BIRATIONAL TRANSFORMATIONS OF CERTAIN QUARTIC SURFACES*

BY

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1. Statement of the problem. It has been shown by Enriques† that if an algebraic surface contains a pencil of elliptic curves and also two curves cutting each curve of the pencil in two groups of points which are not equivalent and which do not possess equivalent multiples, the surface is invariant under a discontinuous infinite series of birational transformations. In a more recent memoir‡ the same author states that, with the exception of the case in which

$$p_g = p_a = P_2 = p^{(1)} = 1,$$

the converse is also true, since transversal curves of each curve of the pencil can always be found. Since the publication of the second theorem the author has discovered that the conclusion must be modified.§ An interesting example of a quartic surface having an infinite discontinuous group of birational transformations but not containing any elliptic curves has been found by Fano.|| In the following paper it is proposed to study the possibility and the nature of birational transformations of quartic surfaces which pass through one or more given curves. The method of the basis of a system of curves on the surface will be employed, as developed by Severi.¶

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† F. Enriques, *Sulle superficie algebriche, che ammettono una serie discontinua di trasformazioni birazionali*, R e n d i c o n t i d e l l a R e a l e A c c a d e m i a d e i L i n c e i, ser. 5, vol. 15 (1906), pp. 665–669.

‡ F. Enriques, *Sulle superficie algebriche con un fascio di curve ellittiche*, R e n d i c o n t i d e l l a R e a l e A c c a d e m i a d e i L i n c e i, ser. 5, vol. 21 (1912), pp. 14–17.

§ A. Rosenblatt, *Sur certaines classes des surfaces algébriques irrégulières et sur les transformations birationnelles de ces surfaces en ellesmêmes*, B u l l e t i n d e l ' a c a d é m i e d e s s c i e n c e s d e C r a c o v i e, Juillet, 1912, pp. 761–810. ˙See footnote 1, page 802.

|| G. Fano, *Sopra alcune superficie del 4. ordine rappresentabili sul piano doppio*, R e n d i c o n t i d e l R e a l e I s t i t u t o L o m b a r d o, ser. 2, vol. 39 (1906), pp. 1071–1086.

¶ F. Severi, *Sulla totalità delle curve algebriche tracciate sopra una superficie algebrica*, M a t h e m a t i s c h e A n n a l e n, vol. 62 (1906), pp. 194–225.

1. Quartic Surfaces Containing a Straight Line

2. General case. If a quartic surface contains a straight line it contains a pencil of plane elliptic cubics. If it is otherwise unrestricted, the surface contains no other rational curves, and consequently no birational transformations into itself.

3. Rational curve on the surface. Let the quartic surface contain the straight line C_1 and a rational curve C_n of order n meeting C_1 in $n - 1$ points. Any plane through C_1 contains a plane cubic curve C_3 on F_4 that meets C_n in one point K. The involution defined by projecting C_3 upon itself from K leaves C_3 invariant. Associated with this involution is a Cremona transformation of the Jonquières type, having the polar conic of K with respect to C_3 for invariant curve. This Jonquières transformation will be understood when we speak of the involution. As K describes C_n the involution defines a Cremona transformation which leaves F_4 invariant and transforms every C_3 of the pencil into itself. When the equations of the surface and of C_1 and C_n are known, the equations of the transformation can be readily derived.

The system of curves C_1, $|C_3|$, and C_n constitute a basis, since we are concerned with the most general surface which satisfies the conditions.[*] The values of $[C_i, C_k]$ are

$$[C_1, C_1] = -2, \quad [C_3, C_3] = 0, \quad [C_n, C_n] = -2,$$

$$[C_1, C_3] = 3, \quad [C_1, C_n] = n - 1, \quad [C_3, C_n] = 1.$$

In terms of the above curves the transformation may be expressed symbolically as follows.

Let C_1, C_3, C_n go into $\overline{C}_1$, $\overline{C}_3$, $\overline{C}_n$ respectively. The equations connecting the old and the new curves are found to be

$$\overline{C}_1 = 2C_1 + 2(n+1)C_3 - 3C_n,$$

$T.$ $$\overline{C}_3 = C_3,$$

$$\overline{C}_n = C_1 + 2(n+1)C_3 - 2C_n,$$

by the following considerations. To obtain $\overline{C}_1$ we determine the image of C_1 as K describes C_n. Draw the lines from K to the three points of intersection of C_1 and the plane cubic curve defined by C_1 and K. As K describes C_n these joining lines generate a ruled surface R containing $\overline{C}_1$. On this ruled surface C_n is a threefold curve, and C_1 is an $(r - 3)$-fold straight line, r being the order of R. The intersection of R with F_4 is a curve of order $4r$, consisting of C_n counted three times, of C_1 counted $r - 2$ times, since the surfaces touch each other at every point of C_1, and of a residual curve, the

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[*] Severi, l. c.

image of C_1. The curve C_n meets C_1 in $n - 1$ points, corresponding to each one of which C_1 counts as a double generator of R, hence $r - 4 = 2n - 2$. The residual curve therefore has the symbol

$$2(n + 1)(C_3 + C_1) - 2nC_1 - 3C_n$$

and is of order $2(n + 1)4 - 2n - 3n$.

It is evident that each plane cubic curve C_3 goes into itself.

The image of C_n is the locus of the first tangential of K as K describes C_n. The tangents to $|C_3|$ at the points of C_n describe a ruled surface of order $2(n + 1)$ passing through C_n and touching F_4 at each point of C_n, and having C_1 as a $(2n + 1)$-fold line. The residual intersection is the required image of C_n.

From K four tangents can be drawn to C_3 besides the one at K. As K describes C_n the four tangents generate a ruled surface R_{2n+6} of order $2n + 6$, having C_n for a fourfold curve and C_1 for a multiple line of order $2n + 2$, as may be seen as follows: The tangent cone to F_4 from any point P on C_1 consists of the tangent plane at P counted twice and of a cone of order 10 on which C_1 is fourfold. This cone intersects the projecting cone of C_n from P in C_1 counted $4(n - 1)$ times and in $6n + 4$ other lines. These $6n + 4$ lines are composed of two groups; those of the first are the $2n + 1$ tangents to C_3 at points of C_n, each counted twice, and the remaining lines belong to the second group. The lines of the second group are generators of the ruled surface in question. The residual intersection of R_{2n+6} with F_4 is a curve C counted twice that remains invariant under the transformation T. It is defined symbolically by the equation

$$2C = (2n + 6)(C_1 + C_3) - 4C_n - (2n + 2)C_1,$$

or

$$C = 2C_1 + (n + 3)C_3 - 2C_n.$$

As K describes C_n its first tangential K' describes a rational curve. By projecting each C_3 from the corresponding K' upon itself, another Cremona transformation T' is defined. Since by T' both $|C_3|$ and C_n remain invariant, and since

$$[\bar{C}_1, \bar{C}_1] = -2, \qquad [\bar{C}_1, \bar{C}_3] = 3, \qquad [\bar{C}_1, \bar{C}_n] = n - 1,$$

we easily obtain the equations

$$T'. \qquad \bar{C}_1 = -C_1 + 2(n + 5)C_3 + 6C_n, \qquad \bar{C}_3 = C_3, \qquad \bar{C}_n = C_n.$$

The determinants of T, T' are each -1, and their product is 1; the operation (TT') is not periodic. Thus the surface is invariant under an infinite number of birational transformations. In any plane the operation (TT')

can be replaced by the product of two projections of the curve upon itself, the center of one of which may be chosen anywhere on the curve. But the operation does not define a continuous group, since the C_3 associated with K has not a constant modulus as K describes C_n.[*]

Moreover, it can be shown by the same method as that employed by Severi[†] that all the effective transformations that leave F_4 invariant can be expressed in terms of T and T'.

4. One double point on the line. Let all the curves C_3 pass through one fixed point P on C_1, the other points of intersection of C_1, C_3 being variable.

The surface is evidently invariant under the projection of the surface upon itself from the point P; the projection is an involutional Cremona transformation of the Jonquières type. The surface is also invariant under the transformation defined by the first tangential of P as in the preceding case. Surfaces of this type necessarily contain rational curves which cut every plane C_3 of the pencil in one point apart from the point P.

Symbolically, the transformation is expressed as follows. For basis we may choose C_1, C_3 and the point P with the following characteristic[‡]

$$[C_1, C_1] = -2, \quad [C_1, C_3] = 2, \quad [C_3, C_3] = 0,$$

$$[C_1, P] = 1, \quad [C_3, P] = 1, \quad [P, P] = -2.$$

The point P is a double point on F_4. The tangent cone to F_4 at P is a quadric passing through C_1; the residual intersection is a rational curve of order 7 which passes through P five times. We have therefore

$$[C_7, P] = 5, \quad [C_7, C_1] = 1, \quad [C_7, C_3] = 1, \quad [C_7, C_7] = -2,$$

so that the equations of the projection from P are easily found to be

$$\bar{C}_1 = C_1,$$
$$S. \qquad \bar{C}_3 = C_3,$$
$$\bar{P} = C_1 + 2C_3 - P.$$

The equations of the transformation defined by the first tangential are similarly found to be

$$\bar{C}_1 = -C_1 + 10C_3 + 4P,$$
$$S'. \qquad \bar{C}_3 = C_3,$$
$$\bar{P} = P.$$

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[*] Picard et Simart, *Fonctions algébriques de deux variables indépendentes*, vol. 2, pp. 517–518.

[†] Rendiconti del Circolo Matematico di Palermo, vol. 30 (1911), pp. 265–288.

[‡] For the idea of using a point as part of a basis, see F. Severi, *La base minima pour la totalité des courbes tracés sur une surface algébrique*, Annales de l'école normale supérieure, ser. 3, vol. 25 (1908), pp. 449–468.

The product (SS') is not periodic, hence the surface is invariant under a group of infinite order.

5. **Line an inflexional tangent.** The cases in which two distinct or coincident points of intersection of C_3 and C_1 are fixed offer no new peculiarities; similarly for the cases of three distinct fixed points or of two coincident and one distinct fixed points. But the case in which all three points are coincident merits special attention. If C_1 is an inflexional tangent for every C_3 passing through it, the point of inflexion must be the same point for every curve of the pencil, as is easily seen by considering the binary cubic whose roots define the point of contact.

An equation of the form

$$\phi(x_2 : x_3) + mf(x_2 : x_3) = 0$$

must be a cube in $x_2 : x_3$ for every value of m; this is possible only when $\phi \equiv kf$, k being a constant.

The equation of the most general quartic surface Φ_4 passing through a line $(x_1 = 0, x_4 = 0)$ and having the property that every plane section (cubic curve) through the line will have a fixed point of inflexion $(0, 0, 1, 0)$ on the line and the line itself for inflexional tangent, is of the form

$$\Phi_4 \equiv x_1 F_3(x_1, x_2, x_3, x_4) + x_4 \phi_3(x_2, x_3, x_4) = 0,$$

where

$$F_3 = \sum_{i=1}^{4} \sum_{k=1}^{4} \sum_{l=1}^{4} a_{ikl} x_i x_k x_l, \qquad \phi_3 = \sum_{i=2}^{4} \sum_{k=2}^{4} \sum_{l=2}^{4} b_{ikl} x_i x_k x_l,$$

and

$$a_{223} = a_{233} = a_{333} = b_{223} = b_{233} = b_{333} = 0.$$

The tangent plane to F_4 at any point of the line $x_1 = 0$, $x_4 = 0$, except the point $(0, 0, 1, 0)$, is $a_{222} x_1 + b_{222} x_4 = 0$, and is a fixed plane throughout the line. It is no restriction to call this plane $x_1 = 0$, so that $b_{222} = 0$.

The equation of the surface now has the simpler form

$$x_1 F_3 + x_4^2 \phi_2 = 0,$$

where

$$\phi_2 \equiv \sum_{i=2}^{4} \sum_{k=2}^{4} b_{ik} x_i x_k.$$

The point $(0, 0, 1, 0)$ is a biplanar point, the equations of the two tangent planes being

$$a_{133} x_1^2 + a_{334} x_1 x_4 + b_{334} x_4^2 = 0.$$

One of these planes may be taken for $x_4 = 0$, by putting $a_{133} = 0$. The surface may be mapped on a double plane $(x_3 = 0)$ by projecting from the double point $(0, 0, 1, 0)$.

The curve of branch points is the intersection of the cone of tangents from the double point. By arranging the equation of the surface in the form

$$u_2 \, x_3^2 + 2u_3 \, x_3 + u_4 = 0,$$

where $u_2 = x_4 (a_{334} \, x_1 + b_{334} \, x_4)$, $u_3 =$ quadratic in x_1, x_4,

$$u_4 = \text{quadratic in } x_1, x_4 + a_{222} \, x_1 \, x_2,$$

the equation of the cone may be written in the form

$$u_3^2 - u_2 \, u_4 \equiv K_6 = p_3 \, x_2^3 + p_4 \, x_2^3 + p_5 \, x_2 + p_6 = 0,$$

where

$$p_3 = a_{222} \, x_1 \, x_4 (a_{334} \, x_1 + b_{33} \, x_4).$$

If now a curve exists on F_4 which meets every plane cubic through the line $x_1 = 0$, $x_4 = 0$, in one point, it must be rational and meet the line in $n - 1$ points.

Projected from $(0, 0, 1, 0)$, the curve lies on the cone $qx_2 - q' = 0$, in which q, q' are binary in x_1, x_4 of orders $n - 1$, n respectively.

Since every generator of the cone meets Φ_4 in one other point, the locus of this residual point is another rational curve lying on Φ_4. The first necessary condition that $qx_2 - q' = 0$ shall meet Φ_4 in two distinct curves is that Q shall be a contact curve of K_6. If this condition is satisfied, the equation obtained by eliminating x_2 between K_6 and Q has only double roots, so that

$$\frac{p_3 \, q'^3 + p_4 \cdot q'^2 \, q + p_5 \, q' \cdot q^2 + p_6 \, q^3}{q^3}$$

is of the form L^2/q^3.

These conditions can be satisfied by properly choosing the constants in q' and q. The second condition is that the cone from $(0, 1, 0, 0)$ through the curve of intersection of Φ_4 and Q shall be composite.

By writing the equation of Φ_4 in the form

$$(u_2 \, x_2 + u_3)^2 = K_6$$

and making it simultaneous with Q, we have

$$q^3 (u_2 \, x_2 + u_3)^2 = L^2.$$

This will be composite when and only when q is a square, hence a further condition is stated in the following

THEOREM. *If a rational curve exists on Φ_4 cutting every cubic curve through $x_1 = 0$, $x_4 = 0$ in one point, it must be of odd order.*

If q is a square, it can contain only $n/2 - 1$ constants, while q' contains $n + 2$ constants, and $\frac{3}{2}(n + 2)$ relations among the coefficients must be satisfied.

Since

$$n + 2 + \frac{n}{2} - 1 = \frac{3n + 2}{2},$$

it follows that there are not enough constants to satisfy the conditions, hence: *The general quartic surface through the line $x_1 = 0$, $x_4 = 0$, having a point of inflexion at $(0, 1, 0, 0)$ and $x_1 = 0$, $x_4 = 0$ for inflexional tangent on the cubic curve in every plane through the line, does not contain any rational curve which cuts every cubic curve once.*

The surface does not have an infinite number of birational transformations under which it is invariant, but only the involution defined by projecting Φ_4 upon itself from the point P.

6. Inflexional tangent obtained by transformation.   More generally, given a surface F_m of order m, having the line $x_1 = 0$, $x_4 = 0$ for an $(m - 3)$-fold line and a rational curve meeting every plane section through the line once, it is possible to transform the surface into a surface of order m' having an $(m' - 3)$-fold straight line on it, and having the further property that every plane cubic section through the line will have the line for inflexional tangent.

At any point P of the rational curve the tangent t and any other line n through P can be chosen in terms of which the equation of the cubic may be expressed in the form

$$lt - n^2 + \cdots = 0, \qquad l(p) \neq 0.$$

The conic $lt - n^2 = 0$ has three point contact with the cubic at P, and meets it in three other points P_1, P_2, P_3.   By using the triangle $P_1 P_2 P_3$ for triangle of quadratic inversion, the cubic will go over into a cubic and the conic into a straight line having at the point P_1' (the image of P) three point contact with the image cubic.   Since a net of conics can be drawn having three point contact with the cubic at P, there are two independent parameters, enough to make the image line any line in the plane.   The coördinates of P_1, P_2, P_3 enter the equations of transformation symmetrically, and all the operations can be expressed rationally.

But it has been seen that, if every C_3 has C_1 for inflexional tangent, the point of contact of C_3 with C_1 is fixed.

It is thus shown that a surface containing a pencil of plane cubic curves and a rational curve cutting each curve of the pencil once can be transformed into another having the given line for inflexional tangent of every plane cubic section through it.   The converse, however, is not true.   In the transformed surface sufficient restrictions are imposed on the coefficients to permit the existence of a rational curve, and hence of an infinite number.   These curves are the transforms of the locus of the successive tangentials of P on the original surface.

2. Quartic Surfaces Containing a Conic

7. F_4 with two coplanar conics. If one plane section of a quartic surface is composed of two conics, the number of linearly independent curves on the surface is two.[*]

Since neither conic is a multiple of the other, the two conics C_2 and C_2' form a basis. The surface is regular and of geometric genus 1, and hence the grade n and the genus π of any curve C on the surface satisfy the relation

$$n = 2\pi - 2 .$$

The curves C are self-adjoint and of dimensions[†] π. In the preceding cases the question of the existence of a birational transformation did not need to be considered, as each transformation could be determined by geometric considerations. In the present case the existence is not so evident, and we shall proceed algebraically.

Since the conics C_2, C_2' are rational and meet in four points, any two curves $\lambda C_2 + \mu C_2'$, $\lambda' C_2 + \mu' C_2'$ meet in i points, defined by

$$i = -2\lambda\lambda' + 4(\lambda\mu' + \lambda'\mu) - 2\mu\mu' ,$$

and the grade n of $\lambda C_2 + \mu C_2'$ is given by

$$n = -2\lambda^2 + 8\lambda\mu - 2\mu^2 .[‡]$$

Hence $\lambda C_2 + \mu C_2'$, which is of order $2\lambda + 2\mu$, meets C_2 in $4\mu - 2\lambda$ points and meets C_2' in $4\lambda - 2\mu$ points. It follows that λ and μ are positive integers and that neither can·exceed twice the other except the cases $\lambda = 1$, $\mu = 0$ and $\lambda = 0$, $\mu = 1$. We shall now show that C_2, C_2' form a minimum basis[§] so that all curves on F_4 are expressible in the form $\lambda C_2 + \mu C_2'$.

Suppose Γ and Γ' constitute a minimum basis. Then

$$C_2 = \alpha\Gamma + \beta\Gamma' ,$$

$$C_2' = \gamma\Gamma + \delta\Gamma' ,$$

in which α, β, γ, δ are integers. If $|\Gamma|$ is of grade a, $|\Gamma'|$ of grade b, and any curve $|\Gamma|$ meets $|\Gamma'|$ in h points, the grade of any curve $l\Gamma + m\Gamma'$ is

$$al^2 + 2hlm + bm^2 .$$

[*] Bagnera and de Franchis, *Le nombre ρ de M. Picard pour les surfaces hyperelliptiques*, R e n d i c o n t i d e l C i r c o l o M a t e m a t i c o d i P a l e r m o, vol. 30 (1911), pp. 185–238. See page 187.

[†] F. Severi, *Complementi alla theoria delle base per la totalità delle curve di una superfice algebrica*, R e n d i c o n t i d e l c i r c o l o m a t e m a t i c o d i P a l e r m o, vol. 30 (1911), pp. 265–288, p. 276.

[‡] F. Severi, M a t h e m a t i s c h e A n n a l e n, loc. cit., see p. 222.

[§] F. Severi, R e n d i c o n t i d i P a l e r m o, loc. cit., p. 277.

Since $\lambda C_2 + \mu C_2'$ may be written in the form

$$(\lambda\alpha + \mu\gamma)\,\Gamma + (\lambda\beta + \mu\delta)\,\Gamma',$$

we have, by making this identical with $l\Gamma + m\Gamma'$,

$$a(\alpha\lambda + \gamma\mu)^2 + 2h(\alpha\lambda + \gamma\mu)(\beta\lambda + \delta\mu) + b(\beta\lambda + \delta\mu)^2$$
$$= -2\lambda^2 + 8\lambda\mu - 2\mu^2.$$

Hence

$$(ab - h^2)(\alpha\delta - \beta\gamma)^2 = -12.$$

Since a, b, and h are integers, it follows that $\alpha\delta - \beta\gamma = \pm 1$ or ± 2. In the former case C_2 and C_2' also form a minimum basis. In any case a and b are even, hence the value ± 2 is impossible, since it requires h^2 to be of the form $4p + 3$, in which p is an integer. We have therefore proved that the conics C_2 and C_2' form a minimum basis. The following table of possible curves on F_4 can now be found.

λ or μ	μ or λ	Order	Genus
1	0	2	0
1	1	4	3
2	1	6	4
2	2	8	9

The curves $\lambda C_2 + \mu C_2'$, $\lambda \geqq \mu$, are the residual intersections of F_4 with surfaces of order λ having C_2' for a $(\lambda - \mu)$-fold curve. All the formulas can be readily verified by the Cayley-Noether equivalence and postulation theorems.* Hence the conics C_2 and C_2' are the only rational curves on the surface; the curves of genus two are composed of the partial series of tangent planes; the only curves of genus three are the plane sections $|C_2 + C_2'|$, etc.

If a birational transformation exists that leaves F_4 invariant, every linear system of curves is transformed into another linear system of the same grade, genus, and dimension. Hence the plane sections must be transformed into plane sections and the two conics either interchanged or left invariant. This is possible only under a linear transformation, and the general quartic surface containing a conic does not admit such a transformation. We may therefore conclude: *The most general quartic surface subjected only to the condition that it must contain a conic is not invariant under any birational transformation.*†

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* M. Noether, *Sulle curve multiple di superficie algebriche*, A n n a l i d i m a t e m a t i c a , ser. 2, vol. 5 (1871), pp. 163–177. See p. 164 and 168.

† L. Godeaux, *Sur la surface du quatrième ordre contenant une conique*, S o c i é t é d e s s c i e n c e s d u H a i n a u t , vol. 62 (1912), pp. 1–8, has considered the problem of possible birational transformations of the quartic surface containing a conic. In consequence of the restriction $\lambda \not> 2\mu$ (in our notation) which he overlooks, the system of rational curves there found are all composite. The proof of the non-existence of involutional transformations is also incomplete because not all possible cases are considered.

3. Quartics Containing One or Two Space Cubic Curves

8. F_4 through a space cubic curve. Let F_4 contain a space cubic curve C_3. From any point P on F_4 can be drawn one bisecant to C_3 which meets F_4 in a fourth point P'. Since the bisecant to C_3 from P' meets F_4 in P, we see that F_4 admits an involutional birational transformation. Through C_3 pass a bundle of quadric surfaces, each meeting F_4 in a residual quintic of genus 2. All the quadrics of the bundle which pass through a fixed point P belong to a pencil whose basis curve consists of C_3 and the bisecant through P. Hence all the quintic curves through P on F_4 also pass through P'.

A basis is composed of C_3 and a plane section C_4. By proceeding as in the preceding cases we obtain the transformation

$$U. \qquad \begin{aligned} \overline{C}_4 &= 9C_4 - 5C_3, \\ \overline{C}_3 &= 16C_4 - 9C_3, \end{aligned}$$

which can be verified as follows:

The image of a plane section C_4 is obtained by constructing the ruled surface of bisecants to C_3 from each point of C_4. The curve C_4 is a simple curve on the surface, and the lines joining the three points of intersection of C_3 and C_4 in pairs are double generators. Hence the surface is of order 10. From any point of C_3 a quadric cone of lines can be drawn meeting C_3 again. This cone meets the quartic cone to C_4 in eight lines, but three of them pass through the points of intersection of C_3 and C_4; hence C_3 is a five-fold curve on the ruled surface. Since $\overline{C}_4$ is the residual intersection of F_4 and the ruled surface, we have

$$\overline{C}_4 = 10C_4 - (C_4 + 5C_3).$$

The image of C_3 is obtained as the residual intersection with F_4 of the ruled surface obtained by connecting each point K of C_3 with the point K' in which the tangent plane to F_4 at K meets C_3 again.

From K can be drawn one generator to K'; from K' can be drawn six generators to points of tangency K. Moreover, the ruled surface and F_4 touch each other at every point K of C_3; hence each generator meets 14 others, and the surface is of order 16. The curve C_3 counts for nine intersections, and hence the residual intersection C_3 is defined by the formula

$$\overline{C}_3 = 16C_4 - 9C_3.$$

The other possible values of λ and μ which satisfy the congruences analogous to those of § 2 lead to non-effective curves. Hence the surface does not admit an infinite number of birational transformations.

The transformation is a Cremona transformation, the invariant surface

being the locus of the double points of the involution determined on C_3 and F_4 by the bisecants of C_3.

   9. *F_4 through two space cubic curves.*   Consider the general quartic surface containing the two space cubic curves C_3 and C_3' which intersect in k points $0 \leqq k \leqq 5$.   A basis may be made of C_3, C_3' and a plane section C_4.   The involution defined by transforming P into the residual point P' of intersection with the surface of the bisecant of C_3 through P may be expressed by the equations

$$\overline{C}_3 = 16C_4 - 9C_3,$$

$V.$ $$\overline{C}_3' = (12 - 2k)C_4 - (6 - k)C_3 - C_3',$$

$$\overline{C}_4 = 9C_4 - 5C_3.$$

The similar involution as to C_3' has the equations

$$\overline{C}_3 = (12 - 2k)C_4 - C_3 - (6 - k)C_3',$$

$V'.$ $$\overline{C}_3' = 16C_4 - 9C_3',$$

$$\overline{C}_4 = 9C_4 - 5C_3'.$$

The product of these two involutions is a discontinuous birational transformation that is not periodic.

   When $k = 5$, the curves C_3 and C_3' lie on a quadric which meets F_4 in a conic C_2.   By the transformation V the curves C_3' and C_2 are interchanged.

   Similarly, the curves C_3 and C_2 are interchanged by the transformation V'.

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