SETS OF METRICAL HYPOTHESES FOR GEOMETRY*

BY

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In this paper is given a set of assumptions, C, concerning point, order † and congruence, which, together with a certain set of order assumptions O,‡ a continuity assumption K,‡ and a very weak parallel assumption P_0,‡ are sufficient for the establishment of ordinary Euclidean geometry. Each assumption of this set is independent of all the remaining ones. Moreover congruence is here an independent symbol in the sense that it would be impossible § to prove that if the order assumptions O, K and P_0 are true, then there must exist a relation satisfying the congruence assumptions C with reference to the points and order in terms of which O, K and P_0 are stated. According to VEBLEN || congruence would not, however, be thus independent if P_0 were replaced even by his still comparatively weak parallel assumption XI.¶

If in this categorical set of assumptions there is substituted for K the assumption that *every segment has a middle point,*** and P_0 is replaced by the somewhat stronger assumption P_2,†† then follows ‡‡ a geometry §§ in which a theory of proportion holds and rigid motion is possible.

Other alternative sets of assumptions are discussed. Among these an interesting set is that obtained by substituting for P_0 the postulate P_2 and for K a postulate I_2 which may be roughly stated as follows: "*If B is within, and B' without, any circle α, then each semicircle with $B'B$ as diameter must have a point in common with the circle α.*" |||| Each assumption in the set composed of P_2, I_2

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† It would be impossible (see Theorem 1 of § 12) to formulate two sets of assumptions, one set, C, being stated in terms of point and congruence alone, and the other, O, being stated in terms of point and order alone, such that from O and C would follow a geometry, Euclidean with respect to the undefined symbols in terms of which O and C are stated.

‡ O, K and P_0 are all stated in terms of point and order.

§ Cf. § 10.

|| These Transactions, vol. 5 (1904), pp. 343–384.

¶ Loc. cit. p. 346.

** Cf. end of § 1 and first part of § 6.

†† See § 6.

‡‡ Cf. Theorem 1 of § 6.

§§ A geometry which is a consequence of this set of assumptions is a consequence of HILBERT's Axiom-groups I–IV and conversely.

|||| Cf. § 7.

and the assumptions C and O is independent of the remaining ones, and moreover this set is sufficient for the establishment of a geometry in which not only is there a theory of proportion and rigid motion but also all ordinary rule and compasses constructions are possible.

In this paper "proof" is used to mean an *indication* of a demonstration. "Hence," "by," "therefore," etc., are intended as suggesting certain relations and not necessarily as describing exactly the logical dependence of one statement upon another.

I wish to thank Professor E. H. Moore and Professor O. Veblen for suggestions and criticisms. Professor Veblen, who suggested the undertaking of this investigation, has not only made numerous suggestions and criticisms, but has given me much help in the way of actual collaboration.

Assumptions for Euclidean geometry.

O. Veblen's order "axioms" I and III–X.*

C. Assumptions concerning point and congruence, sometimes with reference to order, as follows:

C_{1_a}. *If B† is different from C and A' is different from B' then there exists a point C' such that $A'B'C'$‡ and $BC \equiv B'C'$.*§

C_{1_b}. *If B is different from C and A' is different from B' then there is not more than one point C' such that $A'B'C'$ and $BC \equiv B'C'$.*

C_2. *If A is different from B, A' is different from B', A'' is different from B'', $AB \equiv A'B'$ and $A'B' \equiv A''B''$, then $AB \equiv A''B''$.*

C_3. *If ABC, $A'B'C'$, $AB \equiv A'B'$ and $BC \equiv B'C'$, then $AC \equiv A'C'$.*

C_4. *If A, B and C are three non-collinear* ‖ *points and A', B' and C' are three non-collinear points and CAD, $C'A'D'$, $AB \equiv A'B'$, $BC \equiv B'C'$, $CA \equiv C'A'$, $CD \equiv C'D'$, then $BD \equiv B'D'$.*¶

K. For the continuity assumption K may be taken either Veblen's XI or a Dedekind cut assumption stated for the points of a single segment as follows: *If there exists any segment then there exists some segment AB such that if it is*

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* Loc. cit., pp. 344–346. From assumption C_{1_a} below it follows that if A and B are two different points there is a point C different from B such that ABC. Veblen's Axiom II may be seen to be a consequence of this proposition and his Axioms I, III–VIII (cf. a paper presented by me to the American Mathematical Society on October 26, 1907, but not yet published). Thus Axiom II is redundant in the set composed of Veblen's Axioms I–X together with my Axioms C.

† Capital letters are used to designate points.

‡ ABC used as a sentence means A, B and C are in the order ABC.

§ BC as a simple word, unmodified by "line" or "ray," means the segment BC, i. e., the set of all points X such that BXC.

‖ Cf. Veblen, loc. cit., p. 345. Instead of "three non-collinear points" one might, in stating this assumption, use the phrase "three points such that neither ABC, BCA, nor CAB."

¶ Cf. Veronese, *Grundzüge der Geometrie*, pp. 254–260. C_4 and C_3 might be stated together in one assumption.

composed of two sets of points $[M]$* *and* $[N]$, *each set containing at least two points and no point* X *of either set being either the same as* Y_1 *or such that* Y_1XY_2 *where* Y_1 *and* Y_2 *are points of the other set, then there exists a point* C *such that* MCN *for every* M *and* N *different from* C.

With the help of O, from either of these assumptions can be deduced a theorem to the effect that this DEDEKIND cut proposition must apply to every segment as well as to the segment AB (cf. VEBLEN, loc. cit., pages 368 and 369).

P_0. *If there exists any straight line and a point not on it then there exists some straight line* a, *and some point* A, *not on it, such that if* a *and* A *lie in a plane*† β *then in the plane* β *there is not more than one straight line passing through* A *and having no point in common with* a.

§ 1. *Consequences of* O *and* C.

On the basis of O and C we have the following propositions:

Theorem 1. If A' is a point and AB is a segment, then on each ray ‡ starting from A' there is one and only one point B' such that $AB \equiv A'B'$.

Proof. See C_{1_a} and C_{1_b}.

Theorem 2. If ABC and C' is on ray $A'B'$ and $AB \equiv A'B'$ and $AC \equiv A'C'$, then $A'B'C'$.

Proof. There is, according to C_{1_a}, a point C'' such that $A'B'C''$ and $BC \equiv B'C''$. Hence, by C_3, $AC \equiv A'C''$. But, according to hypothesis, $AC \equiv A'C'$. Therefore, by C_{1_b}, C'' is C'. But $A'B'C''$.

Corollary. If B is on ray AC and B' is on ray $A'C''$ and $AB \equiv A'B'$ and $AC \equiv A'C'$, then $BC \equiv B'C'$.

Theorem 3.§ If A is distinct from B, then $AB \equiv AB$.

Proof. According to Theorem 1 there exists, on ray AB, a point B' such that $AB \equiv AB'$. Again there exists, on ray AB, a point B'' such that $AB' \equiv AB''$. But, then, according to C_2, $AB \equiv AB''$ and thus (see Theorem 1) B' is B''. Hence $AB' \equiv AB'$. Therefore $AB \equiv AB'$ and also $AB' \equiv AB'$. But if B is not B' then either ABB' or $AB'B$ and thus, by Theorem 2, $AB'B'$, which is impossible. So B is B'. But $AB \equiv AB'$. Hence $AB \equiv AB$.

Theorem 4. If $AB \equiv A'B'$, then $A'B' \equiv AB$.

Proof. According to Theorem 1 there exists, on ray AB, a point B'' such that $A'B' \equiv AB''$. So, by C_2, Theorem 3 and Theorem 1, B'' must be B. But $A'B' \equiv AB''$. Thus $A'B' \equiv AB$.

* The notation $[M]$ is used to denote a set of elements any one of which may be called M.

† Cf. VEBLEN, loc. cit., p. 345.

‡ If A and B are two distinct points, then the set of all points $[C]$ such that either C is B, or ACB, or ABC, is called a ray or half line starting from A. Such a ray is called the ray AB.

§ Cf. HILBERT's IV, 1 (see page 12 of HILBERT's *The Foundations of Geometry*, TOWNSEND's translation).

Theorem 5.* If $AB \equiv A'B'$ and $AB \equiv A''B''$, then $A'B' \equiv A''B''$.

Proof. This theorem is a logical consequence of Theorem 4 and C_2.

Theorem 6. If ABC and $AC \equiv A'C'$, then there exists one and only one point B' such that $A'B'C'$ and $AB \equiv A'B'$ and $BC \equiv B'C'$.

Proof. According to Theorem 1 there is one and only one point B' on ray $A'C'$ such that $AB \equiv A'B'$. According to Theorem 2 and its corollary, $A'B'C'$ and $BC \equiv B'C'$.

Definition 1. $A'B' < AB$ means: there exists a point P such that APB and $A'B' \equiv AP$. $A'B' > AB$ means $AB < A'B'$.

Theorem 7. If $AB > A'B'$, $AB \equiv CD$ and $A'B' \equiv C'D'$, then $CD > C'D'$.

Proof. According to definition 1 there exists a point P such that APB and $A'B' \equiv AP$. So, according to Theorem 6, there exists P' such that $CP'D$ and $AP \equiv CP'$. By C_2 it follows that $A'B' \equiv CP'$. Therefore, by hypothesis and Theorem 5, $C'D' \equiv CP'$. So $CD > C'D'$, according to Definition 1.

Theorem 8. If A is different from B and A' is different from B', then, of the three statements $AB \equiv A'B'$, $AB > A'B'$, $AB < A'B'$, one and only one is true.

Theorem 9. If $AB > A'B'$ and $A'B' > A''B''$, then $AB > A''B''$.

Definition 2. $AB + CD$ means the segment AE, where E is a point such that ABE and $CD \equiv BE$. If $AB > CD$ then $AB - CD$ means the segment AE where E is a point such that BEA and $CD \equiv BE$.

Theorem 10. If $AB \equiv A'B'$, $CD \equiv C'D'$, then $AB + CD \equiv A'B' + C'D'$. We have the following rules of combination: $AB + CD \equiv CD + AB$, $(AB + CD) + EF \equiv AB + (CD + EF)$, $AB - CD \equiv EF$ if and only if $AB \equiv CD + EF$.

A relation (called congruence) between angles† is introduced by the following definition:

Definition 3.‡ $\measuredangle ABC \equiv \measuredangle A'B'C'$ means that there exist points D, D', E, E', on rays BA, $B'A'$, BC, $B'C'$ respectively, such that $BD \equiv B'D'$, $DE \equiv D'E'$, $EB \equiv E'B'$.

Theorem 11. If $\measuredangle BAC \equiv \measuredangle B'A'C'$, $BA \equiv B'A'$, $CA \equiv C'A'$, then $BC \equiv B'C'$.

Proof. By hypothesis and Definition 3 there exist on rays AB, AC, $A'B'$, $A'C'$, points D, E, D', E' respectively such that $AD \equiv A'D'$, $DE \equiv D'E'$, $EA \equiv E'A'$. By Theorem 1 there exist points F, G, F', G', such that BAF, CAG, $B'A'F'$, $C'A'G'$, $AF \equiv A'F'$, $AG \equiv A'G'$. By hypotheses and C_4 it follows that $EF \equiv E'F'$ and therefore, by Theorem 2 and another application of C_4, it may be proved that $FG \equiv F'G'$. Treating A, B, C, A', B', C',

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* See HILBERT's IV, 2, loc. cit., p. 12.

† An angle is a point together with two non-collinear rays which start from that point.

‡ Cf. VERONESE, loc. cit., p. 257.

with reference to A, G, F, A', G', F' as A, G, F, A', G', F' were treated with reference to A, D, E, A', D', E', one may prove that $BC \equiv B'C'$.

Theorem 12.* $\measuredangle ABC \equiv \measuredangle ABC$.

Proof. Use Definition 3 and Theorem 3.

Theorem 13. If $\measuredangle ABC \equiv \measuredangle A'B'C'$, then $\measuredangle A'B'C' \equiv \measuredangle ABC$.

Proof. Use Definition 3 and Theorem 4.

Theorem 14.† If $\measuredangle ABC \equiv \measuredangle A'B'C'$ and $\measuredangle ABC \equiv \measuredangle A''B''C''$, then $\measuredangle A'B'C' \equiv \measuredangle A''B''C''$.

Proof. Use Definition 3 and Theorems 11 and 5.

Definition 4. Two angles are supplementary to (supplements of) each other if and only if they have a common vertex and one common side and their other sides form two rays of the same straight line.

Two angles are vertical angles if and only if they have a common vertex and their four sides, together with this common vertex, make up two straight lines.

Theorem 15. If $\measuredangle HAK \equiv \measuredangle H'A'K'$ then each angle that is supplementary to $\measuredangle HAK \equiv$ each angle that is supplementary to $\measuredangle H'A'K'$, and the angle that is vertical to $\measuredangle HAK \equiv$ the angle that is vertical to $\measuredangle H'A'K'$.

Proof. See Theorem 11 and proof thereof.

Corollary. Vertical angles are congruent to each other.

Definition 5. $\triangle ABC \equiv \triangle A'B'C'$ means: $AB \equiv A'B'$, $BC \equiv B'C'$, $CA \equiv C'A'$, $\measuredangle ABC \equiv \measuredangle A'B'C'$, $\measuredangle BCA \equiv \measuredangle B'C'A'$, $\measuredangle CBA \equiv \measuredangle C'B'A'$.‡

Theorem 16. $\triangle ABC \equiv \triangle A'B'C'$ if $AB \equiv A'B'$, $BC \equiv B'C'$, $CA \equiv C'A'$.

Proof. See Definitions 3 and 5.

Theorem 17.§ $\triangle ABC \equiv \triangle A'B'C'$ if $AB \equiv A'B'$, $BC \equiv B'C'$, and $\measuredangle ABC \equiv \measuredangle A'B'C'$.

Proof. See Theorem 11 and Definitions 3 and 5.

Theorem 18. If $\measuredangle ABC \equiv \measuredangle A'B'C'$ and D is in || $\measuredangle ABC$, then there exists a point D' in $\measuredangle A'B'C'$ such that $\measuredangle ABD \equiv \measuredangle A'B'D'$ and $\measuredangle DBC \equiv \measuredangle D'B'C'$.

Definition 6. $\measuredangle B'A'C' < \measuredangle BAC$ means: there is a point, P, in $\measuredangle BAC$ such that $\measuredangle B'A'C' \equiv \measuredangle BAP$. $\measuredangle B'A'C' > \measuredangle BAC$ means: $\measuredangle BAC < \measuredangle B'A'C'$.

Theorem 19. If $\measuredangle BAC > \measuredangle B'A'C'$, $\measuredangle BAC \equiv \measuredangle DEF$, $\measuredangle B'A'C' \equiv \measuredangle D'E'F'$, then $\measuredangle DEF > \measuredangle D'E'F'$.

Theorem 20. If $\measuredangle BAC > \measuredangle B'A'C'$ and $\measuredangle B'A'C' > \measuredangle B''A''C''$, then $\measuredangle BAC > \measuredangle B''A''C''$.

Theorem 21. If ABC is a $\triangle$ and $AC \equiv BC$, then $\measuredangle ABC \equiv \measuredangle ACB$.

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* See HILBERT's IV, 4, loc. cit., page 14.

† See HILBERT's IV, 5, loc. cit., page 14. $\triangle ABC$ (triangle ABC) means the three non-collinear points A, B, and C.

‡ See VERONESE, loc. cit., page 254.

§ See HILBERT's, IV, 6, loc. cit., page 15.

|| If ABC is an angle, the two rays, BA and BC, together with the vertex B, divide the plane ABC into two regions. That one of these regions which contains points P such that APC is called the interior of $\measuredangle ABC$. Cf. VEBLEN, loc. cit., pp. 363–365.

Proof. Use Theorem 3 and Definition 3.

Definition 7. "A middle point of the segment AB" means a point M, on the straight line AB, such that $AM \equiv MB$.

Theorem 22. If M is a middle point of AB, then AMB.

Proof. Use Definition 7 and C_{1_b}.

Theorem 23. No segment has two middle points.

Proof. Use Definition 1 and Theorems 22, 7 and 8.

I do not know as yet whether from O and C it follows that every segment has one middle point. This proposition (statement) that *every segment has a middle point* will be denoted by the symbol M.

§ 2. *Two consequences of O, C and M.*

In § 1 a number of propositions were deduced as consequences of O and C. In the present paragraph it will be shown that, if M is assumed in addition to O and C, then follow two propositions concerning angles.

The symbol A_1 will be used to denote the proposition : *If ABC, $AB \equiv BC$, and D is not on the straight line AB, then there is, in the half-plane $D - AB$,* a point E such that $AD \equiv BE$ and $BD \equiv CE$.*†

The symbol A_{-2} denotes the proposition : *If C is not on the straight line AB, then, in the half-plane $C - AB$, there is no point C', different from C, such that $AC \equiv AC'$ and $BC \equiv BC'$.*‡

Theorem 1. From O, C and M follows A_{-2}.

Proof. If this theorem is not true, then there exists a space for which O, C and M are true but which contains four distinct coplanar points A, B, C', C'' such that C' and C'' are on the same side of AB and moreover $BC' \equiv BC''$ and $C'A \equiv C''A$. In case the straight lines $C'C''$ and AB have a point P in common, then with use of Theorems 15, 16, 17 and 3 it may be seen that $\triangle APC' \equiv \triangle APC''$, and thus $PC' \equiv PC''$, but this is impossible according to C_{1_b} and the hypothesis that C' and C'' are on the same side of AB. In case the straight lines $C'C''$ and AB have no point in common, then A and B are on the same side of the straight line $C'C''$. According to M and Theorem 22, there is a point S such that $C''SC'$ and $C''S \equiv SC'$. There exists, as may be seen by use of O, a point P such that PSA and segment PB has a point S' in common with segment $C'C''$. Consider $\triangle$'s ASC' and ASC''. $AS \equiv AS$ by Theorem 3, while, by hypothesis, $AC' \equiv AC''$ and $C'S \equiv SC''$. Hence, by Theorem 16, $\triangle ASC' \equiv \triangle ASC''$. With use of C_2, C_4 and The-

* By a half-plane is meant one of the two regions into which a plane is decomposed by a straight line which lies in it (cf. VEBLEN, loc. cit., pp. 363–365). If the straight line AB thus decomposes a plane into two regions and D lies in one of these regions then this region is called the half-plane $D - AB$.

† Thus, according to Definition 3, $\measuredangle BAD \equiv \measuredangle CBE$ and A_1 is, then, a weak form of the proposition given in the statement of Theorem 4 of §5. Cf. HILBERT's IV, 4, loc. cit., p. 14.

‡ Cf. HILBERT's IV, 4, loc. cit., p. 14.

orem 3 it follows that $PC'' \equiv PC'$. Hence, by hypothesis, Theorem 3 and Definition 3, it follows that $\measuredangle C''BS' \equiv \measuredangle C'BS'$ and, consequently, by means of Theorems 17 and 3 and Definition 5, that $C''S' \equiv S'C'$, but this is impossible according to theorem 23.

*Corollary.** Given an angle BAC and a ray $A'B'$, there is, in a given plane containing $A'B'$ and on a given side of $A'B'$, not more than one ray $A'C'$ such that $\measuredangle BAC \equiv \measuredangle B'A'C'$.

Theorem 2. From O, C and M follows A_1.

Proof. Suppose ABC, $AB \equiv BC$ and D is not on the straight line AC. According to M and Theorem 22, there is a point S such that ASB and $AS \equiv SB$. By C_{1_a} there exists a point E' such that DSE' and $DS \equiv SE'$. According to Theorem 17 and Corollary to theorem 15, $\measuredangle SAD \equiv \measuredangle SBE'$. But by C_{1_a} there is a point E such that $E'BE$ and $AD \equiv BE$. By theorems 13 and 14 and Corollary to Theorem 15, $\measuredangle DAS \equiv \measuredangle CBE$. But $AB \equiv BC$ and $AD \equiv BE$. Hence, by theorem 11, $BD \equiv CE$.

§ 3. *Consequences of O, C and A_{-2}.*

In § 2 it has been shown that if M is assumed in addition to O and C, then there follow the two propositions A_1 and A_{-2}. On the basis of O, C and A_{-2} we have the following propositions.

Theorem 1. $\triangle ABC \equiv \triangle A'B'C'$ if $\measuredangle ABC \equiv \measuredangle A'B'C'$, $\measuredangle BAC \equiv \measuredangle B'A'C'$ and $AB \equiv A'B'$.

Proof. According to Theorem 1 of § 1 there is a point C'' on ray $A'C'$ such that $AC \equiv A'C''$. According to hypothesis and Theorem 17, $\triangle ABC \equiv \triangle A'B'C''$ and thus (see Definition 5) $\measuredangle ABC \equiv \measuredangle A'B'C'$. But $\measuredangle ABC = \measuredangle A'B'C'$. By A_{-2} and Theorem 1 of § 1 it follows that either C'' is C' or, if not, then B' is on the straight line $C''C'$ and, thus, on the straight line $A'C'$, contrary to hypothesis. So C'' is C'. But $\triangle ABC \equiv \triangle A'B'C''$. Hence $\triangle ABC \equiv A'B'C'$.

Theorem 2.† Of the three statements, $\measuredangle BAC < \measuredangle B'A'C$, $\measuredangle BAC \equiv \measuredangle B'A'C'$, $\measuredangle BAC > \measuredangle B'A'C'$, not more than one is true.

Proof. Make use of Definition 5 and A_{-2}. For instance suppose $\measuredangle BAC > \measuredangle B'A'C'$. Then, according to Definition 6, there exists a point D, in $\measuredangle BAC$, such that $\measuredangle B'A'C' \equiv \measuredangle BAD$. So, if $\measuredangle BAC \equiv \measuredangle B'A'C'$, then $\measuredangle BAC \equiv \measuredangle BAD$, and this could not be, in view of A_{-2}; while, if $\measuredangle BAC < \measuredangle B'A'C'$, then there exists, in $\measuredangle B'A'C'$, a point D' such that $\measuredangle BAC \equiv \measuredangle B'A'D'$. Then, according to original hypothesis and Theorem 19, $\measuredangle B'A'D' > \measuredangle B'A'C'$. Hence there exists, in $\measuredangle B'A'D'$, and at the same

* See HILBERT's IV, 4, loc. cit., p. 14.

† Observe that it is not here stated that *one* of these statements *is* true. See Corollary to Theorem 2 of § 4.

time in $\measuredangle B'A'C'$, a point G'' such that $\measuredangle B'A'C' \equiv \measuredangle B'A'G''$, but this could not be according to A_{-2}. So, if $\measuredangle BAC > \measuredangle B'A'C'$, then neither $\measuredangle BAC < \measuredangle B'A'C'$ nor $\measuredangle BAC \equiv \measuredangle B'A'C'$.

Theorem 3. If HGF, and I is not on the straight line HF and A is different from C, and B, D, A and C are coplanar but neither B nor D is on the straight line AC, and $\measuredangle BAC \geqq \measuredangle FGI$ and $\measuredangle ACD \geqq \measuredangle IGH$, then the rays CD and AB do not meet.

Proof. Of course, the rays AB and CD cannot meet unless they are on the same side of the straight line AC. Suppose they are on the same side of AC. According to hypothesis and Definition 6, there exist points P and S such that either ray CS is ray CD or ray CS is in $\measuredangle DCA$ and moreover either ray AP is ray AB or ray AP is in $\measuredangle CAB$ and furthermore $\measuredangle ACS \equiv \measuredangle HGI$ and $\measuredangle CAP \equiv \measuredangle FGI$. Evidently, then, rays CD and AB cannot meet unless rays CS and AP meet. Suppose rays CS and AP do meet in a point M. There exists, according to C_{1_a}, a point M' such that MCM' and $AM \equiv CM'$. Now, by Theorems 15, 13 and 14, $\measuredangle CAM \equiv \measuredangle ACM'$. Hence, by Theorems 3 and 17 and Definition 5, $\measuredangle CAM' \equiv \measuredangle ACM$. But, by Theorems 15, 13 and 14, $\measuredangle ACM \equiv \measuredangle P'AC$, where P' is any point such that $P'AP$. It follows, by means of A_{-2}, that the points M, A and M' are collinear. So the straight lines AP and CS unite in two* points M and M', but this is impossible. Therefore rays AP and CS do not meet.

Corollary. If HGF, and I is not on the straight line HF, and A, B and C are three non-collinear points, and D is in the half-plane $B - AC$, and $\measuredangle BAC \equiv \measuredangle FGI$ and $\measuredangle ACD \equiv \measuredangle IGH$, then the lines CD and AB have no point in common.

Theorem 4. If ABC is a triangle and $BC > AC$, then $\measuredangle A > \measuredangle B$.

Proof. By hypotheses, Theorem 7, and Definition 1, there exists, between B and C, a point D such that $AC \equiv CD$. Since D is between B and C, it follows from Theorem 12 and Definition 6 that $\measuredangle BAC > \measuredangle DAC$. But, by construction and Theorem 21, $\measuredangle DAC \equiv \measuredangle ADC$. Hence, by Theorems 12 and 19, $\measuredangle BAC > \measuredangle CDA$. But, by Theorem 2 and Corollary 2 to Theorem 3 of § 3, $\measuredangle CDA > \measuredangle ABC$. Hence, by Theorems 3 and 19 of § 1, $\measuredangle BAC > \measuredangle ABC$.

Theorem 5. If D is within $\measuredangle BAC$ and D' is within $\measuredangle B'A'C'$ and $\measuredangle BAD \equiv \measuredangle B'A'D'$ and $\measuredangle DAC \equiv \measuredangle D'A'C'$, then $\measuredangle BAC \equiv B'A'C'$.

Proof. It may be easily seen that there exist points E, F, E', G' on rays AC, AD, $A'C'$, $A'D'$ respectively such that $AE \equiv A'E'$, $AF \equiv A'F'$, and EF and $E'F'$ cut rays AB and $A'B'$ in two points G and G' respectively. By hypothesis and Theorem 17, $\measuredangle AFE \equiv \measuredangle A'F'E'$ and $EF \equiv \measuredangle E'F'$. But it is clear that EFG and $E'F'G'$. Hence, by Theorem 15,

* That M and M' are distinct is a consequence of O.

$\measuredangle AFG \equiv \measuredangle A'F'G'$. But $\measuredangle GAF \equiv \measuredangle G'A'F'$ and $AF \equiv A'F'$. Hence, by Theorem 1, $FG \equiv F'G'$. But $EF \equiv E'F'$. Hence, by C_3, $EG \equiv E'G'$. But $AE \equiv A'E'$ and $AG \equiv A'G'$. Hence, by Definition 3, $\measuredangle BAC \equiv \measuredangle B'A'C'$.

Definition 7. A right angle is an angle that is congruent to its supplement. A straight line a is perpendicular to a straight line b if and only if they intersect in a point O and there exist two points, A and B, different from O, A lying on a and B lying on b, such that $\measuredangle AOB$ is a right angle.

Corollary. Every angle which is congruent to a right angle is itself a right angle. If the straight line a is perpendicular to the straight line b, then b is perpendicular to a and if A and B are *any* two points different from O, A lying on a and B lying on b, then $\measuredangle AOB$ is a right angle.

Proof. See Theorems 12, 14 and 15 of § 1.

Theorem 6. In each plane there is a right angle.

Proof. Suppose A, B, C, are three distinct non-collinear points of a given plane. According to Theorem 1 of § 1 there exists, on ray AC, a point B' such that $AB \equiv AB'$. There is a point D such that ADB and, according to Theorem 6 of § 1, there is a point D' such that $B'D'A$, $BD \equiv B'D'$ and $DA \equiv D'A$. According to O, there is a point P such that BPD' and $B'PD$. Now $\measuredangle B'BA \equiv \measuredangle BB'A$ (see Theorem 21). Hence, by Theorems 3 and 17 of § 1, $\triangle BB'D \equiv \triangle B'BD'$. Thus $\measuredangle B'D'B \equiv \measuredangle B'DB$ and therefore, by Theorem 15, it follows that $\measuredangle PD'A \equiv \measuredangle PDA$. Moreover, by Theorems 17 and 12 and Definition 5, $\measuredangle PBD \equiv \measuredangle PB'D'$. Hence $\triangle B'PD' \equiv \triangle BPD$ according to Theorem 1. Thus $PD' \equiv PD$. Now also $DA \equiv D'A$. Hence, by Theorem 3 of § 1 and Definition 3, $\measuredangle B'AP \equiv \measuredangle BAP$. According to O, there is a point M, on AP, such that $B'MB$. By Theorems 17 and 3 and Definition 5, $MB' \equiv MB$ and $\measuredangle AMB' \equiv \measuredangle AMB$. Thus $\measuredangle AMB$ is a right angle (see Definition 7).

Corollary. If BAC is an angle, then there exists, within $\measuredangle BAC$, a point O such that $\measuredangle BAO \equiv \measuredangle OAC$.

Theorem 7. If there is one perpendicular p to a straight line a, then, through each point of a, and lying in the plane pa, there is one and only one straight line perpendicular to a.

Proof. Suppose O is the point at which p intersects a, C is any other point of a, and P is any other point of p. By C_{1_a} there exists a point P' such that POP' and $PO \equiv OP'$. There is also a point P'' such that $P'CP''$ and $PC \equiv CP''$. There is (see Corollary to Theorem 6) a ray CM in $\measuredangle PCP''$ such that $\measuredangle PCM \equiv \measuredangle MCP''$. Now, also, $\measuredangle PCO \equiv \measuredangle P'CO \equiv \measuredangle P''CO'$, where OCO'. Hence (see Theorem 5) $\measuredangle MCO \equiv \measuredangle MCO'$. Hence, by definition 7, MC is perpendicular to a at the point C.

Suppose there is, in the plane pa, another perpendicular to a at the point C. Suppose N is a point of this new perpendicular on the same side of a as M and suppose MCM' and NCN'. Ray CN must lie either in $\measuredangle OCM$ or in $\measuredangle O'CM$. Suppose it lies in $\measuredangle OCM$. Then ray CN' lies in $\measuredangle O'CM'$. According to Definition 7, Theorem 12, and Theorem 15, $\measuredangle MCO \equiv \measuredangle M'CO$ and $\measuredangle NCO \equiv \measuredangle N'CO$. By Theorem 18, there is a point N'' in $\measuredangle M'CO$ such that $\measuredangle OCN \equiv OCN''$. But $\measuredangle OCN \equiv \measuredangle OCN'$ and ray CN' is in $\measuredangle O'CM'$ and thus is different from CN''. But this is impossible according to A_{-2}. Thus CN cannot be in $\measuredangle OCM$. Similarly it cannot be in $\measuredangle O'CM$. There is, then, in the plane pa and passing through C, not more than one straight line perpendicular to a.

Theorem 8. If there is one perpendicular p to a straight line a, and P is any point in the plane pa, then through P, and lying in this plane, there is one and only one perpendicular to a.

Proof. If P is on a cf. Theorem 7. If P is not on a, suppose F is a point of p on the same side of a as P and O is the point where p meets a. If P is on OF, then certainly, through P, there is a perpendicular (OF) to AB. If P is not on OF, it is on one side of it. Suppose H is a point of a that is on the same side of OF as P. Then P lies in $\measuredangle HOF$. There is (by C_{1_a}) a point F' such that FOF' and $OF \equiv OF'$. According to Theorem 18 there is, in $\measuredangle HOF'$, a point P' such that $\measuredangle POH \equiv \measuredangle HOP'$. On ray OP' there is a point P'' such that $OP \equiv OP''$. Since P and P'' are on opposite sides of OH and on the same side of OF, there is a point M of ray OH such that PMP''. According to Theorems 3 and 17 of § 1, $\triangle OMP \equiv \triangle OMP''$. So $\measuredangle PMO \equiv \measuredangle P''MO$ and thus PM is perpendicular to a.

Now suppose that through P there is, in this same plane, another perpendicular to a. Suppose this new perpendicular meets a in the point M'. There is a point p''' such that $PM'P'''$. According to hypothesis, Definition 7 and Theorems 13 and 14, $\measuredangle MM'P \equiv \measuredangle MM'P'''$ and $\measuredangle M'MP \equiv \measuredangle M'MP''$. According to Theorem 17, $\triangle PMM' \equiv \triangle P''MM'$. So $\measuredangle PM'M \equiv \measuredangle MM'P''$. But $\measuredangle PM'M \equiv \measuredangle P'''M'M$. Hence, according to A_{-2}; P'' must lie on ray $M'P'''$ and thus the straight lines MP and $M'P$ intersect in two points P and P'', but this is impossible. So there is through P, and in a given plane containing a, not more than one perpendicular to a.

Theorem 9. If in the plane of two intersecting straight lines there is a perpendicular to one of them, then there is in this plane a perpendicular to the other one.

Proof. Suppose that the straight lines a and b intersect in a point O and there is a perpendicular to a lying in the plane ab. Then, if B is any point of b other than O, it follows, by Theorem 8, that there is a point A on a such that BA is perpendicular to a. In case A is O, then a is perpendicular to b, and the conclusion of Theorem 9 is evidently verified. In case A is not O, then there exist points A' and B', on b and a respectively, such that $OA \equiv OA'$,

$OB \equiv OB'$. By Theorems 12 and 17, $\measuredangle BAO \equiv \measuredangle B'A'O$. But $\measuredangle BAO$ is a right angle. Hence, by the corollary to Definition 7, $\measuredangle B'A'O$ is a right angle. So BA is perpendicular to b.

Theorem 10. If a is any straight line and P is any point, then in a given plane α containing P and a there is, through P, one and only one straight line perpendicular to a.

Proof. By Theorem 6, there exists in α some straight line a' to which there is a perpendicular lying in α. In case a' is not a, suppose A and A' are two points of a and a' respectively. By hypothesis and Theorem 9 there is in α a perpendicular to AA', and therefore, by Theorem 9, there is in α a perpendicular to a. Hence, by Theorem 8, through each point of α there is, in α, one and only one perpendicular to a.

Theorem 11. If a and b are two intersecting straight lines then there exist in the plane ab straight lines a' and b' which are perpendicular to a and b respectively and are such that each angle that a' makes with $a \equiv$ each angle that b' makes with b.

Proof. See Theorem 10 and proof of Theorem 9.

§ 4. *Consequences of O, C, A_{-2} and A_1.*

In § 2 it was shown that A_1 as well as A_{-2} was a consequence of O, C and M. On the basis of O, C, A_1 and A_{-2} we have the following propositions:

Theorem 1. Any two coplanar right angles are congruent to each other.

Proof. If a side of one right angle is collinear with a side of another which is coplanar with it, it may be easily seen, with help of A_1, Definition 7 and Theorems 1, 15, 13 and 14, that they are congruent to each other. From this result and Theorem 11 of § 3, it follows that two coplanar right angles are congruent to each other if the straight line which contains a side of one intersects the straight line which contains a side of the other. From this, O, Theorem 14 of § 1, and Theorem 10 of § 3, it follows that any two coplanar right angles are congruent to each other.

Theorem 2*. If A, B, C are not collinear and D', A', B' are not collinear, but A, B, C, D', A', B' are coplanar, then, in the half-plane $D' - A'B'$, there is a point C' such that $\measuredangle BAC \equiv \measuredangle B'A'C'$.

Proof. According to Theorem 10 of § 3, there exist points, E in the half-plane $C - AB$, and E' in the half-plane $D' - A'B'$, such that $\measuredangle EAB$ and $\measuredangle E'A'B'$ are right angles. According to Theorem 1, $\measuredangle EAB \equiv \measuredangle E'A'B'$. Suppose ray AC lies within $\measuredangle EAB$. Then, according to Theorem 18 of § 1 and Theorem 1 of § 4, there exists, in $\measuredangle E'A'B'$, a ray $A'C'$, such that $\measuredangle BAC \equiv \measuredangle B'A'C'$. Proceed in a similar manner if AC is within $\measuredangle B_1AE$ (where B_1AB).

* Cf. HILBERT's, IV, 4, loc. cit., page 14.

Corollary. If ABC and $A'B'C'$ are two coplanar angles then either $\measuredangle ABC < \measuredangle A'B'C'$, $\measuredangle ABC \equiv \measuredangle A'B'C'$ or $\measuredangle ABC > \measuredangle A'B'C'$.*

Theorem 3. If ABC is a triangle and B is between A and D, then $\measuredangle DBC > \measuredangle CAB$.

Proof. By hypothesis and Theorem 3 of § 3, neither $\measuredangle DBC < \measuredangle CAB$ nor $\measuredangle DBC \equiv \measuredangle CAB$. Hence, by corollary to Theorem 2 of § 4, $\measuredangle DBC > \measuredangle CAB$.

Theorem 4. If ABC is a triangle and $BC > AC$, then $\measuredangle BAC > \measuredangle ABC$ and, conversely, if $\measuredangle BAC > \measuredangle ABC$, then $BC > AC$.

Proof. I. Suppose $BC > AC$. Then, by Definition 1, there exists between B and C a point D such that $AC \equiv CD$. Since D is between B and C, it easily follows, from O and Definition 6, that $\measuredangle BAC > \measuredangle DAC$. But, by construction and Theorem 21 of § 1, $\measuredangle DAC \equiv \measuredangle ADC$. Hence, by Theorems 12 and 19 of § 1, $\measuredangle BAC > \measuredangle CDA$. But, by Theorem 3 of § 4, $\measuredangle CDA > \measuredangle ABC$. Hence by Theorem 20 of § 1, $\measuredangle BAC > \measuredangle ABC$.

2. Conversely, suppose $\measuredangle BAC > \measuredangle ABC$. By Theorem 8 of § 1 either $BC < AC$, $BC \equiv AC$ or $BC > AC$. If $BC < AC$ or $BC \equiv AC$ then, by Theorem 21 of § 1 and first part of present theorem, either $\measuredangle BAC < \measuredangle ABC$ or $\measuredangle BAC \equiv \measuredangle ABC$; and each of these, by Theorem 2 of § 3, is contrary to hypothesis. Hence $BC > AC$.

Theorem 5. If $\measuredangle ABC$ is a right angle, or $\measuredangle ABC >$ a right angle, of the plane ABC, then $AC > AB$.

Proof. From hypotheses, Definitions 6 and 7, Theorems 19, 20 and 12 of § 1, Theorems 3, 2 and 10 of § 3 and Theorem 1 of § 4, it follows that neither $\measuredangle ACB \equiv \measuredangle ABC$ nor $\measuredangle ACB > \measuredangle ABC$. Hence, by the corollary to Theorem 2 of § 4, $\measuredangle ACB < \measuredangle ABC$. Hence, by Theorem 4 of § 4 and Definition 6, $AC > AB$.

Theorem 6. If ABC is a triangle, then $AB + AC > BC$.

Proof. By C_{1_a}, there exists a point D such that CAD and $AB \equiv AD$. By Theorem 21 of § 1 $\measuredangle BDC \equiv \measuredangle DBA$. Hence, by O and Definition 6, $\measuredangle DBC > \measuredangle CDB$. Hence, by Theorem 4 of § 4, $DC > BC$. But, according to Definition 2, $DC = AB + BC$. Hence $AB + AC > BC$.

Corollary. If ABC is a triangle and $AB > AC$, then $AB - AC < BC$.

§ 5. *Euclidean and Bolyai-Lobachevskian geometry.*

It is desired to prove that from O, C, K and P_0 follows Euclidean geometry and from O, C, K and the denial of P_0 follows Bolyai-Lobachevskian geometry.

Definition 8. Two straight lines are said to be parallel to each other if they lie in the same plane and have no point in common.

Theorems 1–5 are based on the assumptions O, C, K.

* Cf. Theorem 2 of § 3.

Theorem 1. Every segment has a middle point.

Proof. Suppose AB is a segment. Certainly there exists a point X, of AB, such that $AX < XB$ and a point Y, of AB, such that $AY > YB$. For suppose P is any point of AB. By Theorem 8 of § 1, either $AB < PB$, $AP \equiv PB$ or $AP > PB$. If $AP > PB$, then, by Theorems 6 and 3 of § 1, there is a point P' of AB such that $AP \equiv BP'$ and $PB \equiv P'A$ and, by Theorem 7 of § 1 and Definition 1, it follows that $AP' < P'B$. Similarly, if $AP < PB$, there exists a point P' such that $AP' > PB$. Finally, if $AP \equiv PB$, any point D between A and P would be such that $AD < DP$, and there would exist a point D' such that $AD' > D'P$. It is true then that there is at least one point X of AB such that $AX < XB$ and at least one point Y of AB such that $AY > YB$. By Theorem 8 of § 1 Y could not be identical with X. The points of AB may then be divided into two classes, the class, $[X]$, of all points such that $AX < XB$ and the class, $[Y]$, of all other points of AB. By Definition 1, O, and Theorems 9 and 8 of § 1 it is clear that no X is between two Y's and no Y is between two X's. Hence, by K, there exists a point M such that XMY for every X and Y which are different from M. Suppose it were true that $AM < MB$. Then, by Theorems 6 and 3 of § 1, there would exist a point N of AB such that $AM \equiv BN$ and $MB \equiv NA$. Clearly M would be between A and N. Hence, by a previous argument, it would follow that there exists a point L, between M and N, such that $ML < LN$. But, from this together with the fact that $AM \equiv BN$, it would follow, from Definition 1 and Theorems 10 and 7 of § 1, that $AL < LB$. But this would be impossible. For then L would be an X and we would thus have an X such that AMX. Similarly it would be impossible that $AM > MB$. Hence $AM \equiv MB$.

This theorem having been established, it is clear that every proposition, given in the preceding part of this paper as a consequence of O, C and M, is also a consequence of O, C and K. Such propositions will therefore be freely referred to in this section. It is to be remembered that A_1 and A_{-2} are propositions of this kind.

Definition 9. If O is a point of a plane β and s is a segment, then the set of all points $[P]$ of the plane β, such that $OP \equiv s$ is called a circle.

Theorem 2. Every circle is a Jordan curve.

Proof. In his article "Theory of plane curves in non-metrical analysis situs,"* VEBLEN has given three sets of conditions as sufficient in order that a simple closed set of points, existing in a space in which K and the plane axioms of the set O hold true, should be a Jordan curve. These conditions are called the conditions of linear order, ordinal continuity, and geometrical continuity respectively.

* These Transactions, vol. 6 (1905), pp. 83–98.

If O is the center of a circle α, then every ray starting from O contains one and only one point of α.

The points A, B, C, D of α are said to be in the order $ABCD$ if the rays OA, OB, OC, OD are in the order $OA-OB-OC-OD$. From O, K and this correspondence, it follows that the points of α satisfy, with reference to this order, the conditions of linear order and ordinal continuity.

The condition of geometrical continuity will evidently be satisfied if the proposition can be established that, given any point P of α and any triangle Δ_P which contains P, then there exist in that triangle two points, P_1 and P_2 of α, such that every point on the arc P_1PP_2 will lie within Δ_P. To establish this proposition, proceed as follows: Given that Δ_P contains the point P of the circle α, consider a segment s which is shorter than each of the perpendiculars from O to the sides of Δ_P. These perpendiculars and this segment exist by Theorem 10 of § 3, Theorems 7, 8 and 9 of § 1, Definition 1 and O. By hypothesis, O, Theorem 5 of §4 and Definition 1 and Theorem 9 of § 1, it follows that every point P', such that $PP' < s$, must be within Δ_P. By Theorem 10 of § 3, there exists, in the plane of α, a straight line a perpendicular to OP at the point P. On a there exist, by Theorem 1 of § 1, two points E and F, one on each side of P, such that $s \equiv PE$ and $s \equiv PF$. Each of the rays OE and OF contains a point of α. Call these points P_1 and P_2 respectively. Let X be any point of α on the arc P_1PP_2. Then either X is P or it is in $\measuredangle POE$ or $\measuredangle POF$. Suppose X is in $\measuredangle POE$. Then the ray OX contains a point M such that PME. By Theorem 5 of § 4, $OM > OP$. Hence $OM > OX$ and therefore OXM. But from the fact that $OP \equiv OX$ and Theorem 21 of § 1 it follows that $\measuredangle OXP \equiv \measuredangle OPX$, and thus (see Theorem 3 of § 3) $\measuredangle OXP <$ a right angle (of the plane α). Hence clearly, $\measuredangle PXM >$ a right angle (of the plane α) and therefore, by Theorem 5 of § 4, it follows that $PX < PM$. But $PM < PE$ and $PE \equiv s$. Hence $PX < s$ and therefore, by what has already been established, X is within Δ_P. This same conclusion would, of course, have been reached had X been in $\measuredangle FOP$. Hence, if X is on the arc P_1PP_2, it lies within the triangle Δ_P. It is true then that every circle is a Jordan curve.

Notation. $\mathfrak{A}_r$ means a circle whose center is A and whose radius is congruent to r.

Theorem 3. Any point P in the plane of the circle $\mathfrak{O}_r$ is within, on, or without $\mathfrak{O}_r$, according as $OP < r$, $OP \equiv r$, or $OP > r$.

Proof. If P is any point in the plane of $\mathfrak{O}_r$ such that $OP > r$, then, with the help of Theorem 5 of § 4, it may be seen that the straight line which lies in the plane of $\mathfrak{O}_r$ and is perpendicular to OP at the point P has no point in common with $\mathfrak{O}_r$. Hence, by the theory of Jordan curves, it follows that P is without $\mathfrak{O}_r$. With the help of O, Definition 7 and Theorem 5 of § 4, it may be seen that all points P such that $OP < r$ lie in one of the regions into which $\mathfrak{O}_r$ divides its plane. But it has already been established that all points such that $OP > r$ lie in the outside region, and moreover the two regions, into which $\mathfrak{O}_r$ divides all those points of its plane which do not lie on it, are non-vacuous and, finally, for every one of these points, either $OP < r$ or $OP > r$. Hence all points P such that $OP < r$ must lie within $\mathfrak{O}_r$.

Theorem 4. If A and A' are two points and r and r' are two segments such that $r + r' > AA'$ but either $r = r'$ or $r - r' < AA'$, then, if $\mathfrak{A}_r$ and $\mathfrak{A}'_r$ are coplanar, they have two points in common, one on each side of AA'.

Proof. $\mathfrak{A}_r$ has two points, D and E, in common with the straight line AA'. It follows, by hypothesis, Theorems of § 1, Definition 9 and Theorem 2, that one of these points is within and the other is without $\mathfrak{A}'_{r'}$. But the points D and E are joined by two distinct segments of the Jordan curve $\mathfrak{A}_r$, one of these segments lying entirely on one side and the other one entirely on the opposite side of DE. Hence, since $\mathfrak{A}'_{r'}$ also is a Jordan curve, $\mathfrak{A}_r$ and $\mathfrak{A}'_{r'}$ intersect in two points, one on each side of DE.

Theorem 5.* If ABC is any angle and $A'B'$ is any ray, then in any plane containing $A'B'$ there exists, on each side of the straight line $A'B'$, a point C' such that $\measuredangle BAC \equiv \measuredangle B'A'C'$.

Proof. On ray $A'B'$ there exists, by Theorem 1 of § 1, a point B'' such that $AB \equiv A'B''$. By hypothesis, Theorem 6 of § 4 and its corollary, and Theorem 4 of § 5, it follows that, on each side of $A'B'$, there is a point C'' such that $AC \equiv A'C''$ and $BC \equiv B'C''$. But $AB \equiv A'B''$. Hence, by Definition 3, $\measuredangle BAC \equiv \measuredangle B'A'C'$.

If P_0 is assumed in addition to O, C, and K, then one has the following two theorems (6 and 7).

Theorem 6.† If b is any straight line of any plane β then there is some point B, in the plane β but not on b, such that through B there is not more than one parallel to b.

Proof. According to P_0, there exists a straight line b' and a point B' such that through B' there is not more than one parallel to b'. On b' there exist two points C' and D'. If b is any straight line in any plane β then on b there exist, by O and C_{1_a}, two points C and D such that $C'D' \equiv CD$. Furthermore, by Theorem 5 of § 5, there exists in the plane β a point B'' such that $\measuredangle D'C'B' \equiv \measuredangle DCB''$. By Theorem 1 of § 1, there exists on ray CB'' a point B such that $C'B' \equiv CB$. Suppose that through B there are two parallels to b. Then one of these has a point E and the other a point F on that side of BC on which D lies. By Theorem 5 of § 5, there exist in the plane $D'C'B'$,

* See HILBERT's IV, 4.

† See VEBLEN's XII. It may here be remarked that VEBLEN's form of statement lacks the proviso that B should not lie on b.

and on that side of $B'C'$ on which D' lies, two points, E' and F', such that $\measuredangle CBE \equiv \measuredangle C'B'E'$ and $\measuredangle CBF \equiv \measuredangle C'B'F'$. By Theorem 13 of § 1, and A_{-2}, the straight line $B'F'$ must be distinct from the straight line $B'E'$. But neither of these straight lines can have a point in common with $C'D'$. For suppose the ray $B'F'$ and the straight line $C'D'$ had a point G' in common. G' would evidently lie on the ray $C'D'$. By Theorem 1 of § 1, there would exist, on the ray CD, a point G such that $C'G' \equiv CG$. Then one would have $C'B' \equiv CB$, $\measuredangle G'C'B' \equiv \measuredangle GCB$ and $C'G' \equiv CG$. Hence, by Theorem 17 of § 1 and Definition 5, it would be true that $\measuredangle C'B'G' \equiv \measuredangle CBG$. But, by construction, $\measuredangle CBF \equiv \measuredangle C'B'G'$. Hence, by Theorem 13 of § 1 and A_{-2} (which by Theorem 1 of § 2 is a consequence of O, C, and M) the ray BF would be the same as the ray BG. Hence ray BF would have a point in common with ray CG, contrary to hypothesis. In a similar manner, with the help of Theorem 15 of § 1, it could be shown that the other ray of the straight line $B'F'$ could not have a point in common with $C'G'$. Hence the straight lines $B'F'$ and $C'D'$ have no point in common. Similarly, $B'E'$ and $C'D$ have no point in common. But then there exist, through B', two parallels to b', and this is contrary to hypothesis. Thus the assumption that through the point B there are two parallels to b would lead to an absurdity. Hence through B there is not more than one parallel to b.

Theorem 7. If a is any straight line and A any point not on it, then through A there is one and only one straight line parallel to a.

Proof. See Theorem 43 of VEBLEN's "*A System of Axioms for Geometry.*"

In view of Theorems 1, 4 and 6 of the present section, C_2, Theorems 1, 3, 5, 12, 14, 17 of § 1 and Theorem 1 of § 2, it is clear that from O, C, K, P_0 follows a geometry in which HILBERT's axioms of groups I–IV hold true. One may then proceed, as is, for example, indicated in HILBERT's *Festschrift* and HALSTED's *Rational Geometry*, to derive a theory of proportion, etc., and then, with the use of K, one may finally develop an analytic geometry exhibiting a one to one correspondence between the points of our geometry and the number triples of the real continuous number system, this correspondence being such as to preserve all relations of congruence and order. Thus would be established the following theorem :

Theorem 8. From O, C, K and P_0, follows Euclidean geometry.

In case the contradictory of P_0 were assumed in addition to O, C and K, then in place of Theorem 7 one would have Theorem 39 of VEBLEN's *A System of Axioms for Geometry* and again a correspondence could be established between points and number triples, this correspondence also being such as to preserve all relations of order and congruence.* The following theorem would then be established :

* In this case congruence and order for this system of number triple would of course be introduced by definitions different from those used in the case of the correspondence with Euclidean geometry.

Theorem 9. From O, C, K and the denial of P_0 follows Bolyai-Lobachevskian geometry.

It is thus seen that if O, C and K are true of a space, then that space must be either Euclidean or Bolyai-Lobachevskian.

§ 6. *Semi-quadratic geometry.*

In this section will be given several sets of assumptions, any one of which sets is sufficient for the establishment of semi-quadratic or plane semi-quadratic geometry. By semi-quadratic geometry is meant the set of propositions which follow from HILBERT's I–IV. In the case of such a geometry rigid motion is possible, there is a coördinate system and a theory of proportion; and, speaking in terms of this theory of proportion, if $a_1, a_2, a_3, \cdots, a_n$ are any finite number of segments and $F(a_1, a_2, a_3, \cdots, a_n)$ is any rational function of these segments, then there exists a segment equal to $F(a_1, a_2, a_3, \cdots, a_n)$ and also a segment equal to $\sqrt{a^2 + b^2}$.*

Use will be made of the following additional notations for propositions:

R_A (equality of right angles). *If ABD, $A'B'D'$, $\measuredangle ABC \doteq \measuredangle CBD$ and $\measuredangle A'B'C' \equiv \measuredangle C'B'D'$, then $\measuredangle ABC \equiv \measuredangle A'B'C'$.*

By R_A^{pl} is meant the proposition R_A with the proviso that the angles ABC and $A'B'C'$ are coplanar.† This superscript, "*pl*," may sometimes be used in notations for other propositions in order that their application may be similarly restricted to the case of coplanar points.

D_{sh} (shortest distance proposition).‡ *If A, B and C are non-collinear, ACB', $CB \equiv CB'$, and B'' is a point on ray AC such that $AB \equiv AB''$, then $AB''B'$.*

P_{er} (perpendicular to a straight line).§ *If ABC, then in any plane containing these points there exists a point D such that $\measuredangle ABD \equiv \measuredangle DBC$.*

A_s.‖ *If A, B and C are three non-collinear points and A', B' and D' are three non-collinear points and $BA \equiv B'A'$, then in the half-plane $D'A'B'$ there exists one and only one point C' such that $AC \equiv A'C'$ and $BC \equiv B'C'$.*

A_G¶. *If $\measuredangle ABC$ and $\measuredangle A'B'C'$ are two angles and $BA \equiv B'A'$, then either there is a point C'' on the ray BC such that $A'C' \equiv AC''$ and $B'C' \equiv BC''$, or there is a point C''' within $\measuredangle ABC$ such that $A'C' \equiv AC'''$ and $B'C' \equiv BC'''$, or there is a point C^{iv} within $\measuredangle A'B'C'$ such that $AC \equiv A'C^{iv}$ and $BC \equiv B'C^{iv}$; but for these given angles no two of these statements can both be true.*

P_2. *In every plane α there is a straight line a such that if A is any point of*

* See HILBERT's *Grundlagen der Geometrie.*

† See Theorem 1 of § 4.

‡ See Theorem 6 of § 4.

§ For a stronger proposition see Theorem 10 of § 3.

‖ Cf. Theorem 2 of § 4 and A_{-2} (see § 2). Cf. also HILBERT's IV, 4.

¶ If $\measuredangle ABC$ and $\measuredangle A'B'C'$ are two angles then either $\measuredangle A'B'C' \equiv \measuredangle ABC$ or $\measuredangle A'B'C' > \measuredangle ABC$ or $\measuredangle A'B'C' < \measuredangle ABC$ but no two of these cases can occur simultaneously. Cf. Definition 6, also Theorem 2 of § 3 and corollary to Theorem 2 of § 4.

a not on a, then through A and lying in α there is not more than one straight line which has no point in common with a.

Theorem 1. From O^{pl}, C, M, and P_2 follows plane semi-quadratic geometry.

Proof. See Theorems 1, 3, 5, 12, 14, 17 of § 1 and Theorems 1 and 2 of § 2 and Theorem 2 of § 4. The proposition that through each point there is one and only one parallel to any given straight line may be proved with the help of P_2 by a method which is suggested in proof of Theorem 6 of § 5.

Theorem 2. From O^{pl}, C, A_{-2}, A_1, P_2, follows plane semi-quadratic geometry.

Proof. In the proof above indicated for Theorem 1, M is used only to demonstrate A_{-2} and A_1.

Lemma 1. From O^{pl}, C and D_{sh} follows A_{-2}.

Proof. Suppose O, C and D_{sh} are true and A_{-2} is not. Then there exist three non-collinear points A, B, C, and a point C', all in the half-plane $C{-}AB$, such that $AC \equiv AC'$ and $BC \equiv BC'$. According to O, there exist points A', B' such that ACA', BCB'. There are five cases.

I. If C' is in $\measuredangle ACB'$, then segments AC and BC' have a point O in common. According to D_{sh}, $AO + OC' > AC'$ and $OC + BO > BC$. Thus $(AO + OC) + (OC' + BO) > BC + AC'$. Hence $AC + BC' > BC + AC'$; this is impossible in view of the hypothesis that $AC \equiv AC'$ and $BC \equiv BC'$.

II. If C' is in $\measuredangle BCA'$, argue in a similar manner.

III. If C' is in $\measuredangle ACB$, then there exists a points D such that $AC'D$, BDC. Now $AC + CD > AD$. Hence $AC + CD + DB > AD + DB$, $AC + CB > AC' + C'D + DB$. But $C'D + DB > C'B$. Therefore $AC + CB > AC' + C'B$, and this is impossible in view of hypotheses.

IV. If C' is in $\measuredangle A'CB'$, proceed as in case III.

V. If C' is on the straight line AC or the straight line BC, proceed with C or C' as in case III with D.

Thus in any case it would be impossible that O, C and D_{sh} should be true and A_{-2} false simultaneously.

Lemma 2. From O, C and P_{er} it follows that through any point there is at least one perpendicular to any given straight line.

Proof. See the proof of Theorem 10 of § 3.

Lemma 3. A_{-2} is a consequence of O, C and P_{er}.

Proof. Suppose C, A and B are three non-collinear points and C' is a point in the half-plane $C - AB$ such that $CA \equiv C'A$ and $CB \equiv C'B$. By Lemma 2, there is, on the straight line AB, a point D such that CD is perpendicular to AB. In case D coincides with A, then $\measuredangle BAC$ is a right angle and therefore, by hypothesis and corollary to Definition 7 of § 3, $\measuredangle BAC$ is a right angle and hence, by P_{er} and C_{1_b}, C' is C. If D does not coincide with A, then, by Theorem 17 of § 1, $\measuredangle ADC \equiv \measuredangle ADC$ and $DC \equiv DC'$. But

$\measuredangle ADC$ is a right angle. Hence, by the corollary to Definition 7 of § 3, $\measuredangle ADC'$ is a right angle, and therefore, by P_{er}, ray DC' coincides with ray DC. But $DC \equiv DC'$. Hence, by C_{1_b}, C' is C.

Theorem 3. From O^{pl}, C, A_1, D_{sh} and P_2 follows plane semi-quadratic geometry.

Proof. See Lemma 1 and Theorem 2.

Theorem 4. From O, C, A_s and P_2 follows semi-quadratic geometry.

Proof. Compare Theorems 1, 3, 5, 12, 14, 17 of § 1 and Definition 3 and A_s with HILBERT's group IV of axioms. Use argument, concerning parallel proposition, suggested in proof of Theorem 1 of § 6.

Theorem 5. From O, C, A_{-2}, R_A and P_2 follows semi-quadratic geometry.

Proof. It may be seen, from the proof of Theorem 2 of § 4, that A_s is a consequence of O, C, A_{-2} and R_A. But, by Theorem 4 of § 6, from O, C, A_s and P_2 follows semi-quadratic geometry.

Theorem 6. From O, C, P_{er}, R_A, P_2 follows semi-quadratic geometry.

Proof. See Lemma 3 and Theorem 5.

Theorem 7. From O, C, D_{sh}, R_A and P_2 follows semi-quadratic geometry.

Proof. See Lemma 1 and Theorem 5.

Theorem 8. From O, C, A_G and P_2 follows semi-quadratic geometry.

Proof. According to Theorem 4, this theorem will be established if it is shown that A_s is a consequence of O, C and A_G.

Suppose A, B and C are three non-collinear points and D', B' and E' are three non-collinear points. On the ray $B'D'$ there is a point A' such that $BA \equiv B'A'$. To prove that in the half-plane $E' - A'B'$ there is one and only one point C' such that* $BC \equiv BC'$ and $AC \equiv A'C'$, argue as follows. According to A_G either (I), there is a point C', on $B'E'$ or in $\measuredangle A'B'E'$, such that $BC \equiv B'C'$ and $AC \equiv A'C'$, or (II), there is a point C'' within $\measuredangle ABC$, such that $\measuredangle ABC'' \equiv \measuredangle A'B'E'$. In this last case there is, on the ray BC'', a point F such that AFC. By Theorem 11 of § 1 there exists on ray $B'E'$ a point F' such that $AF \equiv A'F'$ and $BF \equiv B'F'$. By C_{1_a} there is a point C' such that $A'F'C'$ and $FC \equiv F'C'$. By C_3 and Theorem 11 of § 1, $AC \equiv A'C'$ and $BC \equiv B'C'$.

Suppose there were, in the half-plane $E' - A'B'$, two points C' and C'' such that $AC \equiv A'C'$ and $BC \equiv B'C''$. Then, by C_{1_a} and O, either C'' is within $\measuredangle A'B'C'$ or C' is within $\measuredangle A'B'C''$. Suppose, for instance, C'' is within $\measuredangle A'B'C'$. Then, by hypothesis, Definition 3 and Theorems 11 and 18 of § 1, there exists within $\measuredangle ABC$ a point C''' such that $B'C'' \equiv BC'''$ and $A'C'' \equiv AC'''$. But, by hypothesis, C_2 and Theorem 4 of § 1, $A'C' \equiv A'C'$ and $B'C' \equiv B'C''$. Hence, by C_2, $A'C' \equiv AC'''$ and $B'C' \equiv BC'''$. But C'' is within $\measuredangle A'B'C'$ and $AC \equiv A'C''$ and $BC \equiv B'C''$. Thus the

* Cf. A_s, page 503.

supposition that there are two points C' and C'', in the half plane $E'A'B'$ such that $AC \equiv A'C'$, $BC \equiv B'C'$, $AC \equiv A'C''$ and $BC \equiv B'C''$, would lead to a contradiction with A_G.

Theorem 7 is therefore established.

§ 7. *Geometry of the rule and compasses.*

In this paragraph will be considered several sets of assumptions, from any one of which sets follows a geometry in which all ordinary rule and compasses constructions are possible.

Two propositions concerning intersections of straight lines with circles, or of circles with each other, will be considered and will be referred to by means of the following notations:

I_1.* *If A, B and C are non-collinear points and P is between A and B, then there is, on the straight line DC, a point B' such that $AB \equiv AB'$.*

I_2.† *If APD, PBP', $AD'P'$, $PB \equiv BP'$, $AD \equiv AD'$, and all of these points lie in a plane, and this plane is decomposed by the straight line PB into two regions,‡ then, in each of these regions, there is a point C such that $AC \equiv AD$ and $BC \equiv BP$.*

Lemma 1. M is a consequence of O, C, and I_2.

Proof. Suppose A and B are two different points. From O and C it easily follows that there is a point P, between A and B, such that $AP > PB$. Then there is, by Theorems 3 and 6 of § 1, a point P' such that $AP'B$, $AP' \equiv PB$, and $P'B \equiv AP$. Evidently $AP'P$. Hence, by I_2, there exist two points D and D', on opposite sides of AB, such that

$$BD \equiv AD \equiv BD' \equiv AD' \equiv AP.$$

Segment DD' meets the straight line AB in a point M. According to Definitions 3, 5 and Theorems 3 and 17 of § 1, $MA \equiv MB$. Thus every segment has a middle point.

Lemma 2. I_2 is a consequence of O, C, M, P_2 and I_1.

Proof. According to Theorem 1 of § 6, plane semi-quadratic geometry is a consequence of O^{pl}, C, M, P_2. So a theory of proportion and an analytic geometry may be introduced. Now suppose one extremity of a diameter of $\mathfrak{C}'_{r'}$ is within $\mathfrak{C}_r$ and its other extremity is without $\mathfrak{C}_r$. Take CC' as x-axis and a perpendicular to CC' at the point C as y-axis. Equations of $\mathfrak{C}_r$ and $\mathfrak{C}'_{r'}$, referred to these axes, are $x^2 + y^2 = r^2$ and $(x - a)^2 + y^2 = r'^2$ respectively, where a is the abscissa of C'. It is evident, in view of I_1, that for every value,

* If a straight line lies in the plane of a circle and has a point within that circle then it intersects the circle.

† If a semicircle has one extremity, P, within and the other, P', without a circle with which it is coplanar, then it has a point in common with that circle.

‡ Cf. VEBLEN, loc. cit., pages 363 and 364.

x', of x between $-r$ and $+r$, there is a value, y', of y such that (x', y') satisfies $x^2 + y^2 = r^2$. With help of the fact that $r + r' > a$, $a + r > r'$ and $a + r' > r$, it may be seen that $\frac{1}{2}a^{-1}(r^2 + a^2 - r'^2)$ is between $-r$ and $+r$. Take for x' this value. Then

$$y' = \pm\sqrt{r'^2 - x'^2}$$

and, no matter which sign we take, these values of x' and y' evidently satisfy $(x-a)^2 + y^2 = r'^2$ as well as $x^2 + y^2 = r^2$. Thus $\mathfrak{C}_r$ and $\mathfrak{C}'_{r'}$ intersect in two points, one on each side of CC'.

Lemma 3. I_1 is a consequence of O, C and I_2.

Proof. From O, C and I_2, M follows, according to Lemma 1. Now suppose the straight line p lies in the plane of the circle $\mathfrak{C}_r$ and passes through a point P which is within that circle. If P is the same as C, then, by Theorem 1 of § 1, p evidently has a point in common with $\mathfrak{C}_r$. If P is different from C, it follows from O, C and M (see Theorem 10 of § 3), that there exists on p a point D such that CD is perpendicular to p. By C_{1_a} there is a point C' such that DCC' and $CD \equiv DC'$. Then, evidently, according to hypothesis, Theorem 5 of § 4, Theorems 1 and 2 of § 2, and I_2, $\mathfrak{C}_r$ and $\mathfrak{C}'_r$ have a point F in common. Hence, according to Definitions 3 and 7, FD is perpendicular to CD. Hence, by Theorem 10 of § 3, F lies on the straight line p. So p and $\mathfrak{C}_r$ have a point F in common.

Theorem 1. From O, C, I_2, and P_2 follows a "Geometry of the Rule and Compasses."

Proof. With the help of Theorem 1 of § 6 it may easily be seen that a geometry of the rule and compasses for each plane follows from O, C, M, P_2 and I_2. Our present theorem will then be established if it is proved, (I), that M follows from O, C, P_2 and I_2, and, (II), that a geometry of the rule and compasses follows from I_3, P_2, O, and geometry of the rule and compasses for each plane. Now M follows from O, C, P_2 and I_2, according to Lemma 1; and (II) is evident in view of the fact that Theorem 5 of § 5 is proved as a logical consequence of I_2 and plane semi-quadratic geometry for each plane.

§ 8. *Independence of each assumption in the set composed of K, P_0, C and O.*

The following "independence examples" are constructed to show that each assumption of the set composed of K, P_0, C, and O is independent of the remaining ones.

Example for order Axiom I. Use VEBLEN's K_{I} (loc. cit., page 353).

Example for order Axiom III. Consider four points A, B, C, D in the orders ABC, ACB, ADB, BAD, BCA, BDC, CAD, CBD, CDA, DAC, DBA, DCB. Consider that every segment $\equiv$ every segment.

Example for order Axiom IV. Use VEBLEN's K_{IV}. Consider that every segment $\equiv$ every segment.

Example for order Axiom VI. Points are all integers (including 0). $0K4$ and $4K0$ for every point K except -1, 0, 4 and 5. -1, 0 and 4 are in the orders $-1\ 0\ 4$ and $4\ 0\ -1$. 0, 4 and 5 are in the orders 0 4 5 and 5 4 0. If, in the ordinary sense, $0 < A < 4$, $C > 4$ and $0 < B < 5$, then $\{{ABC \atop CBA}\}$ if and only if $B - A = C - B - 1$. If, in the ordinary sense, $0 < A < 4$, $C < 0$ and $-1 < B < 4$, then $\{{ABC \atop CBA}\}$ if and only if $A - B = B - C - 1$. If D and E are two integers and there is no integer C such that DEC according to these definitions given above, then $\{{DEC \atop CED}\}$ if and only if $E - C = D - E$. Every segment $\equiv$ every segment.

According to this plan one has the following orders:

8 4 1	1 3 4	−4 0 3	3 0 −4
7 4 2	2 4 7	−3 0 2	2 0 −3
6 4 3	3 4 6	−2 0 1	1 0 −2
6 3 1	1 3 6	−2 1 3	3 1 −2
5 4 0	0 4 5	−1 0 4	4 0 −1
5 3 2	2 3 5	−1 1 2	2 1 −1

$0K4$ and $4K0$ for every integer $K \neq -1, 0, 4$ or 5; and in case A and B are two integers which are not respectively the first and second element of any one of these triads, then ABC if and only if $A - B = B - C$.

Example for order Axiom VII. VEBLEN's K_{VII}. Every segment $\equiv$ every segment.

Example for order Axiom VIII. VEBLEN's K_{VIII}. Every segment $\equiv$ every segment.

Example for order Axiom IX. VEBLEN's K_{IX}. Congruence ordinary.

Example for order Axiom X. VEBLEN's K_{10}. Congruence ordinary.

Example for order Axiom XI or K. Some non-Archimedean geometry, for example that of HILBERT (cf. TOWNSEND's translation of HILBERT's *Grundlagen der Geometrie,* page 34).

Example for P_0. Consider any of the ordinary proofs of the compatibility of Bolyai-Lobachevskian geometry.

Example for C_{1_a}. Define points as the points of ordinary Euclidean space that lie on one side only of a given plane, consider these points to be ordered just as in the ordinary sense and two segments to be congruent to each other if and only if they are congruent in the ordinary sense. Then K and P_0 are satisfied and so are all the assumptions of C except C_{1_a}.

Example for C_{1_b}. Consider points to be all the points of ordinary Euclidean space ordered in the ordinary manner, but consider every segment as being congruent to itself and every other segment. Evidently C_{1_b} is the only assumption of K, P_0 and C that is not satisfied in this example.

Example for C_2. Consider points to be all the points of ordinary Euclidean space ordered in the usual manner, but consider that every segment is congruent to every segment which in the ordinary sense is just twice as long.

Example for C_3. Consider points to be the points of ordinary Euclidean space ordered in the usual manner, but consider that there is one segment AB such that $CD \equiv C'D'$ if and only if $C'D'$ is congruent to AB in the ordinary sense.

Example for C_4. Consider points to be the points of ordinary Euclidean space ordered in the usual manner. Consider a certain fixed plane p and regard two segments neither of which is perpendicular to that plane as being congruent to each other if and only if their orthogonal projections upon that plane are congruent to each other in the ordinary sense; in case only one of them is perpendicular to p, regard them as congruent if and only if this one is congruent in the ordinary sense to the projection of the other one upon p; finally, if they are both perpendicular to p, regard them as congruent if and only if they are congruent in the ordinary sense.

As was suggested by Professor E. H. MOORE, the independence example here given for C_2 does not prove C_2 independent of K, C and the negative of P_0. But all of the other independence examples here given (except, of course, that for P_0) do apply if the negative of P_0 is substituted for P_0 and Bolyai-Lobachevskian space is used instead of Euclidean. Of course the negative of P_0 is shown to be independent of O, C and K if point, order and congruence are taken as those of ordinary Euclidean space.

§ 9. *Independence of each postulate in the set composed of* I_3, P_2, C *and* O.

Except for C_{1_a} and I_2 use the same examples as in § 8.

To prove C_{1_a} independent, observe that all the assumptions P_0, C, I_2, except C_{1_a}, are satisfied, either "vacuously" or otherwise, if "points" are all the points of ordinary Euclidean space ordered as usual, no segment, however, being congruent to any segment. C_2, C_3, C_4 and I_2 are, in this case, "satisfied vacuously."

To prove I_2 independent, consider the space obtained by omitting all the points of Euclidean space except those whose coördinates are rationally expressible in terms of expressions of the form $\sqrt{r}$ where r is an integer.

§ 10. *Relation of parallel assumptions to introduction of congruence by definition.*

With use of O, K (or order "Axiom XI"), and VEBLEN's Axiom XII * (concerning parallels), congruence may † be introduced by definition, so that, if O, K and XII are true of a space, then there must exist between the segments of that space a relation satisfying, for example, all of my assumptions C. But

* See footnote † on page 501.

† VEBLEN, loc. cit., page 383.

this is not true if XII is replaced by the weaker postulate P_0, as may be seen with help of the independence example obtained by considering a certain fixed plane in ordinary Euclidean space and regarding as "points" only those points of this space which are on a certain side of this fixed plane, these points being ordered as usual. Here O, K (also XI) and P_0 are satisfied. But there cannot exist among the segments of this space a relation satisfying assumptions C. For in that case this space would be, according to § 5, ordinary Euclidean space and thus, in particular, through each point outside of any given line there would be only one parallel to that line, as is manifestly impossible.

§ 11. *Relation of continuity assumptions to introduction of congruence by definition.*

Theorem 1. It is not possible to prove that if the assumptions* O, P_s, C_{1_b}, C_2, C_3, C_4 (call this the set S), hold true of a space, then there exists a definition for the symbol "$\equiv$" such that the assumptions C and M will hold true with reference to the points and order in terms of which the assumptions of S are stated.

Proof. Select any system (OX, OY, OZ) of three rectangular axes in ordinary Euclidean space and consider the space composed of all points whose coördinates with reference to this system are all rational, congruence and order relations being as usual. Call this space U and let the terms "original order" and "original congruence" be understood here as meaning the order and congruence here indicated for this space. Clearly O, P_s and all of the assumptions of C except C_{1_a} hold true of U with reference to this order and congruence. Suppose a new meaning could be given to the congruence symbol "$\equiv$" such that the assumptions C and M would hold true for the space U with reference to the original order. Then, according to § 6, all the theorems of plane semi-quadratic geometry, in particular a theory of proportion, would hold true, with reference to this new congruence and the original order, for any planes in the space U. There would then exist on OX a point P such that, with reference to this theory of proportion, $OP \times OP = 2OI$ where I is some point of OX (e. g., such that $OI =$ original unit). OP would then clearly not be, with reference to this new congruence, rational in terms of OI. But with use of P_s it may be shown that any two segments which lie on OX, being rational in terms of each other with reference to the original congruence, would necessarily also be so with reference to this new congruence. To prove this let us first suppose that AB and CD are two segments of OX and $AB \equiv CD$ according to original congruence. By P_s there exists a parallelogram $ABEF$ in which AB and EF

* By P_s is meant the strong parallel assumption : If a is any straight line and A is any point not on a then, in the plane aA, there is one and only one straight line which passes through A and has no point in common with a.

are opposite sides. Manifestly $EFCD$ will be a parallelogram in which EF and CD are opposite sides. But it is clear that, in terms of our new as well as in terms of our old congruence, the opposite side of a parallelogram must be congruent to each other. Therefore, in terms of this new congruence, $AB \equiv EF$ and $EF \equiv CD$ and therefore $AB \equiv CD$. It is true then that if AB and CD are segments of OX and $AB \equiv CD$ according to old congruence, then also $AB \equiv CD$ according to our new congruence. Now suppose that the segments AB and CD, of the line OX, are given as rational in terms of one another (instead of simply congruent to one another) in terms of original congruence. Then there exists a segment KL, on OX, such that AB can be divided into m segments and CD can be divided into n segments, all congruent in the original sense, to KL, m and n being positive integers. But these $m+n$ segments of OX, being congruent to each other according to the original congruence, must also be congruent according to our new congruence, and thus AB and CD are rational in terms of one another according to new congruence. But it has been shown that, were such a new congruence possible, then there would exist on OX two segments, OP and OI, which would not be rational in terms of one another according to this congruence. A contradiction would thus be obtained and such a congruence would therefore be impossible.

§ 12. *A question concerning the separation of the assumptions for a geometry into two sets.*

Theorem 1. It would be impossible to formulate a set (I) of assumptions expressed in terms of point and order alone and a set (II) of assumptions expressed in terms of point and congruence alone, such that any geometry satisfying the assumptions I and II must necessarily be ordinary Euclidean geometry with respect to the undefined symbols in terms of which I and II are stated.

Proof. Consider a system of three rectangular axes OX, OY, OZ in an ordinary Euclidean space (E). Consider a paraboloid of revolution (G) whose axis is OZ and whose vertex is O. Let order be as usual but consider that segment $AB \equiv$ segment $A'B'$ if and only if

$$(x_A - x_B)^2 + (y_A - y_B)^2 + (q_A - q_B)^2 = (x_{A'} - x_{B'})^2 + (y_{A'} - y_{B'})^2 + (q_{A'} - q_{B'})^2,$$

the new coördinate, q_P, being defined as 0 if P is on G, and otherwise as $\pm$ the length of SP where S is the point in which G is cut by a parallel to OZ through the point P, and $+$ or $-$ is used according as P is on the same side of the surface G as is Z, or on the other side. Let C' designate this particular congruence. It may be seen that any statement concerning point and order alone that is true for Euclidean space must hold true for the space E with reference to the congruence C'. For if a straight line were defined as the locus of a point P

satisfying two linear equations of the first degree in x_P, y_P, q_P, and, in accordance with this, a new order O' were defined in the usual analytical manner, then one would have a Euclidean geometry concerning the points of the space E, the congruence C' and the order O'. But also, of course, any statement concerning points and order alone which is true of ordinary Euclidean space is true of our original order for the particular space E. But manifestly space E does not satisfy the theorems of ordinary Euclidean geometry with reference to ordinary order and the congruence C'.

§ 13. *Certain queries.*

I do not know as yet whether M* is a consequence of O † and C†. There are several other questions which could not be settled negatively without deciding this question. For example: Is D_{sh} ‡ a consequence of O, C, A_1 § and P_2? ‡ Is A_1 a consequence of O, C, D_{sh} and P_2? Is R_A ‡ a consequence of O, C, A_{-2} § and P_2? Is A_{-2} a consequence of O, C, R_A and P_2? Is A_1 a consequence of O, C, A_{-2} and P_2? Is A_{-2} a consequence of O, C, A_1 and P_2?

Another question is whether Theorem 1 of § 11 would be true if C_{1_a} were substituted for either C_{1_b}, C_2, C_3 or C_4.

In § 6 it was shown that from O, C, M and P_2 follows a geometry for every plane of which all the theorems of plane semi-quadratic geometry hold true. Is then the semi-quadratic geometry of a three-space a consequence of O, C, M and P_2? This could be answered in the affirmative if it could be shown that in every space for which O, C and M are true all right angles are congruent to each other. It can be proved ‖ that in such a space all *coplanar* right angles are congruent to each other.

* See last sentence of § 1.

† See the opening pages of the paper.

‡ See § 6.

§ See § 2.

‖ See Theorem 1 of § 4.