ON CONVEX FUNCTIONS*

BY

HENRY BLUMBERG

A single-valued function $f(x)$ of the real variable x is said to be "*convex,*" if the inequality

$$f\left(\frac{x_1 + x_2}{2}\right) \leqq \frac{f(x_1) + f(x_2)}{2}$$

holds for every pair of real numbers (x_1, x_2) belonging to the region of definition† of $f(x)$. The notion is due to Jensen‡ and has been found useful in various connections. The purpose of this note is to prove two new properties of such functions. In both cases, the proof here given is independent of the results of Jensen or of others. In particular, no use is made of the Cauchy algebraic artifice,§ upon which Jensen's demonstrations are based.

The first property involves the definition of a convex function of two variables. We formulate at once the

Definition of convex function for n-space.|| A single-valued function $f(x_1, x_2, \cdots x_n)$ of the n real variables $x_1, x_2, \cdots x_n$ is said to be "*convex,*" if the inequality

$$f\left(\frac{x_1 + y_1}{2}, \frac{x_2 + y_2}{2}, \cdots \frac{x_n + y_n}{2}\right) \leqq \frac{f(x_1, x_2, \cdots x_n) + f(y_1, y_2, \cdots y_n)}{2}$$

holds for every pair $[A = (x_1, x_2, \cdots x_n), B = (y_1, y_2, \cdots y_n)]$ of points of n-space lying in the region of definition of f. In other words, if M is the midpoint of the segment AB, we have

$$\text{(I)} \qquad f(M) \leqq \frac{f(A) + f(B)}{2},$$

where $f(A)$ stands for $f(x_1, x_2, \cdots x_n)$, and $f(B)$, $f(M)$ have analogous meanings.

THEOREM I. *If a convex function of two variables, defined in the interior and on the boundary of a square, is such that the functional values at the boundary*

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† It will be understood that $f(x)$ must be *finite* to be regarded as defined at x.

‡ Acta Mathematica, vol. 30 (1906), pp. 175–193.

§ Loc. cit., pp. 175–179.

|| Jensen, loc. cit., gives this definition also, but makes no use of it.

*points have a finite upper bound g, then the functional values at interior points have the same upper bound g. Moreover, f is continuous at every interior point.**

If P is a given interior point, there exist two boundary points B_1 and B_2 having P as midpoint. Hence, in virtue of (I), we have

$$f(P) \leqq \frac{f(B_1) + f(B_2)}{2} \leqq g,$$

which proves the first part of the theorem.

To prove the second part, we shall show that no sequence $\{Q_n\}$ of points exists, such that $\lim_{n\to\infty} Q_n = P$, and $\lim_{n\to\infty} f(Q_n) \neq f(P)$. Suppose, on the contrary, there is such a sequence. Letting $f(P) = z$, we may then assume without loss of generality that $\lim_{n\to\infty} f(Q_n) > z$. For let P be the midpoint of the segment Q_nR_n. Then according to (I), we have $2f(P) \leqq f(Q_n) + f(R_n)$, whence $f(R_n) \geqq 2z - f(Q_n)$. If $\lim_{n\to\infty} f(Q_n) < z$, then $\liminf_{n\to\infty} f(R_n) > z$. We may thus obtain in any case a sequence of the desired property. In accordance with $\lim_{n\to\infty} f(Q_n) > z$, we write $f(Q_n) = z + h_n$, $\lim_{n\to\infty} h_n = h > 0$. Let P, $Q_n \equiv Q_{n1}, Q_{n2}, \cdots Q_{nk}$ be a sequence of equally spaced points in the direction from P to Q_n, the distance between two successive points thus being $\overline{PQ}_n$. Then by virtue of (I), we have

$$f(Q_{nk}) - f(Q_{n\overline{k-1}}) > f(Q_{n\overline{k-1}}) - f(Q_{n\overline{k-2}}) > \cdots > f(Q_{n1}) - f(P) = h_n;$$

whence $f(Q_{nk}) > z + kh_n$. If now k is a given integer however large, and ϵ a given positive quantity however small, there exists, on account of $\lim_{n\to\infty} Q_n = P$ and $\lim_{n\to\infty} h_n = h$, an integer ν such that $Q_{\nu k}$ is still in the square and $|h - h_\nu| < \epsilon$. But by a suitable choice of k and ϵ, we see from the relations $f(Q_{\nu k}) > z + kh_\nu > z + k(h - \epsilon)$ that $f(Q_{\nu k})$ may be made large enough to contradict the relation $f(Q_{\nu k}) < g$. Our theorem is thus proved.

It is easy to construct examples to show that f *need not be continuous at boundary points.*

THEOREM II. *A* (Lebesgue) *measurable convex function defined in a given interval is necessarily continuous at every interior point of the interval. In other words, a convex function that is discontinuous at an interior point of an interval where it is defined is necessarily non-measurable.*†

Suppose that $f(x)$ is defined in the interval (a, b) and is discontinuous at

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* See below for generalizations.

† Examples of non-measurable functions have been given by Vitali, Van Vleck, Lebesgue and Hausdorff. See Schoenflies-Hahn, *Entwickelung der Mengenlehre und ihrer Anwendungen* (1913), p. 374. All these examples have been constructed, we might say, for their own sake. On the other hand, non-measurable functions occur in this paper in natural fashion, as a result of the study of convex functions. Cf. Schimmack, *Axiomatische Untersuchungen über die Vektoraddition*, Dissertation (Halle), 1908, p. 14, where the special case $f(x+y) = f(x) + f(y)$ (cf. Corollary below) is treated.

the interior point ξ of (a, b), so that a sequence $\{\xi_n\}$ exists, such that $\lim_{n\to\infty} \xi_n = \xi$ and $\lim_{n\to\infty} f(\xi_n) \neq f(\xi)$. We have here a situation like that in the proof of the second part of Theorem I. By identifying ξ with P and ξ_n with Q_n, we may conclude that $f(Q_{\nu k})$ may be made arbitrarily large, while at the same time, and independently, $Q_{\nu k}$ may be brought arbitrarily near every given point on the left (right) of ξ, if an infinite number of elements of $\{Q_n\}$ lie on the left (right) of ξ. It follows that *the upper bound of the functional values of f in a given subinterval, however small, of (a, b) is ∞*. For suppose f is positively unbounded at every point on the left—a similar argument will hold for the right—of ξ. Let ϵ be a given positive number however small, and n a given positive number however large. Then a positive number $\delta < \epsilon/2$ exists such that $2f(\xi - \delta) - f(\xi - \epsilon) > n$. In virtue of (I), $f(\xi + \epsilon - 2\delta) > 2f(\xi - \delta) - f(\xi - \epsilon) > n$. Since $\xi + \epsilon - 2\delta > \xi$ and vanishes with ϵ, the inequality $f(\xi + \epsilon - 2\delta) > n$ shows that f is positively unbounded also on the right of ξ. From this it follows, in particular, that a sequence $\{R_n\}$ of points entirely on the right of ξ exists such that

$$\lim_{n\to\infty} R_n = \xi, \qquad \lim_{n\to\infty} f(R_n) > f(\xi);$$

hence f is positively unbounded at every point on the right of ξ.*

Let $\mathfrak{S}_n$, where n is a positive integer, represent the set of points x of (a, b) such that $f(x) > n$. If x belongs to $\mathfrak{S}_n$ and d is a positive number such that $x + d$ and $x - d$ are both in (a, b), then either $x + d$ or $x - d$ belongs to $\mathfrak{S}_n$. For otherwise, we would have $f(x - d) \leqq n$, $f(x + d) \leqq n$, whence, according to (I),

$$f(x) \leqq \frac{f(x - d) + f(x + d)}{2} \leqq n,$$

contrary to the assumption that x belongs to $\mathfrak{S}_n$. Let now (α, β) be any subinterval of (a, b). Since $f(x)$ has an infinite upper bound at every point of (a, b), we may select as near $(\alpha + \beta)/2$ as we please a point x belonging to $\mathfrak{S}_n$. Since either $x - d$ or $x + d$ belongs to $\mathfrak{S}_n$, we may, by varying d from 0 to $min(x - \alpha, \beta - x)$, obtain in (α, β) a set of points of $\mathfrak{S}_n$ whose exterior (Lebesgue) measure is as near $(\beta - \alpha)/2$ as we please. As this holds for every interval (α, β), it follows that *the exterior measure of $\mathfrak{S}_n$ is $b - a$*. For it is not difficult to show that, if the exterior measure of a set is less than $b - a$, and if ϵ is a positive number, however small, a subinterval (α, β) of (a, b) exists in which the exterior measure of the set is $< \epsilon(\beta - \alpha)$.

It now follows that $f(x)$ is non-measurable. For if $f(x)$ were measurable,

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* For the purposes of the proof of Theorem II, it is sufficient to know that f is positively unbounded just at ξ.

the complement of $\mathfrak{S}_n$ would have a measure approaching $b - a$ as $n \to \infty$, and hence, in contradiction to the above, $\mathfrak{S}_n$ would have a measure approaching 0 as $n \to \infty$.

If $f(x)$ is a non-measurable function of the type described above, and hence unlimited, then

$$g(x) = \frac{f(x)}{1 + |f(x)|}$$

is a non-measurable function with values lying between -1 and $+1$.

The functional equation*

$$f(x + y) = f(x) + f(y)$$

is a particular case of (I). For it follows from this functional equation that

$$f\left(\frac{x_1 + x_2}{2}\right) = \frac{1}{2}\left[f\left(\frac{x_1 + x_2}{2}\right) + f\left(\frac{x_1 + x_2}{2}\right)\right]$$

$$= \tfrac{1}{2}[f(x_1 + x_2)] = \tfrac{1}{2}[f(x_1) + f(x_2)],$$

whence

$$f\left(\frac{x_1 + x_2}{2}\right) = \frac{f(x_1) + f(x_2)}{2},$$

which is a particular case of (I). We thus obtain the

Corollary. Every discontinuous solution of the functional equation

$$f(x + y) = f(x) + f(y)$$

is non-measurable.

Generalizations of Theorem I. (*a*) Our first extension of Theorem I consists in employing an arbitrary finite planar "*region*" instead of a square. By a planar "region," we shall here understand a point set $\mathfrak{R}$ consisting exclusively of interior points; i. e., such that every given point P of $\mathfrak{R}$ may be enclosed in a neighborhood—dependent on P—of which every point belongs to $\mathfrak{R}$. The "*boundary*" of $\mathfrak{R}$ consists of those points of the plane that do not belong to $\mathfrak{R}$ but in whose every neighborhood there are points of $\mathfrak{R}$. Evidently *the boundary of a region is a closed set of points.* We have

THEOREM I*a*. *If a convex function, defined in the interior and on the boundary of any given finite planar region $\mathfrak{R}$, is such that the functional values at the boundary points have a finite upper bound g, then f has g as upper bound in $\mathfrak{R}$, and it is continuous at every point of $\mathfrak{R}$.*

Manifestly it is sufficient to prove that f has g as upper bound; from this fact, the second assertion of the theorem is seen to follow from Theorem I, by enclosing every point of $\mathfrak{R}$ in a square lying entirely in $\mathfrak{R}$. If P is a given

* Cf. Hamel, Mathematische Annalen, vol. 60 (1905), pp. 459–462; Schimmack, loc. cit.

point of $\mathfrak{R}$, we shall show that there exist two boundary points Q and R having P as midpoint. From this fact, the desired inequality, $f(P) \leqq g$, follows as in the proof of the first part of Theorem I.

To show the existence of Q and R we completely enclose $\mathfrak{R}$ in non-overlapping polygons*—which need not be more than finite in number—such that every boundary point of an enclosing polygon is at a minimum distance $< \epsilon_n$ from the boundary points of $\mathfrak{R}$; here $\{\epsilon_n\}$ is a sequence of numbers such that $\lim_{n \to \infty} \epsilon_n = 0$. For every n we have such a finite set of polygons. Let $\mathfrak{P}_n$ be the polygon belonging to the nth set and containing P. It is obvious in the simplest cases, and susceptible of proof in general, that we may select on the boundary of $\mathfrak{P}_n$ two points Q_n and R_n having P as midpoint. From the sequence $\{Q_n\}$ select a subsequence $\{Q_{\sigma_n}\}$ having the point Q as a limit; then R_{σ_n} will also have a limit, say R, such that P is the midpoint of QR.

Furthermore, Q—and likewise R—belongs to the boundary of $\mathfrak{R}$. For let C_ϵ be the circle of radius ϵ having Q as center. For sufficiently large n, the point Q_{σ_n}, which belongs to the boundary of $\mathfrak{P}_{\sigma_n}$, is at a distance $< \epsilon/2$ from Q. Again, for sufficiently large n, we have $\epsilon_{\sigma_n} < \epsilon/2$, and in particular, there is a boundary point of $\mathfrak{R}$ at a distance $< \epsilon/2$ from Q_{σ_n}. Hence there is a boundary point of $\mathfrak{R}$ at a distance $< \epsilon$ from Q. Since this holds for every ϵ, it must be that Q is a limit of boundary points of $\mathfrak{R}$; and since the boundary of $\mathfrak{R}$ is closed, Q itself is a boundary point of $\mathfrak{R}$.

(*b*) Secondly, it is apparent that *Theorem Ia is directly extensible to n-space.* A "region of n-space" is analogously defined as a set of interior points.

(*c*) THEOREM I*b*. *If a convex function f of n variables, defined in the interior and on the boundary of an n-dimensional cube K, is such that the functional values at the points on the one-dimensional edges of K have a finite upper bound g, then f has g as upper bound in K, and is continuous at every interior point of K.*

For from Theorem I—or I*a*—we conclude that f has the upper bound g in the two-dimensional faces of K; then, from the extension of I*a* to three dimensions, that f has the upper bound g in the three-dimensional cells of K; etc.

Theorem I*b* indicates how a generalization of Theorem I*a*, as extended for n-space, may be effected by demanding the boundedness of g merely in a suitably chosen subset of the boundary of $\mathfrak{R}$, instead of the entire boundary. But we shall not enter here into a more detailed consideration of such a generalization.

* Cf., for example, Hausdorff, *Grundzüge der Mengenlehre* (1914), p. 342.