A NEW INTEGRAL TEST FOR THE CONVERGENCE AND DIVERGENCE OF INFINITE SERIES*

BY

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Introduction†

A new sequence of integral tests for the convergence and divergence of infinite series has been developed by the author. Some of the tests of this sequence, and the principle by which they may be discovered will be set forth by him in another article. In the present paper it is his desire to give a central one of these tests, together with some of its applications. This particular integral test appears to play the same rôle when the ratio of successive terms is explicitly known, that the Maclaurin-Cauchy test plays when the individual term is explicitly known. In testing a series of the form

$$u_0 + u_1 + u_2 + \cdots,$$

du Bois-Reymond‡ called those tests that make use of the test ratio

$$r_n = u_{n+1}/u_n,$$

tests of the second kind to distinguish them from tests using the general term of the series u_n itself, which he called *tests of the first kind*. Similarly, the integral tests developed in this paper, which involve a function $r(x)$ such that $r(n) = r_n$, may be called *integral tests of the second kind;* while the Maclaurin-Cauchy integral test, involving a function $u(x)$ for which $u(n) = u_n$, is an *integral test of the first kind.*

Integral tests of the second kind thus apply to series for which a function is known that for successive integral values of the variable takes on the successive values of the ratio of one term to the preceding term. Such a series can be written in the following *normal form*:

$$c + ca_0 + ca_0 a_1 + ca_0 a_1 a_2 + \cdots,$$

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‡ Journal für Mathematik, vol. 76, p. 61.

where $a_n = a(n)$, $a(x)$ being a known function; and it is to such series that the theorems of this paper apply.

The general method employed to discover the tests of the sequence suggests the test set forth in the first section of the present paper. This test may be considered as the fundamental integral test of the second kind. From it are derived the test given in the second section, and the very simple and useful integral test of the third section. The fourth section gives a test applicable to a series of products of rather general form. The fifth extends the fundamental test to multiple series.

1. Fundamental integral test of the second kind

Theorem I. *Given the series*

$$u_0 + u_1 + u_2 + \cdots \qquad (u_n > 0,\ n \geqq \mu).$$

Let $r_n = u_{n+1}/u_n$, *and suppose that from a certain point* $x = \mu$ *on,* $r(x)$ *is a continuous, positive function such that* $r(n) = r_n$, *and suppose that a constant* m *exists, positive or zero, such that* $r(x') \geqq r(x)$ *when* $x' \geqq x + m$. *Then a necessary and sufficient condition for the convergence of the given series is the convergence of the integral*

$$\int_\mu^\infty e^{\int_\mu^x \log r(x)dx}\, dx .$$

Proof. Under the conditions, from a certain point on either $r(x) > 1$ or $r(x) \leqq 1$. Suppose that $r(x) \leqq 1$, $\mu < x$. We take μ to be an integer. Then

$$\text{(1)} \qquad \log r_{n-m} \leqq \int_n^{n+1} \log r(x)\, dx \leqq \log r_{n+1+m}, \qquad \mu \leqq n.$$

We write

$$\text{(2)} \qquad \int_n^{n+1} e^{\int_\mu^x \log r(x)dx}\, dx = \int_n^{n+1} e^{\overline{\int_\mu^{\mu+1} + \int_{\mu+1}^{\mu+2} + \cdots + \int_{n-1}^{n} + \int_n^x} \log r(x)dx}\, dx,$$

where, as elsewhere in this paper, the bar over the signs of integration indicates that the integrals under it have the same integrand. Therefore, by (1),

$$\text{(3)} \qquad \begin{aligned} \int_n^{n+1} e^{\int_\mu^x \log r(x)dx}\, dx &\leqq \int_n^{n+1} e^{\log r_{\mu+1+m} + \log r_{\mu+2+m} + \cdots + \log r_{n+m}}\, dx \\ &= r_{\mu+1+m} \cdot r_{\mu+2+m} \cdot \cdots \cdot r_{n+m} = \frac{1}{u_{\mu+1+m}} u_{n+1+m}. \end{aligned}$$

Likewise

$$\text{(4)} \qquad \int_n^{n+1} e^{\int_\mu^x \log r(x)dx}\, dx \geqq r_{\mu-m} \cdot r_{\mu+1-m} \cdot \cdots \cdot r_{n-m} = \frac{1}{u_{\mu-m}} \cdot u_{n+1-m}.$$

Similar inequalities hold if $r(x) > 1$, $\mu_1 < x$. Since the integral

$$\int_{\mu}^{x} e^{\int_{\mu}^{x} \log r(x)dx}\, dx$$

cannot oscillate, the theorem follows at once from a comparison of the two series

$$\sum_{n=\mu}^{\infty} u_n \quad \text{and} \quad \sum_{n=\mu}^{\infty} \int_{n}^{n+1} e^{\int_{\mu}^{x} \log r(x)dx}\, dx$$

by means of (3) and (4).

Example. Discuss the convergence of the series

$$\sum_{n=\mu}^{\infty} u_n,$$

where $u_n = e^{\rho_\mu + \rho_{\mu+1} + \cdots + \rho_{n-1}}$,

$$\rho_n = -\frac{l_1 n \cdot l_2 n \cdot \cdots \cdot l_k n + l_2 n \cdot l_3 n \cdot \cdots \cdot l_k n + \cdots + l_k n + p}{n \cdot l_1 n \cdot l_2 n \cdot \cdots \cdot l_k n},$$

and $l_k x = \log(l_{k-1} x)$, $l_1 x = \log x$. Of course μ is to be taken large enough for all terms of the series to be defined.

For this series $r_n = e^{\rho_n}$, and

$$r(x) = e^{\rho(x)} = e^{-\frac{l_1 x \cdot l_2 x \cdot \cdots \cdot l_k x + l_2 x \cdot l_3 x \cdot \cdots \cdot l_k x + \cdots + l_k x + p}{x \cdot l_1 x \cdot l_2 x \cdot \cdots \cdot l_k x}}.$$

This function satisfies the conditions of the theorem, and we have

$$\int_{x_0}^{x} \log r(x)\, dx = \int_{x_0}^{x} -\frac{l_1 x \cdot \cdots \cdot l_k x + \cdots + l_k x + p}{x l_1 x l_2 x \cdot \cdots \cdot l_k x}\, dx$$

$$= c - \log[x l_1 x \cdots l_{k-1} x (l_k x)^p].$$

Therefore

$$\int_{x_0}^{x} e^{\int_{x_0}^{x} \log r(x)dx}\, dx = C \int_{x_0}^{x} \frac{dx}{x \cdot l_1 x \cdot l_2 x \cdot \cdots \cdot l_{k-1} x (l_k x)^p}.$$

Hence the given series converges or diverges according as $p > 1$ or $p \leqq 1$.

The theorem can be extended as follows:

THEOREM II. *Given the series*

$$u_0 + u_1 + u_2 + \cdots \qquad (u_n > 0,\ n \geqq \mu).$$

Let $r(x)$ *be a function with a continuous derivative* $r'(x)$, $x \geqq \mu$, *such that* $r(n) = r_n = u_{n+1}/u_n$. *Suppose that* $0 < A \leqq r(x) \leqq B$, *and that the integral*

$$\int_{\mu}^{\infty} |r'(x)|\, dx$$

converges. Then a necessary and sufficient condition for the convergence of the given series is the convergence of the integral

$$\int_\mu^\infty e^{\int_\mu^x \log r(x)dx}\,dx.$$

Proof. Write

$$(1)\qquad \begin{aligned} d_n &= \int_n^{n+1} \log r(x)\,dx - \log r_{n+1} \\ &= \int_n^{n+1} [\log r(x) - \log r_{n+1}]\frac{d}{dx}(x-n)\,dx. \end{aligned}$$

Integrate by parts. Then we find

$$d_n = -\int_n^{n+1} (x-n)\frac{r'(x)}{r(x)}\,dx.$$

Therefore

$$(2)\qquad |d_n| \leqq \int_n^{n+1} \left|\frac{r'(x)}{r(x)}\right| dx.$$

Now by hypothesis the integral

$$\int_\mu^\infty |r'(x)|\,dx$$

converges, and therefore the integral

$$\int_\mu^\infty \left|\frac{r'(x)}{r(x)}\right| dx$$

converges since

$$\left|\frac{r'(x)}{r(x)}\right| \leqq \frac{|r'(x)|}{A}.$$

Consequently the series

$$\sum_{n=\mu}^\infty d_n$$

converges absolutely, and we can write

$$\sum_{n=\mu}^\infty |d_n| \doteq D.$$

Now

$$\int_n^{n+1} e^{\int_\mu^x \log r(x)dx}\,dx = \int_n^{n+1} e^{\overbrace{\int_\mu^{\mu+1} + \int_{\mu+1}^{\mu+2} + \cdots + \int_{n-1}^{n} + \int_n^x}\log r(x)dx}\,dx$$

$$= \int_n^{n+1} e^{d_\mu + d_{\mu+1} + \cdots + d_{n-1} + \log(r_{\mu+1}\cdot r_{\mu+2}\cdot \cdots \cdot r_n) + \int_n^x \log r(x)dx}\,dx.$$

Since $0 < A \leqq r(x) \leqq B$, we can set $|\log r(x)| \leqq C$. Consequently we have

$$\frac{e^{-C-D}}{u_{\mu+1}} u_{n+1} \leqq \int_n^{n+1} e^{\int_\mu^x \log r(x)dx} dx \leqq \frac{e^{C+D}}{u_{\mu+1}} u_{n+1}.$$

The theorem then follows as before from a comparison of the two series

$$\sum_{n=\mu}^{\infty} u_n \quad \text{and} \quad \sum_{n=\mu}^{\infty} \int_n^{n+1} e^{\int_\mu^x \log r(x)dx} dx.$$

The method used in the following *alternative proof* is sometimes useful. Define two functions $r^{(1)}(x)$ and $r^{(2)}(x)$ such that

$$r^{(1)}(x) \cdot r^{(2)}(x) = r(x),$$

and such that $r^{(1)}(x)$ monotonically increases and $r^{(2)}(x)$ monotonically decreases when x increases. We can do this in the following way. Denote $r^{(1)}(n)$ by $r_n^{(1)}$, and $r^{(2)}(n)$ by $r_n^{(2)}$. Take $\log r_\mu^{(1)} = \log r_\mu$ and $\log r_\mu^{(2)} = 0$; if $r'(x) \geqq 0$ take

$$\frac{d \log r^{(1)}(x)}{dx} = \frac{d \log r(x)}{dx}, \qquad \frac{d \log r^{(2)}(x)}{dx} = 0;$$

if $r'(x) \leqq 0$, take

$$\frac{d \log r^{(2)}(x)}{dx} = \frac{d \log r(x)}{dx}, \qquad \frac{d \log r^{(1)}(x)}{dx} = 0.$$

Then

$$\log r^{(1)}(x) + |\log r^{(2)}(x)| = \int_\mu^x \left| \frac{d \log r(x)}{dx} \right| dx + \log r_\mu.$$

Since the integral

$$\int_\mu^\infty \left| \frac{d \log r(x)}{dx} \right| dx = \int_\mu^\infty \left| \frac{r'(x)}{r(x)} \right| dx$$

converges, $\log r^{(1)}(x)$ and $\log r^{(2)}(x)$ are finite, and we can write

$$0 < a < r^{(1)}(x) < b, \qquad 0 < a < r^{(2)}(x) < b.$$

Now

$$\int_n^{n+1} e^{\int_\mu^x \log r(x)dx} dx$$

$$= \int_n^{n+1} e^{\int_0^1 [\log r(\mu+x) + \log r(\mu+x+1) + \cdots + \log r(x+n-1)]dx + \int_n^x \log r(x)dx} dx$$

$$= \int_n^{n+1} e^{\log [r^{(1)}(\mu+\theta) \cdot r^{(2)}(\mu+\theta) \cdot r^{(1)}(\mu+\theta+1) \cdot r^{(2)}(\mu+\theta+1) \cdots r^{(1)}(n-1+\theta) r^{(2)}(n-1+\theta)] + \int_n^x \log r(x)dx} dx$$

$0 < \theta < 1$. Therefore

$$\int_n^{n+1} e^{\int_\mu^x \log r(x)dx} dx \leqq e^C \cdot r_{\mu+1}^{(1)} \cdot r_\mu^{(2)} \cdot r_{\mu+2}^{(1)} \cdot r_{\mu+1}^{(2)} \cdot \cdots \cdot r_n^{(1)} \cdot r_{n-1}^{(2)}$$

$$= e^C \cdot r_\mu^{(2)} \cdot r_n^{(1)} \frac{u_n}{u_{\mu+1}} \leqq \frac{e^C b^2}{u_{\mu+1}} u_n.$$

Likewise

$$\int_n^{n+1} e^{\int_\mu^x \log r(x)dx} dx \geqq \frac{e^{-C} a^2}{u_{\mu+1}} u_n.$$

The theorem then follows as before.

Instead of testing for the convergence of the integral

$$\int_\mu^\infty |r'(x)|dx$$

it is sometimes convenient to test directly for the convergence of the integral

$$\int_\mu^\infty \left|\frac{r'(x)}{r(x)}\right| dx,$$

which is equivalent.

It is clear that Theorem II is included in the following theorem. We thus have still a third proof of Theorem II.

THEOREM III. *Given the series*

$$u_0 + u_1 + u_2 + \cdots, \qquad (u_n > 0,\ n \geqq \mu).$$

Let $r(x)$ be a continuous function such that, for $x \geqq \mu$,

(a) $$r(n) = r_n = \frac{u_{n+1}}{u_n},$$

(b) $$0 < A \leqq r(x) \leqq B,$$

(c) $$|r(x') - r(x)| \leqq f(x),$$

whenever $0 \leqq (x' - x) \leqq 1$, the series $\sum_{n=\mu}^{n=\infty} f(n)$ being a convergent series. Then a necessary and sufficient condition for the convergence of the given series is the convergence of the integral

$$\int_\mu^\infty e^{\int_\mu^x \log r(x)dx} dx.$$

Proof. By condition (c),

(1) $$r(x') \leqq r(x) + f(x) \leqq r(x)\left[1 + \frac{f(x)}{A}\right],$$

$$0 \leqq x' - x \leqq 1, \qquad x \geqq \mu.$$

$$(2) \qquad r(x') \geqq r(x)\left[1 - \frac{f(x)}{A}\right], \qquad 0 \leqq x' - x \leqq 1, \qquad x \geqq \mu.$$

We have

$$\int_n^{n+1} e^{\int_\mu^x \log r(x)dx} dx = \int_n^{n+1} e^{\log (r(\mu+\theta)\cdot r(\mu+1+\theta)\cdots r(n-1+\theta)) + \int_n^x \log r(x) dx} dx,$$

where $0 < \theta < 1$. As before, we write $|\log r(x)| \leqq \log C$. Then by (1),

$$\int_n^{n+1} e^{\int_\mu^x \log r(x)dx} dx \leqq C\left[r_\mu \left(1 + \frac{f(\mu)}{A}\right) r_{\mu+1}\left(1 + \frac{f(\mu+1)}{A}\right)\right.$$

$$\left. \cdots\cdots r_{n-1}\left(1 + \frac{f(n-1)}{A}\right)\right]$$

$$= \frac{C}{u_\mu} u_n \prod_{n=\mu}^{n-1}\left(1 + \frac{f(n)}{A}\right).$$

Since the series

$$\sum_{n=\mu}^{\infty} f(n)$$

converges, and $f(n) \geqq 0$, the product

$$\prod_{n=\mu}^{\infty}\left(1 + \frac{f(n)}{A}\right)$$

converges to a value F. Therefore

$$\int_n^{n+1} e^{\int_\mu^x \log r(x)dx} dx \leqq \frac{CF}{u_\mu} u_n.$$

Similarly, by (2), the product

$$\prod_{n=\mu}^{\infty}\left(1 - \frac{f(n)}{A}\right),$$

converges, say to F_1, and we get

$$\int_n^{n+1} e^{\int_\mu^x \log r(x)dx} dx \geqq \frac{F_1}{Cu_\mu} u_n.$$

The theorem then follows by comparison.

2. A DERIVED TEST

The following test is based upon Theorems I, II, and III, in much the same way that Ermakoff's test* is based upon the familiar Maclaurin-Cauchy integral test.

* Bulletin des sciences mathématiques vol. 2 (1871), p. 250.

THEOREM IV. *Given the series*

$$u_0 + u_1 + u_2 + \cdots \qquad (u_n > 0,\ n \geqq \mu).$$

Let $r_n = u_{n+1}/u_n$, *and suppose that* $r(x)$ *is a function satisfying the preliminary conditions given in any one of the Theorems I, II, or III, and that* $r(x) \leqq 1$. *Then the series converges if*

$$\frac{[r(e^x)]^{e^x}}{r(x)} < \nu < \frac{1}{e}, \qquad m \leqq x,$$

and diverges if

$$\frac{[r(e^x)]^{e^x}}{r(x)} > \nu > \frac{1}{e}, \qquad m \leqq x.$$

Proof. Since the preliminary conditions hold for one of the earlier tests, the series converges if and only if the integral

$$\int_\mu^\infty e^{\int_\mu^x \log r(x)dx}\, dx$$

converges. Suppose first that

$$\frac{[r(e^x)]^{e^x}}{r(x)} < \nu < \frac{1}{e}, \qquad \mu \leqq m \leqq x.$$

Then

$$[r(e^x)]^{e^x} < \nu \cdot r(x)$$

and

$$e^x \log r(e^x) < \log \nu + \log r(x).$$

It follows that

$$\int_m^x e^x \log r(e^x)\, dx < (x - m) \log \nu + \int_m^x \log r(x)\, dx, \qquad m < x;$$

or, after a change in the variable of integration,

$$\int_{e^m}^{e^x} \log r(x)\, dx < (x - m) \log \nu + \int_m^x \log r(x)\, dx.$$

Consequently

$$\int_{x_0}^x e^x \cdot e^{\int_{e^m}^{e^x} \log r(x)dx}\, dx < \frac{1}{\nu^m} \int_{x_0}^x e^x \cdot \nu^x\, e^{\int_m^x \log r(x)dx}\, dx, \qquad m < x_0 < x.$$

Then, once more changing the variable of integration, we have

$$\int_{e^{x_0}}^{e^x} e^{\int_{e^m}^x \log r(x)dx}\, dx < \frac{1}{\nu^m} \int_{x_0}^x \frac{1}{\lambda^x}\, e^{\int_m^x \log r(x)dx}\, dx, \qquad \lambda > 1.$$

Therefore, since $\log r(x) \leqq 0$,

$$\int_{e^{x_0}}^{e^x} e^{\int_{e^m}^{x} \log r(x)dx}\, dx < \frac{1}{\nu^m}\int_{x_0}^{x} \frac{dx}{\lambda^x}, \qquad m < x_0 < x.$$

But the integral

$$\int_{x_0}^{\infty} \frac{dx}{\lambda^x}$$

converges. Consequently the integral

$$\int_{e^{x_0}}^{\infty} e^{\int_{e^m}^{x} \log r(x)dx}\, dx,$$

and therefore the given series, converge.

Now suppose that

$$\frac{[r(e^x)]^{e^x}}{r(x)} > \nu > \frac{1}{e}, \qquad \mu \leqq m < x.$$

We can take $1/e < \nu < 1$. Then

$$[r(e^x)]^{e^x} > \nu \cdot r(x), \qquad \frac{1}{e} < \nu < 1, \qquad m < x.$$

As before, reversing inequalities, we get

$$\int_{e^{x_0}}^{e^x} e^{\int_{e^m}^{x} \log r(x)dx}\, dx > \frac{1}{\nu^m}\int_{e^{x_0}}^{x} \lambda^x\, e^{\int_{m}^{x} \log r(x)dx}\, dx, \qquad \lambda > 1$$

That is

$$\int_{x}^{e^x} e^{\int_{e^m}^{x} \log r(x)dx}\, dx > \frac{1}{\nu^m}\int_{x_0}^{x} \lambda^x\, e^{\int_{m}^{x} \log r(x)dx}\, dx - \int_{e^{x_0}}^{x} e^{\int_{e^m}^{x} \log r(x)dx}\, dx$$

$$> \frac{1}{\nu^m}\int_{x_0}^{x} \lambda^x\, e^{\int_{m}^{x} \log r(x)dx}\, dx - \int_{e^{x_0}}^{x} \lambda^x\, e^{\int_{m}^{x} \log r(x)dx}\, dx,$$

$$m' < x_0 < x.$$

Then, since $\nu < 1$,

$$\int_{x}^{e^x} e^{\int_{e^m}^{x} \log r(x)dx}\, dx > \int_{x_0}^{e^{x_0}} \lambda^x\, e^{\int_{m}^{x} \log r(x)dx}\, dx = c > 0.$$

This inequality holds for all values of x beyond a certain point. If we take x_1 some number beyond this point, and set

$$x_2 = e^{x_1}, \qquad x_3 = e^{x_2}, \qquad x_4 = e^{x_3}, \qquad \cdots,$$

each term of the series

$$\int_{x_1}^{x_2} e^{\int_{em}^{x} \log r(x)dx} dx + \int_{x_2}^{x_3} e^{\int_{em}^{x} \log r(x)dx} dx + \cdots$$

is greater than the positive constant c, and this series therefore diverges. Therefore the integral

$$\int_{x_0}^{\infty} e^{\int_{em}^{x} \log r(x)dx} dx,$$

and consequently the given series, diverges.

By the transformation $x = e^{x'}$, the test is changed to the following form, sometimes more convenient: *The series is convergent if*

$$\frac{[r(x)]^x}{r(\log x)} < \nu < \frac{1}{e}, \qquad m < x,$$

and is divergent if

$$\frac{[r(x)]^x}{r(\log x)} > \nu > \frac{1}{e}, \qquad m < x.$$

In case

$$\lim_{x=\infty} r(x) = 1,$$

this test is no more than the test of Schlömilch which may be stated as follows: The series converges if $x \log r(x) < \nu_1 < -1$, $m < x$; and diverges if $x \log r(x) > \nu_1 > -1$, $m < x$. From the point of view of the integral tests, Theorem IV may be used as a proof of Schlömilch's test.

3. An integral test involving $(r(x) - 1)$

The following is the most generally useful integral test of the second kind.

Theorem V. *Given the series*

$$u_0 + u_1 + u_2 + \cdots \qquad (u_n > 0,\ n > \mu).$$

Let $r_n = u_{n+1}/u_n$, and suppose that $r(x)$ is a positive, integrable function satisfying the preliminary conditions of one of the Theorems I, II, or III, and the further condition that

$$|r(x) - 1| < a < 1, \qquad \mu_1 < x.$$

Then a sufficient condition for the convergence of the series is the convergence of the integral

$$\int_{x_0}^{\infty} e^{\int_{x_0}^{x} (r(x) - 1)dx} dx.$$

This condition is also necessary for the convergence of the series if from a certain *point on*

$$|r(x) - 1| < \frac{k}{x},$$

where k is some constant.

Proof. We have the expansion

$$(1) \quad \log r(x) = (r(x) - 1) - \tfrac{1}{2}(r(x) - 1)^2 + \tfrac{1}{3}(r(x) - 1)^3 - \cdots.$$

Then

$$\log r(x) \leqq r(x) - 1,$$

so that

$$\int_{x_0}^{x} e^{\int_{x_0}^{x} \log r(x)dx} dx \leqq \int_{x_0}^{x} e^{\int_{x_0}^{x} (r(x) - 1)dx} dx.$$

Therefore, since the preliminary conditions of one of the Theorems I, II, or III are satisfied, so that the given series and the integral

$$\int_{x_0}^{\infty} e^{\int_{x_0}^{x} \log r(x)dx} dx$$

converge or diverge together, the convergence of the integral

$$\int_{x_0}^{\infty} e^{\int_{x_0}^{x} (r(x) - 1)dx} dx$$

is sufficient for the convergence of the given series.

The expansion (1) converges uniformly for $|r(x) - 1| < a < 1$, that is, for $\mu_1 \leqq x$. We can therefore integrate it term by term over any interval $\mu_1 \leqq x \leqq A$. Then

$$\int_{x_0}^{x} \log r(x)dx = \int_{x_0}^{x} (r(x) - 1)dx - \frac{1}{2}\int_{x_0}^{x} (r(x) - 1)^2 dx + \cdots,$$

$$\mu_1 < x_0 < x,$$

and

$$\int_{x_0}^{x} e^{\int_{x_0}^{x} \log r(x)dx} dx$$

$$= \int_{x_0}^{x} e^{\int_{x_0}^{x} (r(x) - 1)dx - \frac{1}{2}\int_{x_0}^{x} (r(x) - 1)^2dx + \frac{1}{3}\int_{x_0}^{x} (r(x) - 1)^3dx - \cdots} dx, \quad \mu_1 < x_0 < x.$$

We have

$$|r(x) - 1| < \frac{k}{x},$$

so that

$$(r(x) - 1) > -\frac{k}{x}, \qquad -(r(x) - 1)^2 > -\frac{k^2}{x^2},$$

$$(r(x) - 1)^3 > -\frac{k^3}{x^3}, \quad \text{etc.,} \quad x > \mu_1.$$

Therefore

$$\int_{x_0}^{x} e^{\int_{x_0}^{x}\log r(x)dx}\,dx$$

$$> \int_{x_0}^{x} e^{\int_{x_0}^{x}(r(x)-1)dx - \frac{1}{2}\int_{x_0}^{x}\frac{k^2}{x^2}dx - \frac{1}{3}\int_{x_0}^{x}\frac{k^3}{x^3}dx - \frac{1}{4}\int_{x_0}^{x}\frac{k^4}{x^4}dx - \cdots}\,dx$$

$$> \int_{x_0}^{x} e^{\int_{x_0}^{x}(r(x)-1)dx - \frac{1}{2}\int_{x_0}^{\infty}\frac{k^2}{x^2}dx - \frac{1}{3}\int_{x_0}^{\infty}\frac{k^3}{x^3}dx - \frac{1}{4}\int_{x_0}^{\infty}\frac{k^4}{x^4}dx - \cdots}\,dx$$

$$= \int_{x_0}^{x} e^{\int_{x_0}^{x}(r(x)-1)dx - \left(\frac{k^2}{2x_0} + \frac{k^3}{3\cdot 2x_0^2} + \frac{k^4}{4\cdot 3x_0^3} + \cdots\right)}\,dx, \quad \mu_1 < x_0 < x.$$

If x_0 is taken greater than k, the series in parentheses converges to a value c_1. Consequently

$$\int_{x_0}^{x} e^{\int_{x_0}^{x}\log r(x)dx}\,dx > c_2 \int_{x_0}^{x} e^{\int_{x_0}^{x}(r(x)-1)dx}\,dx.$$

Therefore when x increases indefinitely, if the second of these two integrals diverges, the first one also diverges, and the given series diverges.

In most cases of interest $r(x)$ is a monotonically increasing function with unity as its limit. In such a case the condition $|r(x) - 1| < k/x$ becomes $(r(x) - 1) > -k/x$. This condition is not a very great restriction. For in case the integral

$$\int_{x_0}^{\infty} e^{\int_{x_0}^{x}(r(x)-1)dx}\,dx$$

diverges, if λ is any constant greater than unity the product of the integrand by λ^x is not finite for x infinite, and it is natural to assume that

$$x^{\lambda}\, e^{\int_{x_0}^{x}(r(x)-1)dx} > 1, \qquad m < x,$$

or

$$\int_{x_0}^{x}(r(x) - 1)\,dx > -\lambda \log x, \qquad m < x.$$

That is, if x_0 is properly chosen,

$$\int_{x_0}^{x}(r(x) - 1)\,dx > -\int_{x_0}^{x}\frac{\lambda}{x}\,dx, \qquad m < x_0 < x.$$

If this condition holds it is clear that for considerable intervals, at least,

$$r(x) - 1 > -\lambda/x.$$

So the condition that $r(x) - 1 > -k/x$, $m < x$, is a condition that we may expect to hold rather commonly when the integral

$$\int_{x_0}^{\infty} e^{\int_{x_0}^{x}(r(x)-1)dx}\,dx$$

diverges.

As a corollary to the theorem we see that if $r(x)$ satisfies the preliminary conditions of the theorem, and if

$$|r(x) - 1| < k/x, \qquad \mu \leqq x, \qquad k \leqq 1,$$

the series diverges. For

$$\int_{x_0}^{x} e^{\int_{x_0}^{x}(r(x)-1)dx}\,dx \geqq \int_{x_0}^{x} e^{-\int_{x_0}^{x}\frac{k}{x}dx}\,dx = \int_{x_0}^{x}\frac{c\,dx}{x^k}.$$

Similarly, if $r(x) - 1 < -k/x$, $k > 1$, the series converges. In this way Raabe's test* may be proved.

The test of Theorem V is stronger than any test of the logarithmic scale, for by means of it we can test the series

$$\sum_{n=\mu}^{\infty}\frac{1}{n\cdot l_1 n\cdot l_2 n\cdot \cdots \cdot l_{\lambda-1} n\cdot (l_\lambda n)^p}.$$

For this series

$$r(x) = \frac{x l_1 x l_2 x \cdot \cdots \cdot l_{\lambda-1} x\,(l_\lambda x)^p}{(x+1)\,l_1(x+1)\,l_2(x+1)\cdot \cdots \cdot l_{\lambda-1}(x+1)\,(l_\lambda(x+1))^p}.$$

In all cases all of the preliminary conditions of the test hold for this series, except perhaps the condition that $|r(x) - 1| < k/x$. Therefore if the integral of the test converges the series is known to converge. Suppose that the integral diverges. The series also will then diverge. For in case that it converged we should have

$$r(x) \geqq \left(\frac{x}{x+1}\right)^p = \left(1+\frac{1}{x}\right)^{-p};$$

and

$$r(x) - 1 > -2p/x, \qquad x > m,$$

which shows that all of the preliminary conditions of the theorem are then satisfied. Consequently, since the integral diverges, the theorem leads to a contradiction of the assumption that the series converges, or the integral and the series always converge or diverge together.

Just as we were able to establish d'Alembert's test and Raabe's first test, we can establish by means of Theorems I, II, III, and V most of the standard tests that use the test-ratio.

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*Journal für Mathematik von Ettinghausen und Baumgartner, vol. 10; Journal de mathématiques, vol. 6 (1841), p. 85; Journal für Mathematik, vol. 11 (1832), p. 309.

Thus consider the series for which

$$r(x) = 1 - \frac{1}{x} - \frac{1}{xl_1 x} - \cdots - \frac{1}{x \cdot l_1 x \cdot \cdots \cdot l_{k-1} x} - \frac{p}{x \cdot l_1 x \cdot \cdots \cdot l_k x}.$$

This series clearly satisfies the conditions for Theorem V.

$$\int_m^\infty e^{\int_m^x (r(x) - 1)dx} dx$$

$$= \int_m^\infty e^{-\int_m^x \left[\frac{1}{x} + \frac{1}{xl_1 x} + \cdots + \frac{1}{x \cdot l_1 x \cdot \cdots \cdot l_{k-1} x} + \frac{p}{x \cdot l_1 x \cdot \cdots l_k x}\right] dx} dx$$

$$= c \int_m^\infty \frac{dx}{xl_1 xl_2 x \cdots l_{k-1} x (l_k x)^p}.$$

Therefore the series converges if $p > 1$, and diverges if $p \leqq 1$.

We thus have the tests of de Morgan* and Bertrand†: *If $r(x)$ can be expressed in the form*

$$r(x) = 1 - \frac{1}{x} - \frac{1}{xl_1 x} - \cdots - \frac{1}{xl_1 x \cdots l_{k-1} x} - \frac{\omega(x)}{xl_1 x \cdots l_k x}$$

and if

$$\lim_{x=\infty} \omega(x) = l,$$

the corresponding series converges when $l > 1$, and diverges when $l < 1$. In stating these tests it is more common to express the reciprocal of $r(x)$ as

$$\frac{1}{r(x)} = 1 + \frac{1}{x} + \frac{1}{xl_1 x} + \cdots + \frac{1}{xl_1 x \cdots l_{k-1} x} + \frac{\omega_1(x)}{xl_1 x \cdots l_k x};$$

then *the series converges if*

$$\lim_{x=\infty} \omega_1(x) > 1,$$

and diverges if

$$\lim_{x=\infty} \omega_1(x) < 1.$$

The same tests are often stated in the following form. *If of the following limits*

$$\lim_{x=\infty} x\left(\frac{1}{r(x)} - 1\right) = k_0,$$

$$\lim_{x=\infty} l_1 x\left(x\left(\frac{1}{r(x)} - 1\right) - 1\right) = k_1,$$

$$\lim_{x=\infty} l_2 x\left(l_1(x)\left(x\left(\frac{1}{r(x)} - 1\right) - 1\right) - 1\right) = k_2, \quad \text{etc.},$$

* *Differential and Integral Calculus*, 1839.

† Journal de mathématiques, vol. 7 (1842), p. 37.

k_m is the first that is not equal to unity, the corresponding series converges if $k_m > 1$, and diverges if $k_m < 1$.

In his study of the hypergeometric series Gauss* gave the following rule. *If*

$$r_n = \frac{n^k + a_1 n^{k-1} + \cdots + a_k}{n^k + b_1 n^{k-1} + \cdots + b_k},$$

where k is a positive integer, the series converges if $(b_1 - a_1) > 1$, and diverges if $(b_1 - a_1) \leqq 1$. This test of course is merely the first test of the logarithmic scale. It is easily established directly by means of Theorem V, through one integration. This gives perhaps the easiest method of testing the hypergeometric series. The same results can be found by applying Theorem V directly to the hypergeometric series, though Gauss's test, so easily established by means of the theorem, is more convenient.

It is not difficult to show that in Theorem V the condition $|r(x) - 1| < k/x$ may be replaced by the condition that the series $\sum_{n=\mu}^{n=\infty}(1 - r_n)$ converges, and $r_n \leqq 1$. Then by the Maclaurin-Cauchy integral test, we see that *the series $\sum_{n=\mu}^{n=\infty} u_n$ diverges if the series $\sum_{n=\mu}^{n=\infty}(1 - r_n)$ converges and $r_n \leqq r_{n+1}$.* Of course this test is very weak. In the case of the series

$$1 + \frac{2^p - 1}{2^p} + \frac{(2^p - 1)(3^p - 1)}{(3!)^p} + \cdots,$$

for example, where

$$r_n = 1 - \frac{1}{n^p},$$

this test, like Raabe's first test, indicates divergence only for $p > 1$, though by Theorem V we see that the series also diverges for $p = 1$.

4. A series of products

We have noticed the normal form to which may be reduced any series capable of being tested by means of a test of the second kind. This normal form suggests a somewhat more general class of series. Suppose that we are given the two-dimensional array of numbers

$$\begin{array}{cccc} a_{00} & a_{01} & a_{02} & \cdots \\ a_{10} & a_{11} & a_{12} & \cdots \\ a_{20} & a_{21} & a_{22} & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{array},$$

and suppose that they form a series as follows:

$$a_{00} + a_{01} \cdot a_{10} + a_{02} \cdot a_{11} \cdot a_{20} + a_{03} \cdot a_{12} \cdot a_{21} \cdot a_{30} + \cdots.$$

* *Gesammelte Werke*, vol. 3, pp. 123–163.

The following theorem then holds

THEOREM VI. *Let* $a(x, y)$ *be a positive, integrable function for* $x \geqq 0$, $y \geqq 0$, *such that*

(*a*) $$a(m, n) = a_{m, n};$$

(*b*) $$0 < c < a(x, y) < k;$$

(*c*) *one of the two relations*

$$a(x', y') \leqq a(x', y''), \qquad 0 \leqq x', \qquad 0 \leqq y' < y''$$

or

$$a(x', y') \geqq a(x', y''), \qquad 0 \leqq x', \qquad 0 \leqq y' < y''$$

always holds; and

(*d*) *such that one of the two relations*

$$a(x', y') \leqq a(x'', y'), \qquad 0 \leqq x' < x'', \qquad 0 \leqq y'$$

or

$$a(x', y') \geqq a(x'', y'), \qquad 0 \leqq x' < x'', \qquad 0 \leqq y'$$

always holds.

Then a necessary and sufficient condition for the convergence of the series

$$\sum_{n=0}^{\infty} u_n \qquad \text{where} \qquad u_n = \prod_{m=0}^{n} a_{m, n-m}$$

is the convergence of the integral

$$\int_0^{\infty} e^{\int_0^{\xi} \log a(x, \xi - x)dx} d\xi .$$

Proof. We can take $c < 1$ and $k > 1$. Take the case that $a(x', y)$ and $a(x, y')$ monotonically decrease as y and x respectively increase. Then

(1) $$\log a(n+1, \xi - n) \leqq \int_n^{n+1} \log a(x, \xi - x)\, dx \leqq \log a(n, \xi - n - 1),$$

and

$$\int_n^{n+1} e^{\int_0^{\xi} \log a(x, \xi - x)dx} d\xi = \int_n^{n+1} e^{\overbrace{\int_0^1 + \int_1^2 + \cdots + \int_{n-1}^n + \int_n^{\xi}} \log a(x, \xi - x)dx} d\xi$$

$$\leqq \int_n^{n+1} e^{\log a(0, \xi - 1) + \log a(1, \xi - 2) + \cdots + \log a(n-1, \xi - n) + \log k} d\xi$$

$$\leqq k a_{0, n-1} \cdot a_{1, n-2} \cdot a_{2, n-3} \cdot \cdots \cdot a_{n-1, 0} .$$

Therefore

(2) $$\int_n^{n+1} e^{\int_0^{\xi} \log a(x, \xi - x)dx} d\xi \leqq k u_{n-1} .$$

Likewise

$$\int_n^{n+1} e^{\int_0^\xi \log a(x,\xi-x)dx} d\xi$$

$$\geqq \int_n^{n+1} e^{\log a(1,\xi)+\log a(2,\xi-1)+\cdots+\log a(n,\xi-n+1)+\log C} d\xi$$

$$\geqq ca_{1,\,n+1}\cdot a_{2,\,n}\cdots a_{n,\,2} = \frac{c}{a_{0,\,n+2}\cdot a_{n+1,\,1}\cdot a_{n+2,\,0}} u_{n+2}.$$

Therefore

$$\text{(3)} \qquad \int_n^{n+1} e^{\int_0^\xi \log a(x,\xi-x)dx} d\,\xi \geqq \frac{c}{k^3} u_{n+2}.$$

Similar inequalities hold if $a(x', y)$ and $a(x, y')$ increase monotonically, or if one decreases and the other increases.

The theorem then follows from a comparison of the given series with the series

$$\sum_{n=0}^{\infty} \int_n^{n+1} e^{\int_0^\xi \log a(x,\xi-x)dx} d\xi$$

by means of the inequalities (2) and (3).

The theorem can readily be extended to series for which the function $a(x, y)$ satisfies conditions similar to the conditions in Theorems II and III. A theorem analogous to Theorem V is also easily deduced.

As special cases this theorem in its extended form includes Theorems I, II, and III.

Example. The following series is conveniently tested by means of Theorem VI.

$$e^{-\beta/2-\alpha/2} + e^{-\beta/2-\beta/3-\alpha/3-\alpha/4} + e^{-\beta/2-\beta/3-\beta/4-\alpha/4-\alpha/6-\alpha/6} + \cdots.$$

This is formed from the array

$$\begin{array}{cccc} e^{-\beta/2-\alpha/2} & e^{-\beta/3-\alpha/3} & e^{-\beta/4-\alpha/4} & \cdots \\ e^{-\beta/2-\alpha/4} & e^{-\beta/3-\alpha/6} & e^{-\beta/4-\alpha/8} & \cdots \\ e^{-\beta/2-\alpha/6} & e^{-\beta/3-\alpha/9} & e^{-\beta/4-\alpha/12} & \cdots \\ \cdots & \cdots & \cdots & \cdots. \end{array}$$

Here

$$a_{m,\,n} = e^{-[\beta/(n+2)]-\alpha/[(m+1)(n+2)]},$$

and

$$a(x, y) = e^{-[\beta/(y+2)]-\alpha/[(x+1)(y+2)]},$$

where $0 \leqq x$, $0 \leqq y$. If we take the series as starting with the term $a_{1,\,1}$ instead of with $a_{0,\,0}$, we can write

$$a(x, y) = e^{-\beta/(y+1)-\alpha/(x(y+1))}, \qquad 1 \leqq x, \qquad 1 \leqq y.$$

Then

$$\int_1^\xi \log a(x, \xi - x)\,dx = \int_1^\xi \left[\frac{-\beta}{\xi + 1 - x} - \frac{\alpha}{(\xi + 1)(\xi + 1 - x)} - \frac{\alpha}{(\xi + 1)x}\right] dx$$

$$= -\left(\beta + \frac{2\alpha}{\xi + 1}\right) \log \xi.$$

Consequently

$$\int_m^\xi e^{\int_1^\xi \log a(x,\, \xi - x)dx}\, d\xi = \int_m^\xi \frac{d\xi}{\xi^{\beta + [2\alpha/(\xi+1)]}}.$$

Therefore the series converges for $\beta > 1$, and diverges for $\beta \leqq 1$.

5. Multiple series

The tests given for simple series are easily generalized for multiple series. Thus for double series we have the following theorem:

Theorem VII. *Given the double series*

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{m,\, n}, \qquad u_{m,\, n} > 0.$$

Let $r_m = u_{m+1,\, 0}/u_{m,\, 0}$, *and* $\rho_{m,\, n} = u_{m,\, n+1}/u_{m,\, n}$. *Suppose that* $r(x)$ *is a continuous function for* $0 \leqq x$, *having the properties that* $r(m) = r_m$, $0 < c < r(x) < k$, *and* $r(x)$ *monotonically increases as* x *increases; suppose also that* $\rho(x, y)$ *is a continuous function for* $0 \leqq x$, $0 \leqq y$, *having the properties that* $\rho(m, n) = \rho_{m,\, n}$, $0 < c < \rho(x, y) < k$, *and* $\rho(x, y)$ *increases monotonically when either* x *or* y *increases; then a necessary and sufficient condition for the convergence of the given double series is the convergence of the double integral*

$$\int_0^\infty \int_0^\infty e^{\int_0^x \log r(x)dx + \int_0^y \log \rho(x,\, y)dy}\, dx\, dy.$$

A proof like that of the first theorem can easily be given.

Example. Test the series

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{m,\, n}$$

where $u_{00} = 1$, and

$$u_{m,\, n} = e^{-p-(p/2)-(p/3)-\cdots-[p/(m+n)]}, \qquad m + n > 0.$$

For this series we have

$$r_m = \frac{u_{m+1,\, 0}}{u_{m,\, 0}} = e^{-[p/(m+1)]}, \qquad \rho_{m,\, n} = \frac{u_{m,\, n+1}}{u_{m,\, n}} = e^{-[p/(m+n+1)]},$$

$$r(x) = e^{-[p/(x+1)]}, \qquad \rho(x, y) = e^{-[p/(x+y+1)]}.$$

Here we have

$$\int_0^x \log r(x)\,dx = \int_0^x \frac{-p}{x+1}\,dx = -p\log(x+1),$$

$$\int_0^y \log \rho(x,y)\,dy = \int_0^y \frac{-p}{x+y+1}\,dy = -p\log(x+y+1) + p\log(x+1),$$

and hence

$$\int_0^\infty \int_0^\infty e^{\int_0^x \log r(x)dx + \int_0^y \log \rho(x,\,y)dy}\,dx\,dy = \int_0^\infty \int_0^\infty \frac{dxdy}{(x+y+1)^p}$$

$$= \int_0^\infty \frac{dy}{(p-1)(y+1)^{p-1}}.$$

Therefore the series converges if $p > 2$, and diverges if $p \leqq 2$.

The theorems of this paper have been stated for constant terms. They can readily be extended to series of functions, not only to test the convergence of a given series, but also to determine whether the convergence is uniform; uniform convergence of an integral implies uniform convergence of the corresponding series.

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