *In the $GF[p^k]$, $p > 2$, any m -ary quadratic form of non-vanishing determinant can be reduced to $\sum_{i=1}^m x_i^2$ or else to $\sum_{i=1}^{m-1} x_i^2 + \nu x_m^2$, ν being a particular not-square.*

4. Let F be the field R of all rational numbers. If $a_1 = \alpha_1/d$, $a_1 x_1^2 = \alpha_1 dy_1^2$, where $y_1 = x_1/d$. Hence we may assume that each a_i in (1) is an integer. If $a_1, \dots, a_4$ are not all negative, the equation

$$(2) \quad \sum_{i=1}^4 a_i b_i^2 = 1$$

can be satisfied by rational values of $b_1, \dots, b_4$. Indeed, there exist† integers $\beta_1, \dots, \beta_4, \sigma$, not all zero, for which $\sum_{i=1}^4 a_i \beta_i^2 - \sigma^2 = 0$. But if $\sigma = 0$ and say $\beta_1 \neq 0, \beta_2 \neq 0$, (2) is satisfied by

$$b_1 = \frac{\beta_1}{2\beta_2} \left(1 - \frac{1}{a_2}\right), \quad b_2 = \frac{1}{2} \left(1 + \frac{1}{a_2}\right), \quad b_3 = \frac{\beta_1}{\beta_1} \beta_3, \quad b_4 = \frac{\beta_1}{\beta_1} \beta_4.$$

It now follows that, if $a_1, \dots, a_4$ are all negative, there exist rational solutions b_i of $\sum a_i b_i^2 = -1$. Hence, by the theorem of § 2, $\sum_{i=1}^4 a_i x_i^2$ can be transformed by a quaternary substitution with rational coefficients into

$$\pm x_1^2 + \sum_{i=2}^4 a'_i x_i^2,$$

the sign being + unless $a_1, \dots, a_4$ are all negative. We thus obtain the

* The reduction of quadratic forms in the $GF[p^k]$ was effected by the writer in a memoir on the linear groups defined by a quadratic invariant, *American Journal of Mathematics*, vol. 21 (1899), pp. 194, 222. (Cf. *Linear Groups*, pp. 158, 197.) For $p = 2$, there exist m -ary quadratic forms of non-vanishing discriminant D only for $m = 2n$, and then the two canonical types are

$$F_\lambda = \lambda x_1^2 + \lambda y_1^2 + \sum_{i=1}^n x_i y_i,$$

where $\lambda = 0$ or a particular one of the marks for which $\lambda x_1^2 + \lambda y_1^2 + x_1 y_1$ is irreducible in the $GF[2^k]$. For $k = 1$, JORDAN states in his memoir cited above that a quadratic form is reducible to F_0 or F_1 according as $(\frac{2}{p}) = +1$ or -1 . This is clearly an oversight since D is the mark 1 and thus can be taken to be an arbitrary odd integer. The same oversight occurs in DE SÉGUIER's *Groupes Abstraites*, p. 51, footnote.

† A. MEYER, *Vierteljahrsschrift der naturforschenden Gesellschaft in Zurich*, vol. 29 (1884), pp. 209-222. He shows that $a_1 x_1^2 + \dots + a_6 x_6^2 = 0$ has integral solutions x_i not all zero if $a_1, \dots, a_6$ are integers $\neq 0$, not all of like sign.

THEOREM. *Within the field of all rational numbers any n -ary quadratic form of non-vanishing determinant can be transformed into one of the forms*

$$- \sum_{i=1}^s x_i^2 + \sum_{i=s+1}^{n-3} x_i^2 + ax_{n-2}^2 + bx_{n-1}^2 + cx_n^2,$$

where $s = 0$ unless a, b, c are all negative.

The difficult problem of ultimate canonical forms is not undertaken here. I pass to hermitian forms and effect a complete reduction to canonical forms.

II. On the reduction of hermitian forms.

5. Let F be any field* for which there is an equation

$$x^2 + ux + v = 0 \quad (\text{roots } \omega, \bar{\omega}, \bar{\omega} \neq \omega) \dagger,$$

belonging to and irreducible in F . Denote the field $F(\omega)$ by Q . Any element of Q may be given the form $e = a + b\omega$, a and b in F . Set $\bar{e} = a + b\bar{\omega}$. Then

$$H_a \equiv \sum_{i,j}^{1,\dots,n} \alpha_{ij} \xi_i \bar{\xi}_j \quad (\text{each } \alpha_{ij} \text{ in } Q, \bar{\alpha}_{ij} = \alpha_{ji})$$

will be called a *hermitian form in the field Q* , and $|\alpha_{ij}|$ will be called the *determinant of H_a* . Under a linear transformation

$$(3) \quad \xi_i = \sum_{k=1}^n \beta_{ik} \eta_k, \quad \bar{\xi}_i = \sum_{k=1}^n \bar{\beta}_{ik} \bar{\eta}_k \quad (i = 1, \dots, n),$$

in which the β_{ik} are elements of Q and $B \equiv |\beta_{ik}| \neq 0$, H_a becomes a hermitian form H_γ whose determinant $|\gamma_{ij}|$ equals $B\bar{B}|\alpha_{ij}|$. ‡

6. **THEOREM.** *By a transformation (3) in Q of determinant unity, any hermitian form H_a in Q of non-vanishing determinant Δ can be reduced to*

$$(4) \quad \sum_{i=1}^n \gamma_i \eta_i \bar{\eta}_i \quad (\text{each } \gamma_i \text{ in } F).$$

We first reduce H_a to a form $H_{a'}$ having $\alpha'_{11} \neq 0$. If $\alpha_{11} = 0$ and $\alpha_{jj} \neq 0$, we apply the transformation $\xi_1 = \eta_j$, $\xi_j = -\eta_1$. If every $\alpha_{ii} = 0$, we may take $\alpha_{12} \neq 0$; under the transformation $\xi_1 = \eta_1$, $\xi_2 = \eta_2 + \mu\eta_1$, H_a becomes $H_{a'}$ with

* In particular, F shall not be the field of all complex numbers $x + yi$, nor a field $F_{m,p}$ defined as the aggregate of the Galois fields of orders $p^m, p^{2m}, p^{4m}, p^{8m}, \dots$. If in a field F of modulus 2 every equation $x^2 + ux + v = 0$ ($u \neq 0$) is reducible, F contains $F_{1,2}$.

† In the field of all rational functions of a variable z with integral coefficients taken modulo 2, the equation $x^2 = z$ is irreducible but has equal roots. We exclude such a case $\bar{\omega} = \omega$, since the problem is then that of bilinear forms in F subject to cogredient transformations.

‡ The simplest proof follows by use of generators of the types

$$\xi_1 = \eta_1 + \lambda\eta_2, \quad \xi_i = \eta_i \quad \text{and} \quad \xi_1 = B\eta_1, \quad \xi_i = \eta_i \quad (i = 2, \dots, n).$$

$\alpha'_{11} = \alpha_{12}\bar{\mu} + \bar{\alpha}_{12}\mu$. Take $\mu = (u + b\omega)/\bar{\alpha}_{12}$; then $\alpha'_{11} \equiv 2a - bu$ can be made different from zero, since we have excluded the case $u = 0$ when F has modulus 2 by assuming that $\bar{\omega} \neq \omega$.

In H_a with $\alpha'_{11} \neq 0$, set $\xi_1 = \eta_1 - \bar{\alpha}'_{12}\alpha'^{-1}_{11}\eta_2$, $\xi_2 = \eta_2$; there results $H_{a''}$ with $\alpha'_{11} = \alpha'_{11}$, $\alpha'_{12} = \alpha'_{21} = 0$. Similarly, we make every α_{ii} and α_{i1} zero ($i > 1$) and reach $H_{a''} = \alpha'_{11}\eta_1\bar{\eta}_1 + f$, where f is a hermitian form on η_i , $\bar{\eta}_i$ ($i = 2, \dots, n$) of determinant $\neq 0$.

7. Let F be the $GF[p^m]$, so that Q is the $GF[p^{2m}]$. For any mark g of F the equation $c\bar{c} = g$, viz., $c^{p^m+1} = g$, is solvable in Q . Hence there is an unique canonical form $\Sigma \xi_i \bar{\xi}_i$.

8. Let F be the field of all real numbers and set $\omega = \sqrt{-1}$. Then $\gamma_i = \pm c_i^2$, so that (4) can be reduced to one of the forms

$$(5) \quad h_r \equiv \sum_{i=1}^r \xi_i \bar{\xi}_i - \sum_{i=r+1}^n \xi_i \bar{\xi}_i \quad (r = 0, 1, \dots, n).$$

Now r is an invariant, i. e., h_r can not be reduced to h_ρ ($\rho \neq r$) by a transformation (3). Indeed, for $\xi_i = x_i + y_i\sqrt{-1}$, (3) becomes a special $2n$ -ary real linear transformation, and (5) becomes

$$(6) \quad \sum_{i=1}^r (x_i^2 + y_i^2) - \sum_{i=r+1}^n (x_i^2 + y_i^2),$$

so that $2r$ is an invariant by § 1. There are $n + 1$ canonical forms (5).

9. Consider for a general field F the possible normalizations of a binary hermitian form $h = a(x\bar{x} + r y\bar{y})$, a and r in F , and each $\neq 0$. Set

$$x = \lambda X + \mu Y, \quad y = \rho X + \sigma Y, \quad D \equiv \lambda\sigma - \mu\rho \neq 0.$$

Then h becomes

$$h' = a(\lambda\bar{\lambda} + r\rho\bar{\rho})X\bar{X} + a(\mu\bar{\mu} + r\sigma\bar{\sigma})Y\bar{Y} + A X\bar{Y} + \bar{A} \bar{X}Y,$$

where $A = a(\lambda\bar{\mu} + r\rho\bar{\sigma})$. We desire that $A = 0$. For $\lambda \neq 0$, we take $\mu = -r\bar{\rho}\sigma/\bar{\lambda}$. Then $D = (\lambda\bar{\lambda} + r\rho\bar{\rho})\sigma/\bar{\lambda}$, and

$$h' = a(\lambda\bar{\lambda} + r\rho\bar{\rho})(X\bar{X} + r\tau\bar{\tau}Y\bar{Y}), \quad \tau \equiv \frac{\sigma}{\lambda}.$$

The same form with $\tau = \mu/r\rho$ results if $\lambda = 0$, whence $\sigma = 0$. Hence r can be changed only by a factor $\tau\bar{\tau}$. By a preliminary unary transformation on y , we can restrict r to the series of multipliers $1, m_1, m_2, \dots$ in a rectangular array of the elements of F with the various distinct elements $1, \kappa_1\bar{\kappa}_1, \kappa_2\bar{\kappa}_2, \dots$ in the first row, where the κ 's are arbitrary in Q . To retain this normalization, we set $\tau = 1$.

THEOREM. *Let $1, m_1, m_2, \dots$ be the multipliers in a rectangular array of elements of F , the elements of the first row being the distinct elements $\kappa\bar{\kappa}$, κ ranging over Q . Any binary hermitian form can be reduced linearly to $a(x\bar{x} + r y\bar{y})$, where a and r belong to the set $1, m_1, m_2, \dots$. Two such reduced forms can be transformed into each other if and only if they have the same r , and the ratio of their a 's is a mark, not zero, expressible in the form $\lambda\lambda + r\rho\bar{\rho}$.*

10. Let F be the field R of all rational numbers. Then $Q = R(\epsilon)$, where $\epsilon^2 = \nu$, ν being a fixed integer $\neq 1$ having no square factor. Thus $\bar{\epsilon} = -\epsilon$. By § 9, we can transform $ax\bar{x} + by\bar{y}$ into $Lx\bar{x} + My\bar{y}$, where $L = a\lambda\bar{\lambda} + b\rho\bar{\rho}$, λ and ρ being any elements of Q for which $L \neq 0$. If $a = \alpha/d$, α and d being integers, then $ax\bar{x} = \alpha dz\bar{z}$, $z = x/d$. Hence we may assume that a and b are integers. Set $\lambda = \alpha + \beta\epsilon$, $\rho = \gamma + \delta\epsilon$, $\alpha, \beta, \gamma, \delta$ in R . Then

$$(7) \quad L \equiv a\lambda\bar{\lambda} + b\rho\bar{\rho} = a(\alpha^2 - \nu\beta^2) + b(\gamma^2 - \nu\delta^2).$$

As* in § 4, we can choose $\alpha, \beta, \gamma, \delta$ in R to make $L = +1$ if ν, a, b are not all negative, and $L = -1$ in the contrary case. Hence if ν is positive, any n -ary hermitian form is reducible to

$$(8) \quad \sum_{i=1}^{n-1} \xi_i \bar{\xi}_i + r \xi_n \bar{\xi}_n;$$

while, if ν is negative, it can be reduced to one of the types

$$(9) \quad f_s \equiv - \sum_{i=1}^s \xi_i \bar{\xi}_i + \sum_{i=s+1}^{n-1} \xi_i \bar{\xi}_i + r \xi_n \bar{\xi}_n \quad (s=0, 1, \dots, n-1).$$

If $r > 0$ and $s > 0$, we can transform $-\xi_s \bar{\xi}_s + r \xi_n \bar{\xi}_n$ into $+\xi_s \bar{\xi}_s + r' \xi_n \bar{\xi}_n$, and hence transform f_s into f_{s-1} . The reduced forms for ν negative are thus

$$(10) \quad f_0 \text{ with } r > 0; f_s (s=0, 1, \dots, n-1) \text{ with } r < 0,$$

falling into $n+1$ types, each characterized by the number of its negative terms. Hence by § 8, a form of one type cannot be reduced to one of a different type by a transformation (3). But by § 5, under a transformation (3) of determinant B , the determinant $(-1)^s r$ of f_s is multiplied by $B\bar{B}$. Hence $f_s(r)$ is reducible to $f_s(\rho)$ if and only if ρ/r is expressible in the form $B\bar{B}$.

THEOREM. *Any n -ary hermitian form in $R(\sqrt{\nu})$ with non-vanishing determinant can be reduced by a linear transformation in $R(\sqrt{\nu})$ to one and but one of the canonical forms:*

*The case $\sigma = 0$ may now be treated more naturally. There are then solutions $\lambda_1 \neq 0, \rho_1 \neq 0$ in $R(\epsilon)$ of $a\lambda_1\bar{\lambda}_1 + b\rho_1\bar{\rho}_1 = 0$. Let $\tau = \rho\lambda_1/\rho_1$. Then $L = 1$ becomes $\lambda\bar{\lambda} - \tau\bar{\tau} = a^{-1}$, and is satisfied by $\lambda = \frac{1}{2}(a^{-1} + 1)$, $\tau = \frac{1}{2}(a^{-1} - 1)$.

(I) for $\nu > 0$, $\sum_{i=1}^{n-1} \xi_i \bar{\xi}_i + r \xi_n \bar{\xi}_n$, where r ranges over the multipliers (integers) $1, m_1, m_2, \dots$, in a rectangular array of all rational numbers with those representable by $x^2 - \nu y^2$ in the first row;

(II) for $\nu < 0$, $\sum_{i=1}^{n-1} \xi_i \bar{\xi}_i + \rho \xi_n \bar{\xi}_n$, $-\sum_{i=1}^s \xi_i \bar{\xi}_i + \sum_{i=s+1}^{n-1} \xi_i \bar{\xi}_i - \rho \xi_n \bar{\xi}_n$, where $s = 0, 1, \dots, n-1$, and ρ ranges over the multipliers (positive integers) in a rectangular array of all positive rational numbers with those representable by $x^2 - \nu y^2$ in the first row.

In the examples,* (k) denotes all primes of the form k ; $q_1, q_2, q_3, \dots$, denote distinct primes; $p_1, p_2, p_3, \dots$, denote distinct primes.

For $\nu = 2$, $r = 1$, $q_1, q_1 q_2, q_1 q_2 q_3, \dots, q_i$ ranging over $(8m+3), (8m+5)$.

For $\nu = 3$, $r = \pm 1, \pm q_1, \pm q_1 q_2, \dots, q_i$ ranging over $(12m+5), (12m+7)$.

For $\nu = 5$, $r = 1$, $q_1, q_1 q_2, \dots, q_i$ ranging over $(20m+3), (20m+7), (20m+13), (20m+17)$.

For $\nu < 0$, $\rho = 1$, $q_1, q_1 q_2, q_1 q_2 q_3, \dots$, where for $\nu = -1$, $q_i = (4m+3)$; for $\nu = -2$, $q_i = (8m+5), (8m+7)$; for $\nu = -3$, $q_i = (3m+2)$; for $\nu = -5$, the only limitations on the primes q_i are that no one is $5, 20m+1$, or $20m+9$, while at most one is chosen from the set $2, (20m+3), (20m+7)$; for $\nu = -6$, no q_i is $24m+1$ or $24m+7$, while at most one is chosen from the set $2, 3, (24m+5), (24m+11)$; for $\nu = -7$, $q_i = (28m+t), t = 3, 5, 13, 17, 19, 27$.

III. The bilinear forms invariant under a given substitution S .

11. Let F be an arbitrarily given field. We seek all bilinear functions $\Phi = \sum_{ij} \gamma_{ij} \xi_i \eta_j$ with discriminant $D \equiv |\gamma_{ij}| \neq 0$ and coefficients in F , such that Φ is invariant under a given substitution, cogredient in the two sets of variables,

$$S: \quad \xi'_i = \sum_{j=1}^n \alpha_{ij} \xi_j, \quad \eta'_i = \sum_{j=1}^n \alpha_{ij} \eta_j \quad (i=1, \dots, n),$$

with coefficients also in F . In the canonical form of S (with the initial variables ξ_i), the new variables fall into as many classes as there are distinct roots of the characteristic equation $\Delta(\rho) = 0$ of S . Each class is composed of one or more series, the variables of any series being transformed by S into linear functions of themselves, as follows:

$$|x_0, x_1, \dots, x_i \quad \rho x_0, \rho(x_1 + x_0), \dots, \rho(x_i + x_{i-1})|.$$

* The integers representable by $x^2 - 2y^2$ are, aside from square factors, $\pm 1, \pm 2, (8n+1), (8n+7)$, and their products two, three, four, $\dots$, at a time. For $x^2 - 3y^2$, they are $-2, -3, (12m+1), -(12m+11)$. For $x^2 + 5y^2$, they are $5, (20m+1), (20m+9), 2p_1, p_1 p_2$, and the products of these expressions two, three, $\dots$, at a time, where $p_i = (20m+3), (20m+7)$. For $x^2 + 6y^2$, they are $(24m+1), (24m+7), p_1 p_2$, and their products, where $p_i = 2, 3, 24m+5$ or $24m+11$.

Similarly, S with the initial variables η_i affects the general series of canonical variables as follows:

$$|y_0, y_1, \dots, y_\tau \quad \rho_1 y_0, \rho_1(y_1 + y_0), \dots, \rho_1(y_\tau + y_{\tau-1})|.$$

In the new variables, Φ becomes a sum of functions each separately invariant, the general one, f , being bilinear in x_i, y_j ($i = 0, \dots, t; j = 0, \dots, \tau$). Let $\alpha x_i y_j$ be any term of the latter, $i + j$ its rank, i being the rank of x_i and j the rank of y_j . Since the increment obtained from any term is of lower rank, the set of terms of maximum rank in the transform of f by S is derived from the set of terms of maximum rank in f by multiplication by $\rho\rho_1$. Unless f is identically zero, $\rho\rho_1 = 1$. But if none of the variables x appeared in the new form of Φ , its discriminant would vanish. Hence there is at least one series of variables y which S_η multiplies by ρ^{-1} . With the class C_x of all the variables which S_ξ multiplies by ρ is associated a class C_y of the variables which S_η multiplies by ρ^{-1} . Then $\Phi = [C_x C_y] + \Psi$, where $[C_x C_y]$ is bilinear in the variables of C_x, C_y , while Ψ does not contain those variables. The discriminant of $[C_x C_y]$ is a factor of D and hence is not zero; this requires that the classes C_x and C_y shall contain the same number of variables. Thus ρ^{-1} must be a root of $\Delta(\rho) = 0$ of the same multiplicity as the root ρ . A first necessary condition for an invariant Φ under S is:

(11) *The characteristic equation of S must be a reciprocal equation.*

A second condition for the existence of $[C_x C_y]$ is (JORDAN, § 13):

(12) *Classes C_x, C_y must be of like type as to number and length of series.*

12. When these conditions on S are satisfied, invariants $[C_x C_y]$ of non-vanishing discriminant exist (JORDAN, § 12, case I alone occurs for bilinear functions), the general one being $\sum_{\alpha, \beta, r} \alpha^{\alpha\beta} f_{\alpha\beta r}$, where f is a definite bilinear function and the α 's are any polynomials in $\rho, \rho_1, \dots, \rho_{\nu-1}$, for which the discriminant of $[C_x C_y]$ is not zero and the following "reality" conditions hold: Since Φ is to be equal to a function of the initial variables with coefficients in F , we must have $\Phi = \Sigma + \Phi_1$, where

$$\Sigma = [C_x C_y] + [C'_x C'_y] + \dots + [C_{x^{\nu-1}} C_{y^{\nu-1}}],$$

$[C'_x C'_y], \dots$, being derived from $[C_x C_y]$ by interchanging ρ and $\rho_1, \dots, \rho$ and $\rho_{\nu-1}$, respectively, $\rho_1, \dots, \rho_{\nu-1}$, being the remaining roots of the same irreducible factor of $\Delta(\rho)$. Thus $\alpha_r^{\alpha\beta}$ must equal a polynomial in ρ . Let these conditions be satisfied. Then (JORDAN,* § 14-16) by a linear transformation leaving the canonical form of S unaltered (not necessarily the same transformation on the y 's as on the x 's), we can reduce Φ to a unique canonical form.

* An obvious correction (not altering the argument) is to be made on p. 237, l. 11. The expression in brackets should be

$$y'_0(x'_m + C_{m-1}^1 x'_{m-1} + \dots + x'_1) + y''_0(x''_m + C_{m-1}^1 x''_{m-1} + \dots + x''_1) + \dots$$

13. Conditions (11) and (12) depend upon quantities irrational in general with respect to the initial field F . It seems desirable to proceed further and exhibit purely rational conditions for the solvability of the problem. We obtain* the

THEOREM. *The necessary and sufficient conditions that a given substitution (α_{ij}) with coefficients in F shall leave invariant one or more bilinear forms with coefficients in F are: (i) the characteristic determinant $\Delta(\rho)$ of S has a decomposition into factors irreducible in F of the type*

$$\Delta(\rho) = S_k S_l'^{b_1} \cdots R_i R_i'^{a_1} \cdots R_s'^{f_1} R_s'^{a_s f_1} \cdots,$$

where $S_k, S_l', \cdots$ are self-reciprocal, viz., $S_k(\rho) \equiv \rho^k S_k(\rho^{-1})$, while R_i and R_i' are reciprocal; (ii) the invariant-factors of the matrix $\Delta(\rho)$ are of the form

$$S_k^{a_1} S_l'^{b_1} \cdots R_i^{c_1} R_i'^{a_1} \cdots R_s'^{f_1} R_s'^{a_s f_1}, S_k^{a_2} S_l'^{b_2} \cdots R_i^{c_2} R_i'^{a_2} \cdots R_s'^{f_2} R_s'^{a_s f_2}, \cdots,$$

a pair of reciprocal factors occurring always to the same power.

When these conditions are satisfied there exist in F bilinear forms $\sum b_{ij} \xi_i \eta_j$ invariant under S_ξ, S_η . All such invariant forms are reducible to a single one by a transformation on the ξ 's commutative with S_ξ , and a (possibly different) transformation on the η 's commutative with S_η .

IV. The quadratic forms invariant under a given substitution S .

14. JORDAN's treatment of the case of a field of order p , a prime, can be extended immediately to any finite field, and with certain essential modifications to any infinite field. Let F be any field.

Let ρ be a root of $f(x) = 0$, where $f(x)$ is a factor of degree k of $\Delta(x)$ and is irreducible in F . If ρ^{-1} is not a root of $f(x) = 0$, the question is essentially the same as that for bilinear forms, discussed above. Let next ρ^{-1} be a root of $f(x) = 0$, $\rho^{-1} \neq \rho$. Then $f(x) = 0$ and $x^k f(x^{-1}) = 0$ are equations belonging to and irreducible in F with a root ρ in common; hence all their roots are common and $f(x) = 0$ is a reciprocal equation. Since no root equals its reciprocal, $k = \text{even} = 2\nu$, and the roots may be designated $\rho \equiv \rho_0, \rho_1, \cdots, \rho_{2\nu-1}$ with $\rho_\nu = \rho_0^{-1}, \rho_{\nu+1} = \rho_1^{-1}, \cdots, \rho_{2\nu-1} = \rho_{\nu-1}^{-1}$. In the canonical form of S all the variables corresponding to a root ρ_i are said to form the class C_i . Thus an invariant quadratic form Φ must equal $[C_0 C_\nu] + \Psi$, where $[C_0 C_\nu]$ is bilinear in the variables of classes C_0 and C_ν , while Ψ does not contain them. Now $[C_0 C_\nu]$ which is itself invariant must have the form

$$(13) \quad [C_0 C_\nu] = \sum_{\alpha, \beta, r} \alpha_r^{\alpha\beta} F_{\alpha\beta r},$$

where $\alpha_r^{\alpha\beta}$ are any constants satisfying the conditions later specified, while $F_{\alpha\beta r}$

* Cf. Transactions, vol. 3 (1902), pp. 290-292.

are perfectly definite bilinear forms derived from those given by JORDAN, pp. 240, 241, by making the following changes. For $b_i, b_i^{\nu}, c_i, c_i^{\nu}, \dots$ write $b_i(\rho), b_i(\rho^{-1}), c_i(\rho), c_i(\rho^{-1}), \dots$, respectively; for e of § 21 write $\rho - \rho^{-1}$. In § 20, take $b_0 = \frac{1}{2}$ if F does not have modulus 2, while for modulus 2 take

$$b_0 = -\frac{1}{\alpha_{\nu}} (\rho^{\nu} + \alpha_1 \rho^{\nu-1} + \alpha_2 \rho^{\nu-2} + \dots + \alpha_{\nu-1} \rho),$$

where the α 's refer to the irreducible reciprocal equation with root ρ :

$$(14) \quad E \equiv y^{\nu} + y^{-\nu} + \alpha_1 (y^{\nu-1} + y^{-\nu+1}) + \dots + \alpha_{\nu-1} (y + y^{-1}) + \alpha_{\nu} = 0.$$

If α_{ν} vanished, there would be a factor $y - y^{-1}$ modulo 2. Since the variables entering $F_{\alpha\beta r}$ are linear functions of the initial variables ξ_i whose coefficients are polynomials in ρ with coefficients in F , $F_{\alpha\beta r}$ can be expressed as a function of ρ and the ξ 's with coefficients in F . In particular $F_{\alpha\beta r}$ is unaltered by any permutation of the roots $\rho_1, \rho_1^{-1}, \dots, \rho_{\nu-1}, \rho_{\nu-1}^{-1}$. The same must be true of $[C_0 C_{\nu}]$, which is composed of all the terms of Φ involving the variables of classes C_0 and C_{ν} (viz., those corresponding to the roots ρ and ρ^{-1}), since Φ is to be equal to a function of the ξ 's with coefficients in F . Hence the a 's in (13) must be symmetric functions of those $2\nu - 2$ roots; but the latter are the roots of $E/(y + y^{-1} - \rho - \rho^{-1}) = 0$, E being given by (14). Hence the coefficients in (13) are rational functions of ρ with coefficients in F . By its construction $F_{\alpha\beta r}$ becomes $F_{\beta\alpha r}$ when ρ is replaced by ρ^{-1} . Hence

$$(15) \quad \alpha_r^{\alpha\beta}(\rho) = \alpha_r^{\beta\alpha}(\rho^{-1}),$$

whence $\alpha_r^{\alpha\alpha}$ is a polynomial in $\rho + \rho^{-1}$.

Finally, since $[C_0 C_{\nu}] + \Psi$ is to be equal to a function Φ of the ξ 's with coefficients in F , Ψ must contain the bilinear forms $[C_1 C_{\nu+1}], \dots, [C_{\nu-1} C_{2\nu-1}]$ derived from $[C_0 C_{\nu}]$ by replacing ρ by $\rho_1, \dots, \rho_{\nu-1}$, respectively. Hence must $\Phi = Q + \Phi_1$, where Φ_1 involves no variable in the classes $C_0, \dots, C_{2\nu-1}$, while

$$(16) \quad Q \equiv \sum_{i=0}^{\nu-1} [C_i C_{\nu+i}] = \sum_{\alpha, \beta, r, i} \alpha_r^{\alpha\beta}(\rho_i) F_{\alpha\beta r}^{(i)},$$

$F_{\alpha\beta r}^{(i)}$ being the same function of ρ_i and the variables of $C_i, C_{\nu+i}$ that $F_{\alpha\beta r}^0 \equiv F_{\alpha\beta r}$ is of ρ_0 and the variables of C_0, C_{ν} . In (16), the a 's are arbitrary rational functions of ρ_i satisfying (15) and making the discriminant of $[C_0 C_{\nu}]$ not zero (requiring that certain determinants of the α 's be $\neq 0$). The same argument is to be repeated for Φ_1 with reference to each new irreducible factor of $\Delta(\rho)$.

15. The next problem is the reduction of $[C_0 C_{\nu}]$ to one or more normal forms by means of a transformation of variables leaving S unaltered. JORDAN shows (§§ 23-28) that a unique normal form results when F is a field of order

p . The same is true for any finite field, but not for an arbitrary infinite field. For an arbitrary field F we can proceed with JORDAN's normalization by observing the following modifications.* The constants $a, a^{\mu\nu}, \lambda, \lambda^{\nu}, \dots$ are to be interpreted as $a(\rho), a(\rho^{-1}), \lambda(\rho), \lambda(\rho^{-1}), \dots$, respectively.

To prove that the argument at the bottom of p. 243 remains valid, we have to show that, for $\alpha_m^{21} \neq 0$,

$$\lambda(\rho)\alpha_m^{21}(\rho) + \lambda(\rho^{-1})\alpha_m^{12}(\rho) + \lambda(\rho)\lambda(\rho^{-1})\alpha_m^{22}(\rho)$$

is not identically zero for every rational function λ . But if the sum vanished for $\lambda(\rho) = 1, -1$, and ρ , then would

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ \rho & \rho^{-1} & 1 \end{vmatrix} \equiv -2(\rho^{-1} - \rho) = 0,$$

which is impossible if F does not have modulus 2. For modulus 2, we employ $\lambda(\rho) = 1, \rho$, and $\rho + 1$, obtaining as the determinant $(\rho^{-1} + \rho)(\rho^{-1} - \rho)$.

The argument to make $\alpha_m^{11} = 1$ (top of p. 244) must now be abandoned. For the present we allow α_m^{11} to remain arbitrary, but $\neq 0$. To make $\alpha_m^{12} = \alpha_m^{21} = 0$, apply the transformation which replaces x'_i and y'_i by $x'_i - a(\rho)x''_i$ and $y'_i - a(\rho^{-1})y''_i$, respectively, for $i = m, \dots, 0$, taking $a(\rho) = \alpha_m^{21}(\rho)/\alpha_m^{11}(\rho)$. By analogous transformations we can make every $\alpha_m^{\alpha\beta} = 0$ ($\alpha \neq \beta$) and reach

$$(17) \quad \phi_a = A \sum_{a=1}^r \{ \alpha_m^{\alpha\alpha} [x_0^\alpha y_m^\alpha + (-1)^m x_m^\alpha y_0^\alpha] \} \quad (\text{each } \alpha_m^{\alpha\alpha} \neq 0, A \neq 0).$$

Consequently we insert the factors $\alpha_m^{\alpha\alpha}$ in the formulæ of §§ 26–28. For λ on p. 248 we now take

$$\lambda(\rho) = -c(\rho^{-1}) \div \{ A(\rho^{-1})\alpha_m^{\alpha\alpha}(\rho) \}.$$

For case 1°, p. 250, the condition to be satisfied is now

$$(-1)^{m'-r'} [\lambda(\rho^{-1}) + \lambda(\rho)] \alpha_m^{\alpha\alpha} + \alpha_r^{\alpha\alpha} = 0,$$

where each a is a rational function of $\rho + \rho^{-1}$. To this end we apply the

LEMMA. *If ρ is a root of an irreducible reciprocal equation (14), we can determine a polynomial $\lambda(\rho)$ such that $\lambda(\rho^{-1}) + \lambda(\rho) = f(\rho + \rho^{-1})$, f being any given polynomial, where the coefficients of λ and f belong to the arbitrary field F .*

In view of (14), we may set

$$f = g_1(\rho^{\nu-1} + \rho^{-\nu+1}) + \dots + g_{\nu-1}(\rho + \rho^{-1}) + g_\nu \quad (g's \text{ in } F).$$

* We do not consider the numerical results of §§ 24, 25, peculiar to finite fields. On p. 245, line 6, the term $c_1^{\nu-1}d_1$ should be deleted. The group considered in § 24 occurs in the literature, *Mathematische Annalen*, vol. 52 (1899) p. 561, and vol. 55 (1902), p. 521, as the hyperorthogonal group in the $GF[p^{2\nu}]$.

We proceed to exhibit a solution λ of the form

$$\lambda(\rho) = c_0 \rho^v + c_1 \rho^{v-1} + \cdots + c_v \quad (c\text{'s in } F).$$

If F does not have modulus 2 we may take

$$c_0 = 0, \quad c_1 = g_1, \quad \cdots, \quad c_{v-1} = g_{v-1}, \quad c_v = \frac{1}{2}g_v.$$

If F has modulus 2, we apply (14) to eliminate $\rho^v + \rho^{-v}$ and get

$$\lambda(\rho^{-1}) + \lambda(\rho) = \sum_{i=1}^{v-1} (c_i - c_0 \alpha_i) (\rho^{v-i} + \rho^{-v+i}) - c_0 \alpha_v.$$

Hence this equals f if we set

$$c_0 = \frac{g_v}{\alpha_v}, \quad c_i = g_i + \frac{\alpha_i g_v}{\alpha_v} \quad (i = 1, \cdots, v-1).$$

For case 2° , page 250, the condition to be satisfied is now*

$$(-1)^{m'-r'-1} e [\lambda(\rho^{-1}) - \lambda(\rho)] \alpha_m^{aa} + \alpha_r^{aa} = 0, \quad e \equiv \rho - \rho^{-1}.$$

Now $(-1)^{m'-r'} \alpha_r^{aa} / \alpha_m^{aa}$ equals a polynomial $f(\rho + \rho^{-1})$ by (15). Set $\mu(\rho) = (\rho^{-1} - \rho)\lambda(\rho)$. The resulting condition $\mu(\rho^{-1}) + \mu(\rho) = f(\rho + \rho^{-1})$ may be satisfied by the Lemma.

Cases 3_1° , 3_2° are analogous to 2° , 1° , respectively.

Hence $[C_0 C_v]$ can be reduced to the semi-normal form

$$(18) \quad \sum_{a=1}^l \alpha_m^a F_{aam} + \sum_{\beta=l+1}^{l+l'} \alpha_m^{\beta\beta} F_{\beta\beta m'} + \sum_{\gamma=l+l'+1}^{l+l'+l''} \alpha_m^{\gamma\gamma} F_{\gamma\gamma m''} + \cdots,$$

where each $a = a(\rho) = a(\rho^{-1}) \neq 0$, while F_{aam} is a bilinear function with fixed coefficients of x_i^a and y_i^a ($i = 0, 1, \cdots, m$; $a = 1, \cdots, l$), $F_{\beta\beta m'}$ a bilinear function of x_i^β and y_i^β ($i = 0, 1, \cdots, m'$; $\beta = l+1, \cdots, l+l'$), etc. Also $y(\rho) = x(\rho^{-1})$, $m > m' > m'' \cdots$.

16. First let F be a finite field, the $GF[p^k]$. Then $F(\rho)$ is the $GF[p^{2vk}]$ and $\rho^{p^{vk}} = \rho^{-1}$. The a 's belong to the $GF[p^{vk}]$. Apply the transformation (commutative with S) which multiplies each x_i^a by λ and each y_i^a by $\lambda^{p^{vk}}$. The new coefficients of F_{aam} is $\alpha_m^{aa} \lambda^{p^{vk}+1}$ and hence can be made unity by choice of λ in the $GF[p^{2vk}]$. For a finite field every a in (18) can be made unity, so that there is a unique normal form.

17. For a general field F the question is not so simple; we shall have to

* By a misprint the wrong sign is given by JORDAN. On p. 249, 5th line from the bottom, $\lambda^{p^v} y_{m-r}^a$ should read $\lambda^{p^v} y_0^a$. On p. 250, 3d line from the bottom, $\lambda^{p^v} + \lambda$ should read $\lambda^{p^v} - \lambda$.

consider normalizations not used or needed in JORDAN's case. Now S affects* the variables entering (18) as follows:

$$(19) \left\{ \begin{array}{ll} x_0^\alpha, x_1^\alpha, \dots, x_m^\alpha, & \rho x_0^\alpha, \rho(x_1^\alpha + x_0^\alpha), \dots, \rho(x_m^\alpha + x_{m-1}^\alpha) \\ y_0^\alpha, y_1^\alpha, \dots, y_m^\alpha, & \rho^{-1} y_0^\alpha, \rho^{-1}(y_1^\alpha + y_0^\alpha), \dots, \rho^{-1}(y_m^\alpha + y_{m-1}^\alpha) \end{array} \right. \quad (\alpha = 1, \dots, l)$$

$$(19) \left\{ \begin{array}{ll} x_0^\beta, x_1^\beta, \dots, x_{m'}^\beta, & \rho x_0^\beta, \rho(x_1^\beta + x_0^\beta), \dots, \rho(x_{m'}^\beta + x_{m'-1}^\beta) \\ y_0^\beta, y_1^\beta, \dots, y_{m'}^\beta, & \rho^{-1} y_0^\beta, \rho^{-1}(y_1^\beta + y_0^\beta), \dots, \rho^{-1}(y_{m'}^\beta + y_{m'-1}^\beta) \end{array} \right. \quad (\beta = l+1, \dots, l+l')$$

analogously for $x_0^\gamma, y_0^\gamma, \dots, x_{m''}^\gamma, y_{m''}^\gamma \quad (\gamma = l+l'+1, \dots, l+l'+l'').$

Any substitution P commutative with (19) is the product of a substitution T on the x 's by the conjugate substitution $\bar{T}$ on the y 's. Further,

$$T = T_m T_{m'} T_{m''} \dots W,$$

where each factor is commutative with S , T_m denoting a substitution of the form

$$(20) \quad \left| x_i^\alpha \quad \sum_{\delta=1}^l b_{\alpha\delta} x_i^\delta \right| \quad (\alpha = 1, \dots, l; i = 0, 1, \dots, m),$$

T_m being cogredient on $x_i^{l+1}, \dots, x_i^{l+l'}$, for $i = 0, 1, \dots, m'$, etc., while W is derived from substitutions, the general one of which replaces x_i^α by $x_i^\alpha + \lambda x_{i-m+r}^\beta$ ($i = m, \dots, m-r$) and leaves fixed x_j^α ($j = m-r-1, \dots, 0$), r being $< m$. Hence† T replaces each x_0^α by a linear function of $x'_0, \dots, x'_0$ only, while

* Here and henceforth it is to be understood that the remaining variables $x(\rho_1), y(\rho_1), \dots, x(\rho_{\nu-1}), y(\rho_{\nu-1})$, conjugate to $x(\rho) \equiv x$, undergo the transformation conjugate to that on x . Thus the complete substitution can be expressed as a substitution on the initial variables with coefficients in F .

† The explicit form of T is not essential to the argument; it is moreover fairly complex (*Linear Groups*, § 218). For $m=2, m'=1, l=2, l'=1, S$ is

$$\left| \begin{array}{ll} x_0^\alpha, x_1^\alpha, x_2^\alpha & \rho x_0^\alpha, \rho(x_1^\alpha + x_0^\alpha), \rho(x_2^\alpha + x_1^\alpha) \\ x_0''', x_1''' & \rho x_0''', \rho(x_1''' + x_0''') \end{array} \right| \quad (\alpha = 1, 2),$$

the explicit form of T is the following (zero coefficients not being entered):

	x'_0	x''_0	x'_1	x''_1	x'_2	x''_2	x'''_0	x'''_1
x'_0	b_{11}	b_{12}						
x''_0	b_{21}	b_{22}						
x'_1	c_1	c_2	b_{11}	b_{12}			e_1	
x''_1	c_3	c_4	b_{21}	b_{22}			e_2	
x'_2	d_1	d_2	c_1	c_2	b_{11}	b_{12}	f_1	e_1
x''_2	d_3	d_4	c_3	c_4	b_{21}	b_{22}	f_2	e_2
x'''_0	g_1	g_2					b_{33}	
x'''_1	h_1	h_2	g_1	g_2			k	b_{33}

$x'_m, \dots, x'_m$ appear in the functions by which T replaces x_i^a only when $i = m$. Analogous remarks on the y 's hold for $\bar{T}$. Hence in the function by which $P \equiv T\bar{T}$ replaces (18), the terms involving the variables x_m^a and y_m^a of maximum rank m come only from the terms of (18) involving these variables. But the latter terms are given by ϕ_a of (17). Hence in order that P shall transform f_a given by (18) into a similar function $f_{a'}$, it is necessary that P shall transform ϕ_a into $\phi_{a'} + \psi$, where the variables of ψ are of rank $< m$. Hence the factor T_m , given by (20), of T must transform ϕ_a into $\phi_{a'}$. The necessary and sufficient conditions for this are

$$(21) \quad \sum_{a=1}^l \alpha_m^{aa} b_{a\delta} \bar{b}_{a\epsilon} = \begin{cases} 0 & \text{if } \delta \neq \epsilon, \\ \alpha_m'^{\delta\delta} & \text{if } \delta = \epsilon. \end{cases}$$

We assume that these conditions on T_m are satisfied. Then T_m replaces $\sum_{a=1}^l \alpha_m^{aa} \alpha_i'^a y_j^a$ by

$$\sum_{\delta, \epsilon=1}^l \left(\sum_{a=1}^l \alpha_m^{aa} b_{a\delta} \bar{b}_{a\epsilon} \right) x_i^\delta y_j^\epsilon = \sum_{\delta=1}^l \alpha_m'^{\delta\delta} x_i^\delta y_j^\delta.$$

Hence T_m replaces $\sum_{a=1}^l \alpha_m^{aa} F_{aam}$ by $\sum_{a=1}^l \alpha_m'^{aa} F_{aam}$. The same reasoning applies to $T_{m'}, T_{m''}, \dots$. Hence* if $T\bar{T}$ replaces f_a by $f_{a'}$, the product $T_m \bar{T}_m T_{m'} \bar{T}_{m'} \dots$ must replace f_a by $f_{a'}$.

THEOREM. *Any possible transformation of (18) into a similar function by means of a substitution commutative with S can be effected by the simple substitutions (20) subject to conditions (21). The normalization of (18) must take place in the individual sums independently.*

18. In the actual normalization of $\sum_a \alpha_m^{aa} F_{aam}$, the plan of § 9 is to apply, instead of a single l -ary substitution (20), a succession of binary substitutions [special cases of (20)]:

$$(22) \quad |x'_i \quad x''_i \quad b_{11}x'_i + b_{12}x''_i, b_{21}x'_i + b_{22}x''_i| \quad (i = 0, 1, \dots, m),$$

where by (21), $\alpha_m^{11} b_{11} \bar{b}_{12} + \alpha_m^{22} b_{21} \bar{b}_{22} = 0$. It follows as in § 9 that $a(F_{11m} + rF_{22m})$ can be multiplied by $\lambda\bar{\lambda} + r\mu\bar{\mu}$ by means of a binary substitution commutative with S , where λ and μ are any rational functions of ρ such that $\lambda\bar{\lambda} + r\mu\bar{\mu} \neq 0$.

19. Finally, for a root $\rho = \pm 1$, the operations take place in the initial field F . We may therefore follow JORDAN's developments † (§§ 33–35), to obtain the most general invariant $[C]$ involving the variables with multiplier ± 1 . But the reduction of $[C]$ to normal forms by means of substitutions commuta-

* For our normalization, we may therefore dispense with the substitutions W . It may be noted in passing that W replaces $f_{a'}$ by $f_{a''}$ only when $f_{a'} \equiv f_{a''}$. This may be proved directly by noting that the first term in JORDAN's expression for F_{aam} appears with the same coefficient after transformation by W .

† On p. 258, l. 7, read $2n - 2$ in the second term; at the end of $G_{a''}$ read $x_{n-k-k'}$.

tive with S requires essential modifications for the generalization to an arbitrary field F . The first step (§ 37) is now impossible in general. Instead,* if F is any field not having modulus 2, we employ the preliminary normalization reducing ϕ to $\sum_{a=1}^l \alpha_n^{aa} x_n^a x_n^a$, each $a \neq 0$. Although α_n^{11} is not necessarily unity, the argument in §§ 38–40 holds after an evident modification, so that we reach

$$[C] = \sum_{a=1}^l \alpha_n^{aa} G_{an} + \Psi \quad (\text{if } m = \text{even} = 2n).$$

No changes are necessary in §§ 41–43, so that there results

$$[C] = \sum_{k=1}^{l/2} f_{2k-1, 2k, 2n-1} + \Psi \quad (\text{if } m = \text{odd} = 2n-1).$$

Indeed, all alternate bilinear forms of determinant $\neq 0$,

$$\sum_{i,k}^{1, \dots, 2m} c_{ik} x_i X_k \quad (c_{ik} = -c_{ki}, \text{ c's in field } F),$$

are reducible by a linear substitution in F , congruent on the x 's and X 's, to a unique normal form $\sum_{k=1}^m (x_{2k-1} X_{2k} - x_{2k} X_{2k-1})$.

Continuing similarly the reduction of Ψ , we obtain as a semi-canonical form of $[C]$ a sum of terms G and f , affecting different variables. The factors α of the G are as yet any marks not zero of F , but otherwise the coefficients in $[C]$ are all fixed constants. As shown above for hermitian forms, so here further normalization must take place in the separate sums

$$(23) \quad \sigma_a \equiv \sum_{\alpha=1}^l \alpha_n^{aa} G_{an}, \quad \sum_{\beta=l+1}^{l+l'} \alpha_n^{\beta\beta} G_{\beta n'}, \dots \quad (n > n' > n'', \dots),$$

the only substitution effective in normalizing σ_a being of the form

$$(24) \quad \left| x_i^a \quad \sum_{\delta=1}^l b_{a\delta} x_i^\delta \right| \quad (a = 1, \dots, l; i = 0, 1, \dots, n).$$

Now $G_{an} = x_n^a x_n^a + \sum c_{ij} x_i^a x_j^a (i \neq j)$. Further, (24) replaces $x_i x_j (i \neq j)$ by a sum of such terms. Hence if (24) replaces σ_a by $\sigma_{a'}$, it must replace $\sum \alpha_n^{aa} x_n^a x_n^a$, the only terms in σ_a with like subscripts to the two x 's, by $\sum \alpha_n^{a'a} x_n^a x_n^a$, the only terms in $\sigma_{a'}$ with like subscripts to the two x 's. Hence, since F does not have modulus 2, we derive the necessary conditions

$$(25) \quad \sum_{\alpha=1}^l \alpha_n^{a\alpha} b_{a\delta}^2 = \alpha_n^{\delta\delta}, \quad \sum_{\alpha=1}^l \alpha_n^{a\alpha} b_{a\delta} b_{a\epsilon} = 0 \quad (\delta, \epsilon = 1, \dots, l; \delta \neq \epsilon).$$

*Some corrections make the reading of §§ 37–39 easier. In §§ 37, 38, read $[C]$ for $[aa]$. In § 38 the use of α_{2n}^{11} instead of α_n^{11} does not conform to the earlier notations. For T in § 39 read

$$T' = |x'_{2n}, \dots, x'_{2n-2k} \quad x'_{2n} + \lambda x'_{2k}, \dots, x'_{2n-2k} + \lambda x'_0|.$$

But these are also sufficient conditions that (24) shall replace σ_a by $\sigma_{a'}$. Indeed, if (25) hold, (24) replaces $\sum_a a_n^{aa} x_i^a x_j^a (i \neq j)$ by

$$\sum_{\delta, \epsilon}^{1, \dots, l} \left(\sum_{a=1}^l a_n^{a\delta} b_{a\delta} b_{a\epsilon} \right) x_i^\delta x_j^\epsilon = \sum_{\delta=1}^l a_n^{\delta\delta} x_i^\delta x_j^\delta.$$

Hence for the purposes of normalization, G_{an} of σ_a may be replaced by its leading term $x_n^a x_n^a$ and the subscript i in (24) restricted to the value n . *The problem is therefore essentially the normalization of an l -ary quadratic form within the group of all l -ary substitutions in the field F .*

20. With regard to fields having modulus 2, we restrict our attention to a finite field, say the $GF[2^k]$. Then every mark is a square. The developments of JORDAN (§§ 46–53) hold for the $GF[2^k]$ after slight changes. In § 46 we must multiply $x_n'^2 + x_n''^2$ by λ , where λ is any fixed mark such that $\lambda x^2 + \lambda y^2 + xy$ is irreducible in the $GF[2^n]$ (cf. *Linear Groups*, § 199). Hence in § 47, $G_{1n} + G_{2n}$ must be multiplied by λ ; similarly in the latter sections. In § 48 the case $b^a \neq 0$ can be reduced to the case $b^a = 1$ by applying the substitution which multiplies x_i^a by $(b^a)^{-\frac{1}{2}}$ for $i = 0, 1, \dots, m$.

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