SINGULAR POINTS OF ANALYTIC TRANSFORMATIONS*

BY

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The object of the present paper is to discuss the nature of an analytic transformation in the neighborhood of a singular point. Let

$$(A) \qquad x_i = \phi_i(u_1, \cdots, u_n) = \frac{g_i(u_1, \cdots, u_n)}{G_i(u_1, \cdots, u_n)} \qquad (i = 1, \cdots, n),$$

where the functions $\phi_i(u_1, \cdots, u_n)$ are meromorphic in the origin, $(u) = (0)$, and at least one of them has a non-essential singularity of the second kind there.

The functions g_i, G_i are all analytic in the origin, and two of these functions corresponding to the same value of i cannot, of course, both vanish identically. We interpret the point (x) in the space of analysis, and thus it is not necessary to exclude the case that a function G_i vanishes identically.

When two functions g_i, G_i both vanish in the origin, it shall be assumed that they have no common factor there.† In particular, then, if a function g_i or G_i vanishes identically, the other function shall be taken as not vanishing, and may be set equal to unity.

In all cases, at least one pair of functions, as g_i and G_i, vanishes in the origin, and the transformation breaks down at this point (and at others, too, in the neighborhood, when $n > 2$).

Let the positive numbers η_i $(i = 1, \cdots, n)$ be chosen arbitrarily small, and certainly small enough so that each function g_i, G_i is analytic throughout the region

$$|u_i| < \eta_i \qquad (i = 1, \cdots, n).$$

Let the points at which two functions g_i, G_i vanish simultaneously be excluded from this region, and let the remaining region be denoted by $\mathfrak{T}$. Then each point (u) of $\mathfrak{T}$ is carried over into a single point of the (x)-space, and these latter points form a certain manifold, M.

As the quantities η_i are taken smaller and smaller and $\mathfrak{T}$ is thus reduced

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† The leading theorems relating to the factorization of functions of several complex variables which are analytic in a point and vanish there are contained in a memoir by Weierstrass, *Werke*, vol. 2, p. 135. For a systematic presentation of the theory cf. a paper by the author in the Annals of Mathematics, ser. 2, vol. 19 (1917), p. 77.

in extent, this manifold M continually loses points. But there are certain points of the (x)-space which are limiting points of M, no matter how small $\mathfrak{T}$ be taken. *The set of these points shall be denoted by* $\mathfrak{M}$.

It is the study of this latter manifold to which this paper is devoted, and a complete solution of the problem is obtained.*

The result of the investigation is embodied in the theorem of § 10. But the reader will find it convenient to begin with the special cases $\mu = 1, 2, 3$ studied in §§ 1, 6, and 8, respectively.

A further problem, which is in a way a continuation of the present one, is the following. Let

$$x_i = \phi_i(u_1, \cdots, u_n) \qquad (i = 1, \cdots, n),$$

where ϕ_i is meromorphic throughout a fixed region, S, of the (u)-space. Let σ be the manifold of points in S, in which at least one ϕ_i has a non-essential singularity of the second kind, and let σ contain at least one point. Imbed σ in a region T of S, remove the points of σ from T, and denote the remainder of T by $\mathfrak{T}$. Then each point of $\mathfrak{T}$ is carried over by the above transformation into a point of the (x)-space. Denote the manifold of the latter points by N.

As $\mathfrak{T}$ is steadily reduced in extent, so that an arbitrary point of S not lying in σ ultimately becomes and remains an exterior point of $\mathfrak{T}$, there are certain points of the (x)-space which always persist as limiting points of N. *The totality of these latter points forms a manifold* $\mathfrak{N}$.

In the simplest case, namely, $n = 2$, σ consists of isolated points, and hence, if S be taken as closed, $\mathfrak{N}$ consists of a finite number of manifolds $\mathfrak{M}$, each of which is made up of a finite number of algebraic plane curves.

The author reserves the further study of this problem for a later occasion.

1. The Case $\mu = 1$. Preparation for the Higher Cases

Let the equations (A) be so arranged that the first μ functions

$$\frac{g_i(u_1, \cdots, u_n)}{G_i(u_1, \cdots, u_n)}, \qquad (i = 1, \cdots, \mu),$$

have a non-essential singularity of the second kind in the origin, the remaining functions, $i = \mu + 1, \cdots, n$, either being analytic there or having at most a pole. Then $1 \leqq \mu \leqq n$.

The Case $\mu = 1$. This case is immediately disposed of, since x_1 can actually assume and retain any preassigned value, as the point (u) approaches the

* This problem has been studied by Autonne, Acta Mathematica, vol. 21 (1897), p. 249, and some of the results of the present paper are there given. But the treatment is rather a sketch than a detailed investigation. It is not possible to read between the lines that which is necessary for rigorous proofs.

origin, $(u) = (0)$, along a suitable path. No matter what the mode of approach, each x_i, $1 < i$, approaches a definite limit, $x_i = a_i$. Hence $\mathfrak{M}$ is seen to consist of the right line

$$|x_1| \leqq \infty, \qquad x_i = a_i \qquad\qquad (i = 2, \cdots, n).$$

The First of Equations (*B*), *and the Regions* $R_1, \Sigma_1, \overline{R}_1$, *etc.* When $\mu > 1$, we set over against equations (*A*) the following equations:

$$(B) \qquad \begin{cases} g_1(u_1, \cdots, u_n) - x_1 G_1(u_1, \cdots, u_n) = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ g_n(u_1, \cdots, u_n) - x_n G_n(u_1, \cdots, u_n) = 0. \end{cases}$$

Every solution of (*A*), where (x) is a point of the space of analysis, is a solution of (*B*). But a solution of (*B*) yields a solution of (*A*) only when, for no value of i, do g_i and G_i vanish simultaneously.

We will begin with the first of the equations (*B*),

$$(1) \qquad g_1(u_1, \cdots, u_n) - x_1 G_1(u_1, \cdots, u_n) = 0.$$

We may assume without loss of generality that

$$g_1(0, \cdots, 0, u_n) \not\equiv 0, \qquad G_1(0, \cdots, 0, u_n) \not\equiv 0,$$

since by means of a suitable linear transformation of the variables $u_1, \cdots, u_n$ these conditions can always be fulfilled.

If, furthermore, Au_n^p and Bu_n^q are the terms of lowest dimension in u_n when the functions $g_1(0, \cdots, 0, u_n)$ and $G_1(0, \cdots, 0, u_n)$ respectively are expanded into power-series in u_n, then we may assume that $p = q$, since a suitable linear transformation of x_1,

$$x_1' = \frac{\alpha x_1 + \beta}{\gamma x_1 + \delta},$$

together with the transformation

$$g_1'(u) = \alpha g_1(u) + \beta G_1(u),$$

$$G_1'(u) = \gamma g_1(u) + \delta G_1(u)$$

will replace (1) by a new equation of the desired type, while on the other hand belonging to the group of the space of analysis.*

The term of lowest dimension in u_n when the function

$$g_1(0, \cdots, 0, u_n) - x_1 G_1(0, \cdots, 0, u_n)$$

is expanded according to powers of u_n is, then,

$$(A - Bx_1)u_n^p, \qquad A \neq 0, \qquad B \neq 0.$$

* This transformation is made for convenience of presentation. Without it, the investigation that follows can be carried through with only formal changes.

The hyperplane obtained by putting the coefficient of u_n^p equal to 0 is the manifold

$$\mathfrak{S}_1: \qquad A - Bx_1 = 0.$$

From the closed (x)-space we remove the points of $\mathfrak{S}_1$ and denote the remaining space by R_1,—

$$R_1: \qquad 0 < |A - Bx_1| \leqq \infty, \qquad |x_j| \leqq \infty \qquad (j = 2, \cdots, n).$$

We shall sometimes have occasion to consider only the space of the variables $(x_1, \cdots, x_k)$, $k < n$, and then, only that part of this space for which these variables satisfy the above relations. This region we shall denote by R_1^k,

$$R_1^k: \qquad 0 < |A - Bx_1| \leqq \infty, \qquad |x_j| \leqq \infty \qquad (j = 2, \cdots, k).$$

Finally, let a neighborhood Σ_1 of $\mathfrak{S}_1$,

$$\Sigma_1: \qquad |A - Bx_1| < h, \qquad |x_j| \leqq \infty \qquad (j = 2, \cdots, n),$$

be removed from the (x)-space, and let the remaining space, which is closed, be denoted by $\bar{R}_1$,

$$\bar{R}_1: \qquad h \leqq |A - Bx_1| \leqq \infty, \qquad |x_j| \leqq \infty \qquad (j = 2, \cdots, n).$$

The regions

$$\Sigma_1^k: \qquad |A - Bx_1| < h, \qquad |x_j| \leqq \infty \qquad (j = 2, \cdots, k);$$

$$\bar{R}_1^k: \qquad h \leqq |A - Bx_1| \leqq \infty, \qquad |x_j| \leqq \infty \qquad (j = 2, \cdots, k),$$

are now self-explanatory.

Solution of Equation (1). Let $(\xi) = (\xi_1, \cdots, \xi_n)$ be any point of $\bar{R}_1$, and restrict (ξ) to begin with to the finite region. Then the roots of (1) which lie in the neighborhood of the point $(u_1, \cdots, u_n, x_1) = (0, \cdots, 0, \xi_1)$ will be given by the equation*

$$u_n^p + A_1 u_n^{p-1} + \cdots + A_p = 0, \tag{2}$$

where

$$A_i = A_i(u_1, \cdots, u_{n-1}, x_1)$$

is analytic in the point $(u_1, \cdots, u_{n-1}, x_1) = (0, \cdots, 0, \xi_1)$ and vanishes there.

Next, if ξ_1' is any point of a certain neighborhood of ξ_1 and the equation (2) is written down for the new point $(u_1, \cdots, u_{n-1}, x_1) = (0, \cdots, 0, \xi_1')$, the new coefficients A_i' will coincide respectively throughout a certain neighborhood of the point $(u_1, \cdots, u_{n-1}, x_1) = (0, \cdots, 0, \xi_1')$ with the coefficients A_i, considered for the same neighborhood.

From this it is seen that the coefficients A_i admit analytic continuation along every finite path of the manifold whose points $(u_1, \cdots, u_{n-1}, x_1)$ are

* Weierstrass, *Werke*, vol. 2, p. 135. Cf. also *Madison Colloquium*, pp. 181–183.

subject to the condition

$$u_k = 0, \qquad k = 1, \cdots, n-1; \qquad h \leqq |A - Bx_1| < \infty .$$

They can, however, also be continued analytically to the point

$$(u_1, \cdots, u_{n-1}, x_1) = (0, \cdots, 0, \infty)$$

along a path in said manifold. For, let G be taken so that

$$G > \left|\frac{A}{B}\right|,$$

and let

$$x_1' = \frac{1}{x_1}.$$

Then, if $|x_1| \geqq G$ and if (1) is replaced by

$$(1') \qquad x_1' g_1(u_1, \cdots, u_n) - G_1(u_1, \cdots, u_n) = 0,$$

the roots of (1′) which lie in a certain neighborhood of the point $(0, \cdots, 0, \xi_1')$, where

$$|\xi_1'| \leqq \frac{1}{G},$$

will be given by an equation

$$(2') \qquad u_n^p + A_1' u_n^{p-1} + \cdots + A_p' = 0,$$

where

$$A_i' = A_i'(u_1, \cdots, u_{n-1}, x_1')$$

is analytic in the point $(u_1, \cdots, u_{n-1}, x_1') = (0, \cdots, 0, \xi_1')$ and vanishes there.

These coefficients A_i' can be continued analytically along any path of the manifold

$$u_k = 0, \qquad k = 1, \cdots, n-1; \qquad |x_i'| \leqq \frac{1}{G}.$$

In the neighborhood of any one of these points $(0, \cdots, 0, \xi_1')$ for which $\xi_1' \neq 0$,

$$A_i'(u_1, \cdots, u_{n-1}, x_1') = A_i(u_1, \cdots, u_{n-1}, x_1),$$

and hence the above statement is seen to be true.

Finally, since h can be taken arbitrarily small, we are led to the following result. *The coefficients* $A_i(u_1, \cdots, u_{n-1}, x_1)$, $i = 1, \cdots, p$, *in* (2) *are single-valued and analytic at every point of the manifold*

$$(3) \qquad (u_1, \cdots, u_{n-1}, x_1) = (0, \cdots, 0, x_1), \qquad 0 < |A - Bx_1| \leqq \infty,$$

and vanish there.

Concerning the algebroid polynomial F that forms the left-hand side of (2) we can assert furthermore that it is *irreducible, i. e.*, that a relation of the form cannot exist:

$$F = (u_n^q + B_1 u_n^{q-1} + \cdots)(u_n^s + C_1 u_n^{s-1} + \cdots),$$

where the coefficients B_j and C_k are functions of $(u_1, \cdots, u_{n-1}, x_1)$ analytic at all points $(0, \cdots, 0, \xi_1)$, ξ_1 being any point of R_1^1. For,

(*a*) F cannot have two essentially distinct factors. Let ξ_1 be any point of R_1^1. Then it is clear that a point $(u_1', \cdots, u_{n-1}')$ can be chosen arbitrarily near the origin and a point ξ_1' arbitrarily near ξ_1 so that, if $u_n^{(1)}$ and $u_n^{(2)}$ denote any two roots of (2) corresponding to the point $(u_1, \cdots, u_{n-1}, x_1) = (u_1', \cdots, u_{n-1}', \xi_1')$, then

$$G_1(u_1', \cdots, u_{n-1}', u_n^{(i)}) \neq 0 \qquad (i = 1, 2).$$

For, the condition that F and G_1 have a common root is given by the vanishing of the resultant R of (2) and (2′), the latter equation being written for $x_1' = 0$:

$$R(u_1, \cdots, u_{n-1}, x_1) = 0.$$

This function is analytic in the point $(0, \cdots, 0, \xi_1)$ and does not vanish identically, since g_1 and G_1 have no common factor in the origin.

Next, it is possible to join the points $(u_1', \cdots, u_{n-1}', u_n^{(1)})$ and $(u_1', \cdots, u_{n-1}', u_n^{(2)})$ by a curve lying in the neighborhood of the origin, in no point of which does G_1 vanish nor does $g_1/G_1 = A/B$. Hence a path on the configuration (2) is determined, along which $u_n^{(1)}$ is carried over into $u_n^{(2)}$.

(*b*) The function F cannot have a multiple factor,

$$P = u_n^q + B_1 u_n^{q-1} + \cdots + B_q.$$

For then the function $g_1 - x_1 G_1$ would also admit this factor in a point $(u_1, \cdots, u_n, x_1) = (0, \cdots, 0, \xi_1)$, where ξ_1 is in R_1^1:

$$g_1 - x_1 G_1 = P^l Q.$$

Hence

$$\frac{\partial}{\partial x_1}(g_1 - x_1 G_1) = P^{l-1} Q_1 = - G_1.$$

Thus $g_1 - x_1 G_1$ and G_1 would have a common factor in $(0, \cdots, 0, \xi_1)$, and consequently g_1 and G_1 would have a common factor in the origin. But this is contrary to hypothesis.

2. The Functions Ω_i

We turn now to the equations (*B*), for which $i = 2, \cdots, \mu$. Let

$$\omega_i(u_1, \cdots, u_n, x_i) = g_i(u_1, \cdots, u_n) - x_i G_i(u_1, \cdots, u_n),$$

and consider these functions on the manifold (2). It follows from the hypotheses that no G_i vanishes identically. Form the functions

$$\Omega_i(u_1, \cdots, u_{n-1}, x_1, x_i) = \prod_{k=1}^{p} \omega_i(u_1, \cdots, u_{n-1}, u_n^{(k)}, x_i) \quad (i = 2, \cdots, \mu),$$

where $u_n^{(k)}$ are the roots of (2). Then Ω_i is an algebroid polynomial in x_i, and furthermore it is of degree p, as we will now show.

Equation (2) has been shown to be irreducible. Its left-hand member was denoted by F. Let $(u_1^0, \cdots, u_{n-1}^0, x_1^0)$, where $(u_1^0, \cdots, u_{n-1}^0)$ lies in the neighborhood of the origin and x_1^0 lies in R_1^1, be a point in which all the roots of F are distinct from one another. Then

$$F = \prod_{j=1}^{p} (u_n - f_j),$$

where $f_j(u_1, \cdots, u_{n-1}, x_1)$ is analytic in the point $(u_1^0, \cdots, u_{n-1}^0, x_1^0)$ and actually contains x_1.

The coefficient of x_i^p in the expression for Ω_i is

$$\chi(u_1, \cdots, u_{n-1}, x_1) = (-1)^p \prod_{k=1}^{p} G_i(u_1, \cdots, u_{n-1}, u_n^{(k)}),$$

and we wish to show that this function does not vanish identically. If it did, then one of its factors, considered in the neighborhood of the point $(u_1^0, \cdots, u_{n-1}^0, x_1^0)$, must vanish identically. Let $k = 1$ and $j = 1$ correspond to this factor:

$$u_n^{(1)} = f_1(u_1, \cdots, u_{n-1}, x_1),$$

and let $u_n^0 = f_1(u_1^0, \cdots, u_{n-1}^0, x_1^0)$. Then $G_i(u_1, \cdots, u_n)$ must vanish whenever the function

$$u_n - f_1(u_1, \cdots, u_{n-1}, x_1),$$

which is analytic in the point $(u_1^0, \cdots, u_{n-1}^0, u_n^0, x_1^0)$ and vanishes there, and moreover, is irreducible there, vanishes. Hence $G_i(u_1, \cdots, u_n)$ must be divisible by this function at the point in question. But this is impossible, since f_1 actually contains x_1, while G_i does not.

The functions Ω_i are readily seen to be analytic at every point of the manifold

$$u_k = 0, \qquad k = 1, \cdots, n-1; \qquad 0 < |A - Bx_1| \leqq \infty, \qquad |x_j| \leqq \infty \quad (j = 2, \cdots, n),$$

except at those points in which $x_i = \infty$, and in the neighborhood of such a point the function

$$\Omega_i'(u_1, \cdots, u_{n-1}, x_1, x_i') = x_i'^{\,p}\, \Omega_i(u_1, \cdots, u_{n-1}, x_1, x_i) \quad \left(x_i' = \frac{1}{x_i}\right),$$

is seen to be analytic. For, in a point $(u_1, \cdots, u_{n-1}, x_1)$ in which each irreducible factor of (2) has all its roots distinct, Ω_i is evidently analytic. The excepted points form a manifold of a lower order of dimensions, and in its points Ω_i is continuous. Hence, by the extension of Riemann's theorem for removable singularities of functions of a single variable,* Ω_i is analytic in these points also.

3. The Function $\Phi(x_1, x_2)$

It may happen that the function $\Omega_2(u_1, \cdots, u_{n-1}, x_1, x_2)$ is divisible in the point $(0, \cdots, 0, \xi_1, \xi_2)$, where (ξ_1, ξ_2) is a point of the region R_1^2, by a function $f(u_1, \cdots, u_{n-1})$. Let the other factor, which shall not be divisible by such a function, be developed into a series of homogeneous polynomials in $(u_1, \cdots, u_{n-1})$, the coefficients being functions of (x_1, x_2) analytic in the region R_1^2; and let r be the order of the first of these terms. Then, on making if necessary a linear transformation of $(u_1, \cdots, u_{n-1})$, we can ensure the presence of the term in u_{n-1}^r. Let its coefficient be $\Phi(x_1, x_2)$:

$$\Omega_2(u_1, \cdots, u_{n-1}, x_1, x_2)$$
$$= f(u_1, \cdots, u_{n-1})\{\Phi(x_1, x_2)u_{n-1}^r + (u_1, \cdots, u_{n-1})\},$$

where the parenthesis, when developed into a power series in $u_1, \cdots, u_{n-1}$, contains no terms of lower than the rth order, and the term in u_{n-1}^r is lacking.

The function $\Phi(x_1, x_2)$ is a polynomial in x_2 which does not vanish identically, and the coefficients are each analytic in R_1^1, the degree of Φ in x_2 being $\lambda \leqq p$. Hence $\Phi(x_1, x_2)$ itself is analytic at all points of R_1^2 for which $|x_2| < \infty$, and $x_2^{-\lambda}\Phi(x_1, x_2)$ is analytic at the remaining points of R_1^2.

A necessary condition for a simultaneous solution of the first two equations (A) is the following:

$$\Phi(x_1, x_2)u_{n-1}^r + (u_1, \cdots, u_{n-1}) = 0. \tag{4}$$

For, if $(u_1', \cdots, u_n', x_1', x_2')$ be such a solution, then $\Omega_2(u_1', \cdots, u_{n-1}', x_1', x_2') = 0$. Suppose $f(u_1', \cdots, u_{n-1}') = 0$. It is possible to find a point $(u'') = (u_1'', \cdots, u_n'')$ indefinitely near to (u') for which $f(u_1'', \cdots, u_{n-1}'') \neq 0$. The corresponding values $x_1 = x_1''$, $x_2 = x_2''$ lie near to x_1', x_2', and for $(u_1'', \cdots, u_{n-1}'', x_1'', x_2'')$ equation (4) is satisfied. From the continuity of the left hand side of (4) it follows, then, that (4) is also satisfied in $(u_1', \cdots, u_{n-1}', x_1', x_2')$.

We distinguish two cases: Case 1, $r = 0$; Case 2, $r > 0$.

Conversely, every solution of (4) yields a solution of the first two equations (B), but not necessarily a solution of the corresponding equations (A). For, it makes $\Omega_2 = 0$, and hence

$$\omega_2(u_1, \cdots, u_{n-1}, u_n^{(k')}, x_1, x_2) = 0,$$

* Cf. *Madison Colloquium*, p. 163.

and (2) is also satisfied by $u_n = u_n^{(k')}$. Indefinitely near to a given solution of (4) lies, however, a second solution of (4) which does yield a solution of (A).

4. Case 1, $r = 0$

In this case we have

$$\Omega_2(u_1, \cdots, u_{n-1}, x_1, x_2) = f(u_1, \cdots, u_{n-1})\{\Phi(x_1, x_2) + (u_1, \cdots, u_{n-1})\},$$

and (4) becomes

$$\Phi(x_1, x_2) + (u_1, \cdots, u_{n-1}) = 0, \tag{5}$$

where the ( ) vanishes for $u_1 = 0, \cdots, u_{n-1} = 0$.

If $(x) = (\xi)$ is a point not lying on $\mathfrak{S}_1$ and such that

$$\Phi(\xi_1, \xi_2) \neq 0,$$

then it is clear that a neighborhood of (ξ) and a region $\mathfrak{T}$ can be so chosen that, when (x) lies in the first region and (u) in the second, equation (5) cannot be satisfied. Hence for no such pair of points (x), (u) can (A) be satisfied, and consequently (ξ) is not a point of $\mathfrak{M}$. It follows, then, that the manifold $\mathfrak{M}$ is contained in the hyperplane $\mathfrak{S}_1$ and the points of the cylinder

$$x_1 = \xi_1, \qquad x_2 = \xi_2, \qquad |x_j| \leqq \infty \qquad (j = 3, \cdots, n), \tag{6}$$

where

$$\Phi(\xi_1, \xi_2) = 0.$$

In case $\xi_2 = \infty$, $\Phi(x_1, x_2)$ is to be replaced by

$$\Phi'(x_1, x_2') = x_2^{-\lambda}\Phi(x_1, x_2) \qquad \left(x_2' = \frac{1}{x_2}\right).$$

We will now show that certain elements (6) of the cylinder

$$\Phi(x_1, x_2) = 0 \tag{7}$$

contain at least one point of $\mathfrak{M}$. To do this we will show that the degree λ of Φ in x_2 is positive.

Let ξ_1 be chosen arbitrarily. Ω_2 is a polynomial in x_2 of degree p, the coefficient of x_2 being $\chi(u_1, \cdots, u_{n-1}, x_1)$. We can, then, find in an arbitrary neighborhood of ξ_1 a point ξ_1', and, independently of the choice of ξ_1', in an arbitrary neighborhood of the origin a point $(u_1', \cdots, u_{n-1}')$ such that

$$\chi(u_1', \cdots, u_{n-1}', \xi_1') \neq 0,$$

and hence

$$f(u_1', \cdots, u_{n-1}') \neq 0.$$

Let $\phi(x_1)$ be the coefficient of x_2^λ in $\Phi(x_1, x_2)$. We restrict ξ_1' furthermore so that

$$\phi(\xi_1') \neq 0.$$

Let x_2' be a root of the equation

$$\Omega_2(u_1', \cdots, u_{n-1}', \xi_1', x_2) = 0,$$

and hence of the equation

$$\Phi(\xi_1', x_2) + (u_1', \cdots, u_{n-1}') = 0. \tag{8}$$

We now infer that $\lambda > 0$. For, if λ were $= 0$, we should have

$$\Phi(\xi_1', x_2) = \phi(\xi_1') \neq 0,$$

and by choosing $(u_1', \cdots, u_{n-1}')$ sufficiently near the origin, the second term in (8) could be made indefinitely small. This leads to a contradiction.

The roots of the equation

$$\Phi(\xi_1', x_2) + (u_1, \cdots, u_{n-1}) = 0 \tag{9}$$

when $(u_1, \cdots, u_{n-1})$ is taken at the origin, coincide with the λ roots of the equation

$$\Phi(\xi_1', x_2) = 0. \tag{10}$$

Hence, for all points $(u_1, \cdots, u_{n-1})$ of a certain neighborhood of the origin, the roots of the former equation lie near those of the latter.

Let the positive numbers η_i, $i = 1, \cdots, n$, be chosen at pleasure, and let ξ_1' be a point of the neighborhood of ξ_1 such that $\phi(\xi_1') \neq 0$. Then ξ_1' and a point $(u_1^0, \cdots, u_{n-1}^0)$ can be so determined that the corresponding roots u_n^0 of (2), x_1 being set $= \xi_1'$, will be distinct, and each will yield a point for which

$$|u_i^0| < \eta_i \qquad (i = 1, \cdots, n).$$

Moreover, if ξ_2' be an arbitrary root of equation (10) and an arbitrarily small neighborhood of ξ_2' be chosen, the point $(u_1^0, \cdots, u_{n-1}^0)$ can then be so taken that a root of (9) will lie in this neighborhood.

Furthermore, the point $(u_1^0, \cdots, u_{u-1}^0)$ can be so determined that, no matter what root of (2) be associated with it, the functions g_1, G_1, and likewise the functions g_2, G_2, will not both vanish in $(u^0) = (u_1^0, \cdots, u_n^0)$.

Lastly, the point $(u_1^0, \cdots, u_{n-1}^0, \xi_1')$ can be so taken that, for all points $(u_1, \cdots, u_{n-1}, x_1)$ in its neighborhood, the roots x_2 of (9), as well as the roots u_n of (2), can be grouped together so as to constitute functions each analytic in $(u_1^0, \cdots, u_{n-1}^0, \xi_1')$.

We are led, then, to the following result.

In the region of the space of the variables $(u_1, \cdots, u_{n-1}, x_1)$ for which $(u_1, \cdots, u_{n-1})$ lies in the neighborhood of $(u_1^0, \cdots, u_{n-1}^0)$ and x_1 in the neighborhood of ξ_1', the p roots u_n of (2) *are analytic functions.*

Each of these, when substituted in the second of the equations (*B*) *or* (*A*) *yields a function x_2 likewise analytic in the point $(u_1^0, \cdots, u_{n-1}^0, \xi_1')$.*

The systems of values $(u_1, \cdots, u_n, x_1, x_2)$ thus found satisfy the first two of *the equations* (A) *and* (B), *and they exhaust all such systems for which* $(u_1, \cdots, u_{n-1})$ *lies in the neighborhood of* $(u_1^0, \cdots, u_{n-1}^0)$ *and* x_1 *in the neighborhood of* ξ_1'.

Equation (5) is an algebroid equation in x_2. Its left-hand side cannot have two distinct irreducible factors, each being of the form

$$E_0 x_2^l + E_1 x_2^{l-1} + \cdots + E_l, \qquad (0 < l),$$

where $E_k = E_k(u_1, \cdots, u_{n-1}, x_1)$ is analytic in each point $(0, \cdots, 0, \xi_1)$ for which ξ_1 lies in R_1^1. For, the roots, x_2, of this equation are precisely the values that the function

$$x_2 = \frac{g_2(u_1, \cdots, u_n)}{G_2(u_1, \cdots, u_n)}$$

takes on in the points of the configuration (2), and since the latter is irreducible, there must be a path in the $(u_1, \cdots, u_{n-1}, x_1)$-space along which a given root of (5) is carried over into any second root of (5).

The locus (7) consists conceivably in part of right lines,—or hyperplanes, if interpreted in the (x)-space,—

$$x_1 = \text{const.},$$

and these may conceivably cluster about the line $x_1 = A/B$,—or the hyperplane $\mathfrak{S}_1$.

The remainder of the locus consists of a finite number of monogenic analytic configurations

$$C_1: \qquad x_2 = \psi(x_1),$$

where $\psi(x_1)$ is a finitely multiple-valued function of x_1 having at most ordinary branch-points and poles in R_1^1. If $\Phi_1(x_1, x_2)$ be the product of the distinct irreducible factors of $\Phi(x_1, x_2)$ regarded as an algebroid polynomial in x_2, each factor being taken as primitive, then the totality of the configurations C_1 is also represented by the equation

$$\Phi_1(x_1, x_2) = 0.$$

From the foregoing we infer that *if* (ξ_1, ξ_2) *be any point of* C_1, *then the element*

$$x_1 = \xi_1, \qquad x_2 = \xi_2, \qquad |x_j| \leqq \infty \qquad (j = 3, \cdots, n),$$

contains at least one point of $\mathfrak{M}$.

For, this element is a closed manifold in the space of analysis, and in every neighborhood of this element there are points of M, no matter how far $\mathfrak{T}$ be restricted. We have shown, namely, that a point (u) can be found indefinitely near the origin, for which the first two functions $\phi_i(u_1, \cdots, u_n)$ of (A) are defined and yield values of x_1, x_2 indefinitely near ξ_1, ξ_2 respectively. This point (u) can then be slightly modified, if necessary, so that all the functions ϕ_i will be analytic there.

The Algebraic Character of the Configurations C_1. Let x_1 and x_2 interchange their roles, *i. e.*, let the second equation (B) be solved for u_n, thus giving an equation

$$(2') \qquad u_n^{p'} + A_1'(u_1, \cdots, u_{n-1}, x_2)\, u_n^{p'-1} + \cdots = 0 .$$

By the foregoing considerations we are led to a function

$$\Omega_2'(u_1, \cdots, u_{n-1}, x_2, x_1) = f'(u_1, \cdots, u_{n-1})\{\Phi'(x_2, x_1)\, u_n^{r'} + (u_1, \cdots, u_{n-1})\},$$

and we infer at once that $r' = 0$. For otherwise it would be possible to choose a point (x_1, x_2) not on $\mathfrak{S}_1$ or the locus (7), and also not on

$$\mathfrak{S}_1': \qquad A' - B' x_2 = 0$$

or on the locus

$$(11) \qquad \Phi'(x_2, x_1) = 0,$$

and then determine a point (u) arbitrarily near the origin, for which the first two of the equations (A) admit a simultaneous solution. But here is a contradiction.

From the foregoing theory, the locus (11) may consist in part of right lines $x_2 = \text{const}$. It will surely consist in part of a finite number of monogenic analytic configurations

$$C_1': \qquad x_1 = \psi_1(x_2),$$

where ψ_1 is finitely multiple-valued and has at most ordinary branch-points and poles in the region $R_1'^{\,1}$.

Combining these two results we see that a given configuration C_1 may reduce to a straight line $x_2 = \text{const}$. If this is true of every configuration C_1, then each configuration C_1' must reduce to a straight line $x_1 = \text{const}$.

If, however, there is a configuration C_1 not such a right line, then C_1 and a certain C_1' must be the same monogenic analytic configuration; for, each element (6), (ξ_1, ξ_2) being a point of C_1, contains a point of $\mathfrak{M}$, and if $x_2 = \xi_2 \neq x_2'$ is not one of the lines which form part of the locus (11), then the above element must lead to a point (ξ_1, ξ_2) of some C_1'.

The configuration C_1 is algebraic. The curve is known to have at most an ordinary singularity* at any one of its points save, perhaps, the point (x_1', x_2'), in which the lines

$$A - Bx_1 = 0, \qquad A' - B' x_2 = 0$$

* A curve, surface, etc., is said to have an *ordinary singularity* at a point A if in the neighborhood of A it consists of one or more of the manifolds which are given by a set of equations of the form

$$G_k(x_1, \cdots, x_n) = 0 \qquad (k = 1, \cdots, \nu < n),$$

where G_k is analytic at A and vanishes there, but does not vanish identically.

intersect. For values of x_1 in the neighborhood of x_1', but distinct from this point, the equation of C_1 is

$$\Phi_0(x_1, x_2) = 0,$$

where Φ_0 is an algebroid polynomial in x_2 whose coefficients are analytic in the neighborhood of x_1' with the possible exception of this point. Moreover, on making if necessary a linear transformation of x_2, all the roots x_2 of this equation will remain finite. For, if $\xi_2 \neq x_2'$ be a finite cluster-point of roots x_2 when x_1 approaches x_1', then the point $(x_1, x_2) = (x_1', \xi_2)$ is a point of C_1', and C_1' is given near this point by the equation

$$\Phi_0'(x_2, x_1) = 0,$$

Φ_0' being analytic at $(x_2, x_1) = (\xi_2, x_1')$. Hence the coefficient of the highest power of x_2 in $\Phi_0(x_1, x_2)$ can be reduced to unity, and the other coefficients remain finite at x_1'. They have, therefore, at most removable singularities there, and thus the statement is proven.

The above investigation includes the case that the ( ) in (5) vanishes identically. Let (u^0) be a point of the neighborhood of the origin, in which neither G_1 nor G_2 vanishes. Then x_1 and x_2, computed for any point (u) in the neighborhood of (u^0), will necessarily make $\Phi(x_1, x_2)$ vanish. Consequently, if x_1 and x_2 are replaced in $\Phi(x_1, x_2)$ by g_1/G_1 and g_2/G_2 respectively, the resulting function of $(u_1, \cdots, u_n)$ vanishes identically in those variables. Hence g_1/G_1 and g_2/G_2 are connected by an algebraic relation.

More generally, if the left-hand side of (5) is divisible by an irreducible algebroid polynomial in x_2 with coefficients in x_1 alone, analytic in R_1^1, or by a power of such an algebroid polynomial, the other factor is of the form

$$\theta(x_1) + (u_1, \cdots, u_{n-1}),$$

where $\theta(x_1)$ is analytic in R_1^1 and the ( ) vanishes at the origin. For, this factor is an algebroid polynomial in x_2 of degree 0.

The Manifold $\mathfrak{C}_1$. The points (x_1, x_2) which lie on the manifolds C_1 and C_1' shall constitute the manifold $\mathfrak{C}_1^2$, and $\mathfrak{C}_1$ *shall consist of the points*

$$\mathfrak{C}_1: \qquad x_1 = \xi_1, \qquad x_2 = \xi_2, \qquad |x_j| \leqq \infty \qquad (j = 3, \cdots, n),$$

where (ξ_1, ξ_2) traces out $\mathfrak{C}_1^2$.

Those points of the cylinder $\mathfrak{C}_1$, for which (ξ_1, ξ_2) is a fixed point form an *element* of $\mathfrak{C}_1$.

THEOREM. *In Case* 1, $r = 0$, $\mathfrak{M}$ *lies on* $\mathfrak{C}_1$, *and each element of* $\mathfrak{C}_1$ *contains at least one point of* $\mathfrak{M}$.

For, no point of $\mathfrak{M}$ can lie in R_2 or R_2'. The only remaining points are those which lie on $\mathfrak{C}_1$ and those for which (x_1, x_2) is at the intersection of one of the exceptional lines $x_1 = \text{const.}$ (i. e., $\mathfrak{S}_1$ or $\phi(x_1) = 0$) with a similar line, $x_2 = \text{const.}$ But $\mathfrak{M}$ is a connected manifold, and hence this case is excluded.

5. Case 2, $r > 0$

The Manifold $\mathfrak{S}_2$ and the Regions R_2, $\bar{R}_2$, R_2^k, $\bar{R}_2^k$. The manifold $\mathfrak{S}_2$ shall be defined as consisting of those points of R_1 in which Φ vanishes,

$$\mathfrak{S}_2: \qquad \Phi(x_1, x_2) = 0.$$

The remaining points of R_1 constitute the region R_2.

Let $\mathfrak{S}_2$ be imbedded in a region Σ_2 and let the points of Σ_2 which lie in $\bar{R}_1$ be removed from this region. The remainder of $\bar{R}_1$ forms the region $\bar{R}_2$, which shall be taken as *closed.*

The definitions of R_2^k, $\bar{R}_2^k$ are now given precisely as in the earlier case for R_1, $\bar{R}_1$.

The Equation

$$\Phi(x_1, x_2)u_{n-1}^r + (u_1, \cdots, u_{n-1}) = 0. \tag{12}$$

Let (ξ_1, ξ_2) be a point of R_2^2. Then equation (12) is equivalent to the following:

$$u_{n-1}^r + B_1 u_{n-1}^{r-1} + \cdots + B_r = 0, \tag{13}$$

$$B_k = B_k(u_1, \cdots, u_{n-2}, x_1, x_2), \qquad B_k(0, \cdots, 0, x_1, x_2) = 0,$$

where (x_1, x_2) lies, to begin with, in the neighborhood of (ξ_1, ξ_2).

It is shown as in the case of equation (2) that $B_k(u_1, \cdots, u_{n-2}, x_1, x_2)$ is analytic at every point of the manifold

$$u_k = 0, \qquad k = 1, \cdots, n-2; \qquad (x_1, x_2) \text{ in } R_2^2.$$

Let (ξ_1, ξ_2) be an arbitrary point of R_2^2. It is then possible to find a point (u^0) within an arbitrarily preassigned neighborhood of the origin, for which the first two of the equations (B), written for $x_1 = \xi_1$, $x_2 = \xi_2$, are satisfied. Thus (u^1) can be chosen near (u^0) so that no pair of functions g_i, G_i vanish in (u^1), the corresponding point (ξ_1', ξ_2') lying near (ξ_1, ξ_2). Hence for (u^1) equations (A) all have a meaning, and *thus it appears that there is at least one point of $\mathfrak{M}$ on the manifold*

$$x_1 = \xi_1, \qquad x_2 = \xi_2.$$

Since $\mathfrak{M}$ is perfect, the restriction that (ξ_1, ξ_2) be a point of R_2^2 can now be removed, and *the result just obtained is seen to hold when the point (ξ_1, ξ_2) is wholly arbitrary.*

6. The Case $\mu = 2$

When $\mu = 2$, the last $n - 2$ equations (A) each give for x_i a definite limiting value a_i, no matter how (u) approaches the origin, and we are thus led to the following result:

THEOREM. *When $\mu = 2$ and $r = 0$, the manifold consists of the intersection of the (reducible or irreducible) algebraic cylinder $\mathfrak{C}_1$ with the hyperplanes $x_j = a_j$, $j = 3, \cdots, n$.*

In particular, if $n = 2$, r is necessarily 0, and $\mathfrak{M}$ is the algebraic plane curve $\mathfrak{C}_1^2 = \mathfrak{C}_1$.

When $\mu = 2$ and $r > 0$, $\mathfrak{M}$ consists of the linear manifold

$$x_j = a_j \qquad (j = 3, \cdots, n).$$

7. THE GENERAL METHOD

We have discussed all cases in which $\mu = 1$ and $\mu = 2$, and the treatment of the general case, $\mu = \mu$, is already clearly indicated. When $\mu > 2$, however, there are still points requiring further development.

The argument here is as follows: Let (ξ_1, ξ_2) be a point of R_2^2, and let $r > 0$. Then (12) can be solved for u_{n-1} by (13). Moreover, a point (ξ_1', ξ_2') can be found near (ξ_1, ξ_2) and a point $(u_1', \cdots, u_{n-2}')$ near the origin such that one branch of (13),

$$u_{n-1} = f(u_1, \cdots, u_{n-2}, x_1, x_2)$$

will be analytic at $(u_1', \cdots, u_{n-2}', \xi_1', \xi_2')$ and

$$\frac{\partial f}{\partial x_2} \neq 0$$

there. Hence the last equation can be solved for x_2:

$$x_2 = \lambda(u_1, \cdots, u_{n-1}, x_1),$$

where λ is analytic at $(u_1', \cdots, u_{n-1}', \xi_1')$.

It is now possible to find a point $(u_1'', \cdots, u_{n-1}'', \xi_1'')$ near $(u_1', \cdots, u_{n-1}', \xi_1')$ such that, no matter what root u_n'' of (2) be associated with it, (A) will be defined in (u''). On the other hand, (ξ_1'', ξ_2'') lies near (ξ_1, ξ_2).

We begin as in the case $\mu = 2$ by forming the functions Ω_i and $\Phi(x_1, x_2)$, and we distinguish the two cases, $r = 0$ and $r > 0$. For $r = 0$, the discussion of § 4 is complete.

It may happen that for every pair of the μ equations in question, $r = 0$. Then $\mathfrak{M}$ lies at once on each of a set of (reducible or irreducible) algebraic cylinders

$$\Phi_k(x_k, x_1) = 0, \qquad (k = 2, \cdots, \mu),$$

and has at least one point in each element of such a cylinder. Moreover,

$$x_j = a_j, \qquad (j = \mu + 1, \cdots, n).$$

Hence $\mathfrak{M}$ lies on a finite number of irreducible algebraic curves in the (x)-

space. That $\mathfrak{M}$ consists of all the points of a certain number of these can be shown by a linear transformation. The details are indicated in the treatment of the special case of § 8.

When, on the other hand, for some pair of the μ equations $r > 0$, let these be the first two equations. Form the functions

$$X_j(u_1, \cdots, u_{n-2}, x_1, x_2, x_j) = \prod_{k=1}^{r} \Omega_j(u_1, \cdots, u_{n-2}, u_{n-1}^{(k)}, x_1, x_j) \qquad (j = 3, \cdots, n),$$

where $u_{n-1}^{(k)}$ denotes a root of (12) or (13). On making, if necessary, a linear transformation of $u_1, \cdots, u_{n-2}$ we can write

$$X_3(u_1, \cdots, u_{n-2}, x_1, x_2, x_3)$$
$$= f_1(u_1, \cdots, u_{n-2})\{\Psi(x_1, x_2, x_3)u_{n-2}^s + (u_1, \cdots, u_{n-2})\},$$

where the $\{\ \}$ admits no factor in $u_1, \cdots, u_{n-2}$ alone, and the $(\ )$, when developed into a power series, contains only terms of at least the sth dimension, the term in u_{n-2}^s not appearing.

This equation is the precise analogue of the earlier equation $\Omega_2(u_1, \cdots, u_{n-1}, x_1, x_2) = 0$, and the treatment follows exactly the same lines.

First, X_3 is seen to be a polynomial in x_3 of degree pr, the coefficients being analytic in the points $(u_1, \cdots, u_{n-2}, x_1, x_2) = (0, \cdots, 0, \xi_1, \xi_2)$, where (ξ_1, ξ_2) is any point of R_2^2. $\Psi(x_1, x_2, x_3)$ is also a polynomial in x_3 with coefficients which are analytic in R_2^2.

A necessary condition that the first three equations (A) admit a simultaneous solution $(u_1, \cdots, u_n, x_1, x_2, x_3)$ is that

$$\Psi(x_1, x_2, x_3)u_{n-2}^s + (u_1, \cdots, u_{n-2}) = 0.$$

Again, we distinguish two cases: Case 1, $s = 0$; Case 2, $s > 0$.

In Case 1 it is shown as before that Ψ is of positive degree in x_3. The algebroid polynomial Ψ may admit factors which depend on x_1 and x_2 only. If these are suppressed and the remaining factor is denoted by Ψ_1, then the equation

$$\Psi_1(x_1, x_2, x_3) = 0 \tag{14}$$

defines a finite number of monogenic analytic configurations

$$C_2: \qquad x_3 = \psi(x_1, x_2),$$

where ψ denotes a finitely multiple-valued function having at most ordinary singular points* when (x_1, x_2) lies in R_2^2, as is shown by the equation (14).

Denote the coefficient of the highest power of x_3 in $\Psi(x_1, x_2, x_3)$ by $\phi(x_1,$

* A *function* is said to have an *ordinary singular point* if the corresponding analytic configuration has an ordinary singular point.

x_2). Then ϕ is analytic at every point of R_2^2. Denote the locus $\phi = 0$ in the space of the variables $(x_1, \cdots, x_n)$ by

$$S_2: \qquad \phi(x_1, x_2) = 0,$$

and in the $(x_1, \cdots, x_k)$-space by S_2^k. S_2 is regular at every point of R_2.

Let (ξ_1, ξ_2, ξ_3) be a point of C_2. Then the element

$$x_i = \xi_i, \qquad i = 1, 2, 3; \qquad |x_j| \leqq \infty \qquad (j = 4, \cdots, n),$$

contains at least one point of $\mathfrak{M}$. Furthermore, if (ξ_1, ξ_2) lies in R_2^2, but not on S_2^2, the only points of $\mathfrak{M}$ are those which belong to C_2.

If $\psi(x_1, x_2) =$ const., then all points of the corresponding hyperplane, $x_3 =$ const., which is surely algebraic, belong to $\mathfrak{M}$, when $n = 3$, and lead to points of $\mathfrak{M}$ when $n > 3$.

Suppose ψ actually involves at least one of the letters x_1 and x_2, say x_2. We will now allow x_2 and x_3 to permute their rôles. We get, then, a function X_3' with an s', and it is seen as in the earlier case that s' must vanish. Thus we are led to a configuration

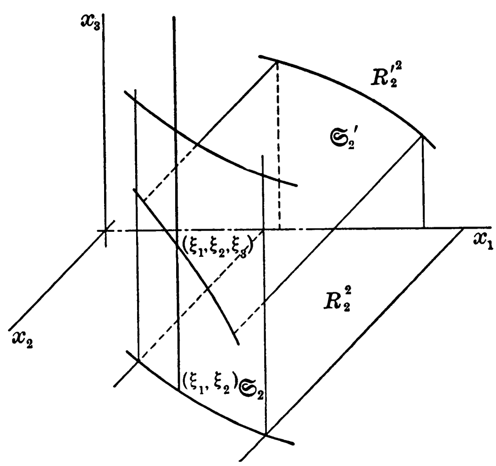
$$C_2': \qquad x_2 = \psi'(x_1, x_3),$$

and this must be the same as C_2.

It remains to show that C_2 is algebraic. We know already that C_2 has at most an ordinary singularity at any one of its points which does not lie on $\mathfrak{S}_1$ or $\mathfrak{S}_2$ and $\mathfrak{S}_1'$ or $\mathfrak{S}_2'$. Assume to begin with that $n = 3$. Let (ξ_1, ξ_2, ξ_3) be a (finite or infinite) point of intersection of C_2 with $\mathfrak{S}_2$ and $\mathfrak{S}_2'$ (hence not lying in $\mathfrak{S}_1$ or $\mathfrak{S}_1'$), and let $\xi_1 \neq \alpha$, where

$$(15) \qquad x_1 = \alpha$$

is the equation of one of the planes, if such exist, which form part of the surface $\mathfrak{S}_2$ or $\mathfrak{S}_2'$. The line



$$(16) \qquad x_1 = \xi_1, \qquad x_2 = \xi_2$$

meets the surface $\mathfrak{S}_2'$ in p' points. Surround each of these points by a neighborhood, $\sigma_1, \sigma_2, \cdots$. The line may meet C_2' outside of these neighborhoods. If so, each such point of intersection will be at most an ordinary singular point of C_2'. In general, the number of such points of intersection will be finite; in particular, the whole line (16) may lie in C_2'.

We may assume in the first case that all the points of intersection of the line in question with the surfaces $\mathfrak{S}_2'$ and C_2' lie in the finite region, for a suitable transformation of the group of the space of analysis will ensure this result.

Consider, then, the x_3 coördinate of C_2, regarded as a function of x_1 and x_2 in the neighborhood of the point $(x_1, x_2) = (\xi_1, \xi_2)$. This function is p-valued, and it has at most ordinary singularities, except possibly for the points (x_1, x_2) which lie on $\mathfrak{S}_2$. Moreover, all its determinations remain finite. Hence it satisfies an equation of the form

$$(17) \qquad x_3^p + C_1(x_1, x_2)x_3^{p-1} + \cdots + C_p(x_1, x_2) = 0,$$

where $C_k(x_1, x_2)$ is analytic at all points of the neighborhood of (ξ_1, ξ_2) except possibly at such as lie on $\mathfrak{S}_2$ and where, furthermore, $C_k(x_1, x_2)$ remains finite in this region. It follows, then, from the extension of Riemann's theorem relating to removable singularities that $C_k(x_1, x_2)$ will be analytic throughout the entire neighborhood of (ξ_1, ξ_2) if properly defined there. Consequently C_2 or C_2' has only an ordinary singularity at (ξ_1, ξ_2, ξ_3).

If, on the other hand, the whole line (16) lies in C_2', we infer at once that it must be an isolated line of this nature. For, at any one of its points (ξ_1, ξ_2, ξ_3') not on $\mathfrak{S}_2'$, C_2' is given by an equation of the form

$$G(x_1, x_2, x_3) = 0,$$

where G is analytic at this point and vanishes there and is irreducible there, and moreover

$$G(\xi_1, \xi_2, x_3) \equiv 0.$$

Hence if G be developed into a power series in $x_3 - \xi_3'$, each coefficient will vanish at the point $(x_1, x_2) = (\xi_1, \xi_2)$. But the coefficients will not admit a common factor, and hence for no other point (ξ_1', ξ_2') of the neighborhood of (ξ_1, ξ_2) will $G(\xi_1', \xi_2', x_3)$ vanish identically.

In the neighborhood, then, of $(x_1, x_2) = (\xi_1, \xi_2)$ the function x_3 is given by an equation of the form (17), whose coefficients $C_k(x_1, x_2)$ are meromorphic in the neighborhood of (ξ_1, ξ_2) with the possible exception of this one point. According to a theorem proved by Hartogs* such a function $C_k(x_1, x_2)$ must

*Mathematische Annalen, vol. 70 (1911), p. 217. Cf. also *Madison Colloquium*, p. 165.

be meromorphic at (ξ_1, ξ_2), also, and hence the foregoing result is extended to all points of intersection of C_2 with $\mathfrak{S}_2$ and $\mathfrak{S}_2'$ with the sole exception of such as lie in one of the planes (15).

If $\psi(x_1, x_2)$ does not depend on x_1, C_2 is a cylinder whose elements are parallel to the x_1-axis, and hence C_2 has at most ordinary singularities in the points where it meets the planes (15) and $\mathfrak{S}_1, \mathfrak{S}_1'$. Thus all points of the space of analysis are accounted for, and C_2 is seen to be algebraic.

If, however, $\psi(x_1, x_2)$ actually involves x_1, then the roles of x_1 and x_2 (or x_3) can be interchanged, and reasoning similar to the foregoing will show that $\psi(x_1, x_2)$ first has only ordinary singularities in R_1^2 except for isolated points. Secondly, these points are eliminated by Hartogs's theorem.

Finally, the points of intersection of the plane $\mathfrak{S}_1$ with the corresponding planes of the permuted variables are disposed of in a similar manner, and thus we see that the configuration C_2, in the extended space of analysis, has at most ordinary singular points. It is, therefore, algebraic.

The intersection of the part of $\mathfrak{S}_2$ distinct from the planes (15) with the analogous part of $\mathfrak{S}_2'$ may comprise curves not lying on C_2, and we are not able to say from the foregoing analysis whether these belong to $\mathfrak{M}$ or not. We shall return to this question in § 9.

8. The Final Theorem for the Case $\mu = 3$, $n = 3$

Beginning with the first two of the equations (B), we are led first to distinguish between the cases $r = 0$ and $r > 0$.

In Case 1, $r = 0$, the manifold $\mathfrak{M}$ lies wholly on an algebraic cylinder, which may be reducible.

On permuting the variables x_1, x_2, x_3 and the corresponding equations (B), and applying the results which have been obtained, it is seen that one must again distinguish two cases, according as the new r' is $= 0$ or > 0.

In the first case it appears that $\mathfrak{M}$ lies on a second algebraic cylinder, whose elements are perpendicular to those of the first, and hence $\mathfrak{M}$ lies on a finite number of irreducible algebraic space curves.

That the points of $\mathfrak{M}$ coincide with the totality of those of a certain number of these curves is seen from the fact that, on making if necessary a linear transformation

$$x_i' = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 \qquad (i = 1, 2, 3),$$

and introducing new functions g_i', G_i' such that

$$\frac{g_i'}{G_i'} = a_{i1}\frac{g_1}{G_1} + a_{i2}\frac{g_2}{G_2} + a_{i3}\frac{g_3}{G_3},$$

a point of $\mathfrak{M}$ will go over into a point of $\mathfrak{M}'$, and furthermore an element of one of the cylinders containing $\mathfrak{M}'$ in the transformed space will meet $\mathfrak{M}'$ in general in only one point. But at least one point of $\mathfrak{M}$ must lie on each element of such a cylinder.

There will remain at most a finite number of elements of the cylinder in question which meet the curves on which $\mathfrak{M}$ lies in more than one point, but only in a finite number of points. These points also belong to $\mathfrak{M}$, since this manifold from its nature is perfect.

If, on the other hand, at least one of the numbers r, r' is positive, let it be r. Since we have assumed $n = 3$, s will always be $= 0$. The points of $\mathfrak{M}$ then coincide with the totality of those of a finite number of irreducible surfaces. For, if (ξ_1, ξ_2) is a point of R_2 not lying on S_2, the line

$$x_1 = \xi_1, \qquad x_2 = \xi_2, \qquad |x_3| \leqq \infty$$

meets the surfaces C_2 in a finite number of points, each of which is a point of $\mathfrak{M}$, and these are the only points of $\mathfrak{M}$ on such a line.

If, however, the above line lies in $\mathfrak{S}_1$ or $\mathfrak{S}_2$ or S_2, let two of the x's, as x_2 and x_3, be interchanged. A point (ξ_1, ξ_2, ξ_3) of the above line for which (ξ_1, ξ_3) lies in $R_2'^2$, but not on S_2', will belong to $\mathfrak{M}$ if, and only if, it lies on the surfaces in question.

Thus doubt remains only for such points as lie at once on $\mathfrak{S}_1$ or $\mathfrak{S}_2$ or S_2 and $\mathfrak{S}_1'$ or $\mathfrak{S}_2'$ or S_2'; i. e., on certain manifolds of a lower number of dimensions. The proof that no such points belong to $\mathfrak{M}$ unless they lie on the surfaces in question will be given in § 9.

Had we taken $\mu = 3$, $n = 4$, it might have happened that neither r nor s is 0. In that case, the points of $\mathfrak{M}$ have their first three coördinates wholly arbitrary, and the fourth is a constant:

$$x_4 = a_4 = \frac{g_4(0, 0, 0, 0)}{G_4(0, 0, 0, 0)}.$$

Thus $\mathfrak{M}$ is the hyperplane $x_4 = a_4$.

9. Proof of the Non-Existence of Manifolds of Lower Order of Dimensions*

Restricting ourselves still to the case of § 8, namely, $\mu = 3$, $n = 3$, we begin by disposing of any manifolds S_2 which may be present. For this purpose it is enough to apply a suitable linear transformation to x_1, x_2, x_3, replacing the functions ϕ_1, ϕ_2, ϕ_3 by new functions through the same transformation. Thus a cylinder S_2 goes over either into one of the new surfaces C_2 or else its points are seen not to belong to $\mathfrak{M}$, with the exception of such as lie on the curves which it is the object of this paragraph to investigate.

* This paragraph was written in June, 1918.

We have, then, a finite number of algebraic surfaces, whose totality is denoted by C_2 and all of whose points are points of $\mathfrak{M}$. No other points of the (x)-space can belong to $\mathfrak{M}$ except possibly those in which an $\mathfrak{S}_1$ or an $\mathfrak{S}_2$ cuts an $\mathfrak{S}_1'$ or an $\mathfrak{S}_2'$, and we proceed to show that these points are never points of $\mathfrak{M}$.

Suppose, then, that $\mathfrak{M}$ contained a point (x^0) lying on the intersection of $\mathfrak{S}_2$ with $\mathfrak{S}_2'$, but not on C_2, and suppose (x^0) to be an ordinary point of such a curve. Then we can arrange things so, by means of a linear transformation of the sort just considered, that (x^0) lies in the finite region and the plane

$$x_3 = x_3^0 \tag{18}$$

meets this arc in only one point near (x^0), and the same will be true of the intersection of every other plane

$$x_3 = \lambda, \tag{19}$$

where

$$|\lambda - x_3^0| < h,$$

h being a suitably chosen positive number.

Let us cut the (x)-space by the plane (19), considering now those points (x) given by (A) which lie in this plane. Analytically this means that we restrict (u) to lying on the surface

$$g_3(u_1, u_2, u_3) - \lambda G_3(u_1, u_2, u_3) = 0, \tag{20}$$

but not at a point of any one of the three curves

$$g_i(u_1, u_2, u_3) = 0, \qquad G_i(u_1, u_2, u_3) = 0 \qquad (i = 1, 2, 3). \tag{21}$$

For a non-specialized choice of the coördinate system in the (u)-space equation (20) is equivalent to an algebroid equation,

$$u_3^\nu + \Gamma_1 u_3^{\nu-1} + \cdots + \Gamma_\nu = 0, \tag{22}$$

$$\Gamma_k = \Gamma_k(u_1, u_2, \lambda), \qquad \Gamma_k(0, 0, \lambda) = 0.$$

We are thus led to a transformation

$$x_1 = \phi_1(u_1, u_2, u_3), \qquad x_2 = \phi_2(u_1, u_2, u_3), \tag{A'}$$

where x_1, x_2 are single-valued functions on the algebroid configuration (20), in general analytic; and we are interested in certain points of the manifold $\mathfrak{M}'$ belonging to (A').

It is impossible for x_1 and x_2 both to remain finite in the neighborhood of a point $\lambda = \lambda_0$:

$$|\lambda - \lambda_0| < \delta,$$

where λ_0 has a value near x_3^0:

$$|\lambda_0 - x_3^0| < h' < h.$$

For then x_1 and x_2 would both approach limits

$$x_1 = \psi_1(\lambda), \qquad x_2 = \psi_2(\lambda),$$

and hence $\mathfrak{M}$, for all values of x_3 near x_3^0, would be restricted to the one-dimensional locus

$$x_1 = \psi_1(x_3), \qquad x_2 = \psi_2(x_3).$$

But $\mathfrak{M}$ contains at least the points of intersection of the plane (19) with C_2, and this intersection is a curve.

Let $\mathfrak{M}$ be encased in a neighborhood Σ so chosen that the cross-section σ_1 of Σ by (19) near (x^0) is exterior to the remainder σ of the total cross-section. Let ϵ be so chosen that, when (u) lies in a region $\mathfrak{T}$, for whose points

$$(23) \qquad |u_k| < \epsilon \qquad (i = 1, 2, 3).$$

(x) lies in Σ.

If, now, (x^0) be a point of $\mathfrak{M}$, it is possible to find a point (x') near (x^0) such that a point (u') of the above $\mathfrak{T}$ yields (x') through (A); and moreover $u_1'\ u_2'$ can here be restricted to being arbitrarily small:

$$(24) \qquad |u_1'| < \epsilon_1, \qquad |u_2'| < \epsilon_1.$$

Furthermore, if λ be chosen arbitrarily within a suitable region

$$(25) \qquad |\lambda - \lambda'| < \eta, \qquad \lambda' = x_3';$$

and if u_1, u_2 be taken near u_1', u_2', then equation (20) will have a root u_3 near u_3', and (A) will be defined in the new point (u), the corresponding point (x) lying near (x').

Let S be a region of the (x_1, x_2)-plane including in its interior the region σ_1 which corresponds to a λ in (25), but not including any point of the σ which corresponds to such a λ. These conditions can be met by restricting Σ and h suitably at the outset. Then we can find a λ_1 in (25) and a point (u) for which (A) is defined, such that

$$(i) \qquad |u_1| < \epsilon_1, \qquad |u_2| < \epsilon_1;$$

$$(ii) \qquad (x_1, x_2) \text{ lies outside of } S.$$

And the same will be true if λ be chosen arbitrarily in a certain neighborhood of λ_1:

$$(26) \qquad |\lambda - \lambda_1| < \eta_1,$$

which region lies wholly in (25).

Next, we can find a λ_2 in (26) and a point (u) for which (A) is defined, such that (*i*) $$|u_1| < \epsilon_2, \qquad |u_2| < \epsilon_2;$$

(*ii*) $$(x_1, x_2) \text{ lies outside of } S.$$

And the same will be true if λ be chosen arbitrarily in a certain neighborhood of λ_2:

(27) $$|\lambda - \lambda_2| < \eta_2,$$

which region lies wholly in (26).

Repeating the step indefinitely, we obtain a set of nested regions in the λ-plane, which have at least one point λ'' in common. Giving to λ this value, $\lambda = \lambda''$, we see that the plane

(28) $$x_3 = \lambda''$$

contains a set of points (x_1, x_2) all lying outside of S and corresponding to points (u) for which (A) is defined and $\lim (u) = (0)$. These points (x) have at least one point of condensation which lies outside of S, and this point belongs both to $\mathfrak{M}$ and to the $\mathfrak{M}'$ corresponding to $\lambda = \lambda''$.

We can, then, find a point (u'') on (20) arbitrarily near the origin such that (A) is defined in it and the point (x'') $(x_3'' = \lambda'')$ lies in Σ but outside the S corresponding to $\lambda = \lambda''$, and hence in the σ which corresponds to $\lambda = \lambda''$.

Returning to (x'), we now modify its choice so that $x_3' = \lambda''$; the point (x') still lying in σ_1 and (u') being restricted as before.

We now are able to deduce a contradiction. For, (u') and (u'') can be joined by a path lying on (20), at each point of which (A) is defined, and furthermore this path can be taken to lie wholly in (23). As (u) describes this path, (x) goes from (x') to (x''), always remaining in the plane (28), and hence (x) passes outside of σ_1 and σ; i. e., outside of Σ. But this is impossible, since each point (u) of (23) for which (A) is defined, yields a point (x) of Σ.

It follows, then, that $\mathfrak{M}$ can have no point which is an ordinary point of intersection of $\mathfrak{S}_2$ and $\mathfrak{S}_2'$ not lying on (15). Other points of intersection of $\mathfrak{S}_2$ and $\mathfrak{S}_2'$ not lying on (15), being isolated, cannot belong to $\mathfrak{M}$, either.

To dispose of the points of (15), it is sufficient to allow x_1 to interchange its rôle with x_2 or x_3; and the points of $\mathfrak{S}_1, \mathfrak{S}_1'$ are disposed of in like manner.

The method admits extension to the higher cases. Thus, when $n = 4$ and $\mu = 4$, the manifold $C = C_3$ consists of a reducible or irreducible algebraic hypersurface (three-dimensional). Here, we cut by the pair of hyperplanes,

$$x_3 = \lambda_3, \qquad x_4 = \lambda_4.$$

If, on the other hand, $n = 4$ and the manifold of the maximum number of dimensions consists of a surface (two-dimensional), the case is analogous to that of the curve treated in § 8, or else to the linear manifold treated at the end of that paragraph.

10. The General Theorem

We can now state the result in the general case, the foregoing treatment applying without let or hindrance to the proof in that case.

THEOREM. *Let*

$$x_i = \frac{g_i(u_1, \cdots, u_n)}{G_i(u_1, \cdots, u_n)} \qquad (i = 1, \cdots, n),$$

be such a transformation as is defined in the introductory paragraph. The manifold $\mathfrak{M}$ *is then made up of a finite number of algebraic manifolds of the following kind.*

In the space of the first μ *variables,* $(x_1, \cdots, x_\mu)$*, where* $1 \leqq \mu \leqq n$*, there exists a manifold* $\mathfrak{K}$ *formed by a finite number of irreducible algebraic curves* $(k = 1)$*, surfaces* $(k = 2)$*, or hypersurfaces of order* $k < \mu$*, this number* k *being the same for all; or finally, when* $\mu < n$*,* $\mathfrak{K}$ *may include all the points of the space in question, and we set here* $k = \mu$.

Then $\mathfrak{M}$ *consists of the points* $(x_1, \cdots, x_n)$*, where* $(x_1, \cdots, x_\mu)$ *is an arbitrary point of* $\mathfrak{K}$*, and*

$$x_j = a_j = \frac{g_j(0, \cdots, 0)}{G_j(0, \cdots, 0)} \qquad (j = \mu + 1, \cdots, n).$$

We note that, if $\mathfrak{M}$ be imbedded in an arbitrarily restricted neighborhood $\mathfrak{U}$, then $\mathfrak{T}$ can be so chosen that the images (x) of all points (u) of $\mathfrak{T}$ will lie in $\mathfrak{U}$. For, if the points of $\mathfrak{U}$ be removed from the (x)-space, the remaining region will be closed. To each of its points (x') corresponds a definite positive η such that, if $\mathfrak{T}$ lie in the region

$$|u_k| < \eta \qquad (k = 1, \cdots, n),$$

(x') will not be the image of any point (u) of $\mathfrak{T}$. And now, if each η is chosen as large as possible, it is shown by familiar reasoning that the lower limit of the η's for the closed region in question is positive.

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