SOME TWO-DIMENSIONAL LOCI CONNECTED WITH CROSS RATIOS*

BY

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1. Introduction. The writer has recently published a proof of the following theorem:†

THEOREM I. *If the points z_1, z_2, z_3 vary independently and have circular regions as their respective loci, then the locus of the point z_4 defined by the real constant cross ratio*

$$\lambda = (z_1, z_2, z_3, z_4)$$

is also a circular region.

It is the purpose of the present paper to consider generalizations of and other results related to Theorem I, primarily the determination of the locus of the point z_4 defined as in Theorem I, when z_1, z_2, z_3 vary independently so as to have certain prescribed loci. Thus one may raise the following question: If two variable points lie respectively on two fixed circles, where does the mid-point of their segment lie? The answer is contained in Theorem VI. Or again, if the loci of the points z_1, z_2, z_3 are regions each bounded by a number of circles, what can be said of the locus of z_4? The answer is given by Theorem VII.

All the results proved concerning such loci as these can be interpreted in terms of the roots of the jacobian of two binary forms. Such an interpretation is given in §12.

The chief method used in the present paper to determine the locus of z_4 in any given case is that suggested in I (pp. 102, 103, footnote), namely the determination of the locus of z_4 when z_1 and z_2 are held fast and z_3 varies over its locus; the determination of the locus of this locus of z_4 when z_1 is kept fixed but z_2 varies over its locus; and finally the determination of the locus of this new locus of z_4 when z_1 varies over its prescribed locus. In the present paper this method is carried through geometrically. The same method has been used by Professor A. B. Coble‡ to determine analytically the locus of z_4 in Theorem I,

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† These Transactions, vol. 22 (1921), pp. 101–116; this paper will be referred to as I. We shall also have occasion to refer to another of our papers, using the letter S: *Comptes Rendus du Congrès International des Mathématiciens*, Strasbourg, 1920, pp. 349–352.

The term *locus* used in Theorem I of the present paper replaces the term *envelope* used in I.

‡ Bulletin of the American Mathematical Society, vol. 27 (1920–1921), pp. 434–437.

but analytic determination in the more general case is less illuminating and probably more difficult than the geometric determination.

Throughout the present paper, as throughout I, circles play a central rôle.

2. Theorem II, a property of the boundaries of the loci. We suppose once for all that the loci of the points z_1, z_2, z_3 are closed regions bounded by a finite number of regular curves.* We admit, however, the possibility of having these loci either points or regular curves instead of two-dimensional continua bounded by such curves.

For loci of this sort we prove a result which corresponds to, but is more general than Lemma III (I, p. 104); the prescribed loci of z_1, z_2, z_3 are denoted by T_1, T_2, T_3, respectively, and the locus of z_4 defined as in Theorem I is denoted by T_4.

THEOREM II. *If the point z_4 is on the boundary of T_4, then any set of points z_1, z_2, z_3 corresponding lie on the boundaries of their respective regions T_1, T_2, T_3; if none of these four points is at a vertex of its proper region, the circle C through the points z_1, z_2, z_3, z_4 cuts the boundaries of T_1, T_2, T_3, T_4 all at angles of the same magnitude, and if C is transformed into a straight line, the lines tangent to these boundaries at the points z_1, z_2, z_3, z_4, respectively, are parallel. If one or more of the points z_1, z_2, z_3, z_4 is at a vertex of its proper region, C cuts the boundaries of the other regions at angles of the same magnitude, and if C is transformed into a straight line the lines tangent to these boundaries at the corresponding points are parallel.*

If we consider the defining relation to be

$$(z_1, z_2, z_3, z_4) = \lambda,$$

if $\lambda = 0, 1$, or ∞, and if T_4 coincides with one of the original regions (see I, p. 103), only one of the points z_1, z_2, z_3 is effectively concerned with the location of z_4, and Theorem II is not true. Whenever T_4 is the entire plane, the locus of z_4 has no boundary and the theorem has no meaning. These possibilities are henceforth excluded.

We phrase the proof to deal with the case that each of the regions T_1, T_2, T_3 is a two-dimensional continuum, but a change in wording rather than of reasoning is all that is required if one or more of these regions is a point. We discuss later the possibility of curves instead of two-dimensional continua.

When any two of the points z_1, z_2, z_3, z_4 are kept fixed, the relation defining z_4 becomes a linear relation between the other two points. Thus, when z_2 and z_3 are kept fixed, motion of z_1 over a two-dimensional continuum moves z_4 over a two-dimensional continuum, so if z_4 is on the boundary of its locus, z_1 is also on the boundary of its locus, and similarly for z_2 and z_3. Under such a linear correspondence, moreover, a circle corresponds to a circle. For example, when z_3

* For the definition of a regular curve, see Osgood, *Funktionentheorie* (second edition), pp. 51, 324, 150.

and z_4 are considered fixed and when z_1 moves along C, z_2 also moves along C. When z_1 moves from C along the boundary of T_1, z_2 moves from C and makes the same angle with C as the angle between C and the boundary of T_1. If the boundary of T_2 does not make this same angle with C, and if neither z_1 nor z_2 is at a vertex of the corresponding boundary, this motion of z_1 in one sense or the other from C will cause z_2 to move into the *interior* of T_2. Then we keep z_1 and z_3 fixed. Motion of z_3 over a two-dimensional continuum all of which is interior to T_2 will cause z_4 to move over a two-dimensional continuum all of which is interior to T_4, so z_4 cannot be on the boundary of T_4.

Thus the boundaries of T_1, T_2, T_3 all make the same angle with C, and if we keep z_1 and z_2 fixed, moving z_3 and z_4, we see by consideration of the angles at these latter two points that the boundaries of T_3 and T_4 make the same angle with C. The *direct conformality* of the angles in every case here shows that when C is transformed into a straight line the tangents to the boundaries of T_1, T_2, T_3, T_4 at z_1, z_2, z_3, z_4 are parallel if these points are not at vertices of those boundaries. If one or more of the points z_1, z_2, z_3, z_4 is at a vertex of the corresponding boundary, the result always holds for such of those points as are not at vertices of those boundaries.

If one of the original regions T_1, T_2, T_3 is a curve instead of a point or a two-dimensional continuum, further proof is necessary. If for example T_1 and T_2 are curves and if C does not cut them at z_1 and z_2 at equal angles, let us transform z_3 to infinity. Denote by C'_2 the curve of all points z_4 which correspond to a particular point z_1 of T_1 and to z_3, as z_2 traces T_2. The angle between C and C'_2 at z_4 is the same as the angle between C and C_2 at z_2. When z_1 now moves on T_1, points of C'_2 which correspond to particular points z_2 of T_2 move in the direction of motion of z_1. If C does not cut T_1 and T_2 at the same angle, this motion of z_1 on T_1 will therefore cause C'_2 to sweep out an entire two-dimensional continuum in the neighborhood of z_4 and hence z_4 cannot be on the boundary of T_4.

To compare the angles in which C cuts T_1 and the boundary of T_4, we hold z_2 and z_3 fast. Motion of z_1 from C along T_1 causes z_4 to move from C along a curve through z_4 either tangent to C or extending on both sides of C. But all points of this curve are points of T_4, and the curve cuts C at the same angle as T_1. Hence either T_4 has a vertex at z_4 or its boundary cuts C at the same angle as does T_1. We have now given all the essential reasoning in the proof of Theorem II.

We remark incidentally that curves related as are the curves bounding these regions T_1, T_2, T_3, T_4 have interesting geometric properties.* Thus when the

* In three-dimensional space we may consider the analogous problem of finding the locus of points z_4 determined by their real constant cross ratio with the points z_1, z_2, z_3 whose loci are respectively either three surfaces S_1, S_2, S_3 or three regions bounded by these three surfaces; the term *cross ratio* is to be interpreted precisely as in the plane—the four points z_1, z_2, z_3, z_4 are to be concyclic and the cross ratio can be defined by any complex coördinate system in

region T_3 is simply the point at infinity, the bounding curves C_1, C_2, C_4 are so related that C (in this case a line) sets up a correspondence between those curves, so that tangents to C_1 and C_2 at corresponding points are parallel and the corresponding points z_1, z_2, z_4 on the line C divide C in a constant ratio. Whenever three curves C_1, C_2, C_4 have this property, it is true that lines tangent to these curves at corresponding points z_1, z_2, z_4 are parallel, the centers of curvature of C_1, C_2, C_4 at z_1, z_2, z_4 are collinear and the ratios of the segments cut on the line joining them equal the ratios of the corresponding segments of the line C. If the curves C_1 and C_2 are parallel, C_4 is parallel to them.

3. Theorems III and IV; cases where the loci are curves. There are two particularly interesting types of curves T_1, T_2, T_3 which may be chosen so that T_4 shall also be a curve.

THEOREM III. *Let C_1 and C_2 be two curves which are the loci of the points z_1 and z_2 respectively; denote by C the locus of the point*

$$z = \frac{m_2 z_1 + m_1 z_2}{m_1 + m_2}, \quad m_1 m_2 \neq 0,$$

which divides the segment (z_1, z_2) in the real constant ratio $m_1 : m_2$. A necessary and sufficient condition that C be a curve is that C_1 and C_2 be segments of two parallel lines.

The sufficiency of the condition is immediate; all points z lie on a certain line parallel to C_1 and C_2. The locus C is connected since the loci C_1 and C_2 are connected (see I, p. 103). Hence C is a curve, a line segment.

To prove the necessity of the condition we resort to the type of reasoning just used in the proof of Theorem II. Let P be an arbitrary point of C_1, and C'_2 the curve obtained from C_2 by similarity transformation with center P. When P varies on C_1 and takes a position P', every point Q of C'_2 varies and takes a position Q' such that QQ' is parallel to PP'. According to our hypothesis on C, of which C'_2 is a part, and when we allow PP' to become smaller and smaller, we find that the tangent to C'_2 at Q is parallel to the tangent to C_1 at P. But Q is an arbitrary point of C'_2; then C'_2 (and consequently C_2) is a curve whose tangent has but a single direction and hence is a straight line segment. On the other

the plane of their circle. If under these conditions the locus of z_4 is not the whole of space, which is always the case if the arbitrary surfaces S_1, S_2, S_3 are sufficiently small and sufficiently remote from each other, simple consideration of the boundary of the locus of z_4 as in Theorem II gives us the following theorem of pure geometry which seems to be new:

The three surfaces S_1, S_2, S_3 define a congruence of circles C every circle of which cuts all those surfaces at points z_1, z_2, z_3 at the same angle, and such that if C is inverted into a straight line the planes tangent to S_1, S_2, S_3 at z_1, z_2, z_3 are all parallel. That is, any sphere through C cuts S_1, S_2, S_3 at z_1, z_2, z_3 at equal angles. If any surface S_4 is defined as the locus of points z_4 defined by a real constant cross ratio with z_1, z_2, z_3, the isogonal property holds for all the surfaces S_1, S_2, S_3, S_4.

The congruence C is a generalization of the well known normal congruence, for which the cross ratio property is also well known.

hand, C_1 is a curve whose tangent at an arbitrary point P is parallel to the line C_2. Hence C_1 and C_2 are segments of parallel lines.

Theorem III is not true for the degenerate values $m_1=0$ or $m_2=0$. The following theorem is likewise false for the degenerate values of the cross ratio:

THEOREM IV. *Let three curves C_1, C_2, C_3 be the loci of points z_1, z_2, z_3, respectively. Denote by C_4 the locus of the point z_4 defined by the real constant cross ratio*

$$\lambda=(z_1, z_2, z_3, z_4) \qquad \lambda \neq 0,\ 1,\ \textit{or}\ \infty .$$

A necessary and sufficient condition that C_4 be a curve is that C_1, C_2, C_3 be arcs of a single circle having no point common to all.

The sufficiency of the condition is immediate. If two of the points z_1, z_2, z_3 coincide, z_4 coincides with them and hence is on the circle of which C_1, C_2, C_3 are arcs. If no two of the original points coincide, z_4 is concyclic with those three points and hence also on that circle. The locus C_4 is connected (proved as in I, p. 103) and hence is an arc of that circle.

To demonstrate the necessity of the condition, we notice that C_1, C_2, C_3 must be circular arcs; otherwise we could choose suitably the point at infinity on C_1, for example, and C_2 and C_3 would become two curves not parallel lines, which would be in contradiction to Theorem III. If C_1, C_2, C_3 are not arcs of a single circle there is a point P which belongs to C_1, for example, but which does not belong to both of the circles of which C_2 and C_3 are arcs. When P is transformed to infinity, C_2 and C_3 are not both straight lines, so we have again a contradiction of Theorem III. If the arcs C_1, C_2, C_3 of a single circle have a point common to, all the locus of z_4 is the entire plane instead of a curve. The proof of Theorem IV is thus complete.

4. Theorem V, the interest of regions bounded by circular arcs. Theorems I-IV all indicate the central position in this study occupied by circles and circular arcs; this is heightened by the invariance of circles and circular arcs under linear transformation. The following theorem results from Theorem II:

THEOREM V. *Let the regions T_1, T_2, T_3, each bounded by a finite number of circular arcs, be the loci of points z_1, z_2, z_3, respectively. Then the region T_4 which is the locus of the point z_4 defined by the real constant cross ratio:*

$$\lambda=(z_1, z_2, z_3, z_4)$$

is also bounded by a finite number of circular arcs.

Every point of the boundary of T_4 corresponds to points z_1, z_2, z_3 on the boundaries of T_1, T_2, T_3 and such that the circle C through the four points cuts the boundaries of T_1, T_2, T_3 at vertices or such that the angles between C and the boundaries of T_1, T_2, T_3 are equal in magnitude. Corresponding to each set of circles, one a boundary (or in part forming part of the boundary) of each of the original regions, there are four circles of the type of C_4 (Lemma IV, I, p. 105) according as C cuts all three circles at the same angle or a definite one at an angle supplementary to the angle cut on the other two.* Corresponding to each vertex of one of the original regions and two circles, one a boundary of each of the other of the original regions, there are two circles of the type of C_4 (Lemma IV, I, p. 105), and corresponding to two vertices of two separate original regions and a circle bounding the other region there is but one circle of this type. Since there are but a finite number of circular arcs and of vertices in the boundaries of each of the regions T_1, T_2, T_3, we have in all but a finite number of circles whose points can be boundary points of T_4; every boundary point of T_4 lies on at least one of these circles. The theorem follows at once. It is also true that if T_1, T_2, T_3 are bounded by at most a countable infinity of circular arcs, then T_4 is bounded by at most a countable infinity of circular arcs.

Regions of the sort considered in Theorem V are particularly interesting if they are bounded by *entire* circles instead of arcs of circles. We shall study in some detail the loci connected with such regions. The simplest case is that of circular regions, and has already been treated. The next simplest case is that of annular regions. We define an *annular region* to consist of the points common to two circular regions whose boundaries either have no point in common, or are tangent, or coincide.† An annular region is thus a closed region; every circular region is an annular region.

5. Theorem VI, the locus resulting from three circles; case of a single null circle. A general result concerning loci generated by regions whose boundaries are entire circles is to be proved later (Theorem VII). We now prove a preliminary result:

THEOREM VI. *If the loci of the points z_1, z_2, z_3 are the circles S_1, S_2, S_3, respectively, then the locus S_4 of the point z_4 defined by the real constant cross ratio*

$$\lambda = (z_1, z_2, z_3, z_4)$$

is an annular region.

For the values $\lambda = 0$, 1 or ∞, we must have at least two of the points z_1, z_2, z_3, z_4 coincident. Thus if $\lambda = 0$ the locus of z_4 is the entire plane or S_3 according as S_1 and S_2 have or have not a point in common. Similar facts hold for the other degenerate values of λ; compare I, p. 103. In the future we assume λ to have none of these values; then no two of the points z_1, z_2, z_3, z_4 coincide unless three of them coincide. Furthermore we place ourselves in what we may call the *general*

* This is true even if the three circles are coaxial; see the detailed proof of Lemma IV, I, p. 105 ff.

† It is possible to define annular regions so as to exclude the possibility of tangency of the bounding circles. But under this possible definition Theorem VI is no longer true; compare the results of §5.

situation, where none of the circles S_1, S_2, S_3 is a null circle, where no two of these circles are tangent, and where the three circles are not coaxial. We may immediately treat the case that the three original circles have a common point; the points z_1, z_2, z_3 may all be considered to lie at that point, z_4 may lie anywhere in the plane, so its locus is the entire plane.

The fact that S_4 is a two-dimensional continuum appears from the results of §3; the fact that the locus S_4 is connected appears by the reasoning used in I, p. 103.

Choose two arbitrary but distinct points P_1 and P_2 on S_1 and S_2, respectively, and let us consider the locus of z_4 defined by

$$(P_1, P_2, z_3, z_4) = \lambda$$

when the locus of z_3 is S_3. If we transform P_1 to infinity, z_4 is a point which divides the segment $P_2 z_3$ internally or externally in a constant ratio. Hence the locus of z_4 is a non-null circle S_4'. If P_1 and P_2 are chosen coincident, z_4 must coincide with them and its locus is a null circle. Let now P_2 trace the circle S_2, while P_1 remains fixed. If we determine the locus of all points of the variable circle S_4' we shall have precisely the locus of z_4 defined by

$$(P_1, z_2, z_3, z_4) = \lambda$$

when z_2 and z_3 have as their loci the circles S_2 and S_3.

We shall first assume that S_2 and S_3 are not coaxial with the null circle P_1 and that neither passes through P_1. The family of circles α through P_1 and cutting S_2 and S_3 at equal angles forms a coaxial family; compare the theorem quoted in I, p. 106. If P_1 is the point at infinity, this family passes through the external center of similitude of S_2 and S_3 if S_2 and S_3; are equal in size and P_1 is the point at infinity, this family is a family of parallel lines. Consider the circle α through the point P_2 and on α the point A determined by the cross ratio

$$(P_1, P_2, z_3, A) = \lambda,$$

where z_3 is the intersection of α with S_3 such that the tangent to S_3 at z_3 becomes parallel to the tangent to S_2 at P_2 when the circle α is transformed into a straight line. The point z_3 is uniquely determined by these conditions except in the particular case that α is orthogonal to S_2 and S_3; in this case we determine z_3 by continuity.

When the circle α varies, the point A traces a circle σ_1, by Lemma IV (I, p. 105). Moreover, the point A is continually on the circle S_4' corresponding to P_1 and P_2, and when α is transformed into a straight line, the line tangent to S_4' at A is parallel to the line tangent to S_3 at z_3. Hence as α varies, S_4' remains constantly tangent to σ_1; there is a circle S_4' tangent to σ_1 at every point of σ_1.

The family of circles β through P_1 and cutting S_2 and S_3 at supplementary angles also forms a coaxial family. When P_1 is the point at infinity the circles β all pass through the internal center of similitude of S_2 and S_3. When β varies, the procedure just employed for α can be used to show that the circle S_4' remains constantly tangent to a second circle σ_2 at a point of the circle β through P_2; there is a circle S_4' tangent to σ_2 at every point of σ_2.

The circles σ_1 and σ_2 are distinct. In fact, let us consider P_1 still as the point at infinity, and denote by A_2 and A_3 the centers of S_2 and S_3. Then the reasoning used in the proof of Lemma IV (I, p. 105) shows that σ_1 and σ_2 have as common center the point z_4 defined by

$$(P_1, A_2, A_3, z_4) = \lambda.$$

The external center of similitude of S_2 and S_3 (and which may be the point at infinity) is the external center of similitude or the internal center of similitude for *both* the pairs of circles σ_1, S_2 and σ_1, S_3; the internal center of similitude of S_2 and S_3 is the external center of similitude for one of the pairs of circles σ_2, S_2 and σ_2, S_3 and the internal center of similitude for the other pair. Neither of the circles σ_1 and σ_2 can coincide with either of the circles S_2 and S_3, for the common center of σ_1 and σ_2 is distinct from A_2 and A_3. It follows from this fact and the fact that the internal center of similitude of S_2 and S_3 differs from their external center of similitude that σ_1 and σ_2 are distinct.

The locus of the points of the variable circle S_4' is now apparent. Suppose for convenience in phraseology that σ_2 is interior to σ_1. The circle S_4' moves continuously so as to touch both σ_1 and σ_2 and touches every point of both circles. Either S_4' includes σ_2 or S_4' does not include σ_2, but in either case the locus of the points of S_4' is the annular region between and bounded by σ_1 and σ_2. In particular σ_2 (or σ_1) may be a null circle, but the locus of the points of S_4' is never the entire plane.

We have left aside the case that S_2 and S_3 have a common center M when P_1 is transformed to infinity. This situation will not be discussed in detail, but is entirely analogous to the case already treated. The two circles σ_1 and σ_2 have the common center M; if S_4 is chosen in any position and rotated about M, the two circles whose common center is M and which are tangent to S_4' are seen to play the rôle of σ_1 and σ_2. These two circles are the boundaries of the annular region which is the locus of the points of S_4'.

The case where S_2 or S_3, say for definiteness S_3, is a straight line remains to be considered. The circle S_4' is in every position a straight line parallel to S_3. When P_2 varies over S_2, the locus of the points of S_4' is seen to be a strip of the plane bounded by two lines parallel to S_3, which is an annular region. The theorem is then proved for the case that S_1 is a null circle. We notice that in no case can σ_1 and σ_2 simultaneously be null circles, that they never coincide, and that they are tangent when and only when P_1 lies on S_2 or S_3.

6. Theorem VI, the locus resulting from three circles; general case. Our theorem, thus completely proved when S_1 is a null circle P_1, will be proved in the general case by determining the locus of the points of the annular region S_4'' bounded by σ_1 and σ_2 as the point P_1 traces the circle S_1. We shall suppose for the present that the three circles S_1, S_2, S_3 are not coaxial and that no two of them are tangent. It will follow that in Lemma IV (I, p. 105) we can say that as z_1 is made to vary continuously and in one sense on its locus, the points z_1, z_2, z_3, z_4 all vary continuously and in one sense on their loci. It should be remarked in connection with Lemma IV (I, p. 105) that it is essential to suppose the variable circle C and the points z_1, z_2, z_3 to vary continuously or that at least a suitable convention be made as to the choice of the points z_1, z_2, z_3 on C when C is orthogonal to C_1, C_2, C_3. For in that position of C, the condition that when C is transformed into a straight line the lines tangent to the three circles at z_1, z_2, z_3 are all parallel does not determine sufficiently these three points so that z_4 shall lie on C_4; and there are in general some extraneous points which enter into our locus if some convention concerning them is not made.

When P_1 varies and traces S_1, σ_1 varies also so as to remain constantly tangent to two fixed circles τ_1 and τ_2. For consider the circle C' through P_1 and which cuts S_1, S_2, S_3 all at the same angle. Choose on C' the points z_2 on S_2 and z_3 on S_3 such that when C' is transformed into a straight line the lines tangent to S_1, S_2, S_3 at P_1, z_2, z_3 are parallel. Then the point z_4 lies on σ_1 and also lies on the circle τ_1 of Lemma IV (I, p. 105) corresponding to the variable circle which cuts the three original circles at equal angles. Moreover when C' is transformed into a straight line the lines tangent to S_1 at P_1, S_2 at z_2, S_3 at z_3, σ_1 at z_4, τ_1 at z_4 are all parallel. Hence σ_1 is tangent to τ_1 at z_4.

In a precisely similar way it also appears that σ_1 is tangent to the circle τ_2 generated as described in Lemma IV (I, p. 105) corresponding to a variable circle C'' which cuts S_1 at an angle supplementary to the angles cut on S_2 and S_3. The variable circle σ_2 remains always tangent to two fixed circles ω_1 and ω_2 corresponding to a variable circle C''' which cuts S_2 at an angle supplementary to that cut on S_1 and S_3, and a variable circle C'''' which cuts S_3 at an angle supplementary to that cut on S_1 and S_2.

If the variable circle σ_1, is never a null circle, τ_1 and τ_2 do not intersect and are not tangent; for definiteness suppose τ_2 interior to τ_1. Whether σ_1 moves so as to include τ_2 or so as to exclude τ_2, σ_1 passes through every point of the plane between τ_1 and τ_2, so every such point is a point of S_4. A precisely similar remark obtains with reference to σ_2.

If σ_2 is ever a null circle, we shall prove that no arc of either of the circles

ω_1 and ω_2 can be a part of the boundary of S_4. The circles ω_1 and ω_2 must either intersect or be tangent. Suppose σ_2 never to pass through the point at infinity. When σ_2 is a null circle, σ_1 entirely surrounds σ_2. As σ_2 varies, immediately exterior to it there are always points of the final locus S_4, for σ_1 and σ_2 can never coincide and they vary continuously. It may occur that σ_1 and σ_2 become tangent, but this can happen only when P_1 is on S_2 or S_3 and thus only at the most at four isolated points. Except possibly at these four points, ω_1 and ω_2 are entirely embedded in points of S_4. A similar remark is evidently true for σ_1 and the circles τ_1 and τ_2, if σ_1 is ever a null circle.

The boundary of S_4 consists entirely of points of the circles τ_1, τ_2, ω_1, ω_2, by Theorem IV. The region S_4 is closed and its boundary cannot consist in whole or in part of isolated points. We may therefore suppose that at least one of the circles σ_1 or σ_2, say for definiteness σ_1, is never a null circle. If σ_1 passes through every point of the plane, S_4 is surely the whole plane. If σ_1 does not pass through every point of the plane, we suppose that it does not pass through the point at infinity. If σ_2 is ever exterior to σ_1, it always remains exterior; if ever interior, it always remains interior; for definiteness suppose that σ_2 always remains interior to σ_1. If σ_1 surrounds neither τ_1 nor τ_2, σ_1 passes through every point between τ_1 and τ_2, and every point of S_4'' is a point between those two circles, so S_4 is the annular region bounded by the non-intersecting circles τ_1 and τ_2.

If τ_2 is interior to τ_1 as well as to σ_1 (σ_2 constantly interior to σ_1), τ_1 is one boundary of S_4. But no arc of τ_2 can be a part of the boundary of S_4, since except at most for four isolated positions there are points of S_4'' interior and adjacent to σ_1 and hence there are points of S_4 interior and adjacent to τ_2. That is, not more than one of the circles τ_1 and τ_2 can be a part of the boundary of S_4; no arc of either circle is a part of that boundary unless that entire circle is a part of the boundary. The corresponding statement is true for ω_1 and ω_2.

In any case, then, the boundary of S_4 is composed of at most two of the circles τ_1, τ_2, ω_1, ω_2; it follows quite easily from our previous reasoning that these two circles do not intersect, so S_4 is an annular region.

The foregoing reasoning in proof of Theorem VI is not essentially altered and need not be further considered in detail if any of the circles τ_1, τ_2, ω_1, ω_2 is a null circle or if either of the circles σ_1 or σ_2 remains fixed during the motion of P_1. But on the other hand, it is necessary to note that we have essentially two distinct pairs of circles τ_1 and τ_2, and ω_1 and ω_2, or at least that when we speak of the circle σ_1 as tangent to τ_1 at z_4' and tangent to τ_2 at z_4'' we are not dealing with two points z_4' and z_4'' which always coincide and at the same time with two circles τ_1 and τ_2 which always coincide or are tangent at z_4' $(=z_4'')$. We prove this by choosing P_1 of such a nature that C' does not cut S_1 orthogonally and is not tangent to S_1, and such that σ_1 is not a null circle; such choice is possible under our restrictions on S_1, S_2, S_3. Then C' and C'' cannot cut S_1 at the same angle and hence are distinct. The lines tangent to τ_1 (also to σ_1) at z_4' and to τ_2 (also to σ_1) at z_4'' cannot coincide, by the property of those tangent lines when C' and C'' are transformed into straight lines. Moreover σ_1 was chosen a non-null circle, so we are led to the conclusion that z_4' and z_4'' are distinct. This remark is similarly applicable to the points on the circles ω_1 and ω_2, so Theorem VI is completely proved under our assumption that S_1, S_2, S_3 are not coaxial and that no two of them are tangent.

For the more general case which makes no such assumption concerning S_1, S_2, S_3,* we consider variable auxiliary circles which approach the three given circles, and are such that at no stage are these three variable circles coaxial nor are any two of them tangent. The locus corresponding to the variable circles is always an annular region; it approaches uniformly the locus corresponding to the three given circles; this latter locus is therefore an annular region.

7. Theorem VII, circular boundaries lead to circular boundaries; case of a single null circle. We shall now make use of Theorem VI and its method of proof in the demonstration of our general theorem concerning loci whose boundaries are entire circles.

THEOREM VII. *If the loci of the points z_1, z_2, z_3 are, respectively, T_1, T_2, T_3, regions each bounded by a finite number of non-intersecting circles, the locus of z_4 defined by the real constant cross ratio*

$$\lambda = (z_1, z_2, z_3, z_4)$$

is a region T_4 also bounded by a finite number of non-intersecting circles. The number of circles bounding T_4 is not greater than the greatest number of circles bounding any of the regions T_1, T_2, T_3. In particular if T_1, T_2, T_3 are annular regions, T_4 is also an annular region; if T_1, T_2, T_3 are circular regions, T_4 is also a circular region.

In counting the number of boundaries of a region, any region T_i which consists merely of the points of a single circle is to be considered as having two boundaries. This is a natural convention, for we may think of T_i as the limit of a proper annular region as the two bounding circles approach each other. Some such convention is desirable so that the present theorem shall accord with Theorem VI.

It is of course true that T_4 may have as many bounding circles as the greatest number of circles bounding any of the regions T_1, T_2, T_3. This always occurs

* Theorem VI is extremely easy to prove for three coaxial circles. For three circles through two fixed points or all tangent at a single point the locus is the entire plane. For the other case the circles can be transformed so as to be made concentric. From symmetry about the common center and from the connectedness of the locus (as in I, p. 103) it follows that S_4 is an annular region.

($\lambda \neq 0, 1,$or ∞) if for example T_1 and T_2 are points; T_3 is the region or at least one of the regions with the greatest number of bounding circles. If T_1 and T_2 are sufficiently small, the number of their bounding circles has no effect; T_3 and T_4 have the same number of bounding circles.

The cases $\lambda = 0, 1,$ or ∞ or that one of the given regions is the entire plane are to be treated as in I, p. 103, and will not be further considered. We shall further suppose for the present, that no two bounding circles of any one of the given regions are tangent, that no three circles, boundaries respectively of the three given regions, are coaxial, and that none of the three given regions is a point or a circle.

The boundary of the locus T_4 is composed of circles S_4 or arcs of these circles which are generated by the point z_4 determined by its cross ratio with the points z_1 on a circle S_1 which is a boundary of T_1, z_2 on a circle S_2 which is a boundary of T_2, and z_3 on a circle S_3 which is a boundary of T_3, while the circle S through these four points cuts S_1, S_2, S_3 all at the same angle or cuts one at an angle supplementary to the angle cut on the other two. We shall prove that *if an arc of any one of these circles S_4 is part of the boundary of T_4, the entire circle S_4 is part of the boundary of T_4.* For definiteness suppose S to cut S_1, S_2, S_3 all at the same angle so that the circle S_4 with which we are concerned is τ_1.

Since no two of the circles bounding T_1 intersect, there are points of T_1 adjacent to and all along one side of S_1, and similarly for the circles S_2 and S_3. We shall prove that even if we consider the entire circular region T_1' lying on that particular side of and bounded by S_1 as the locus of z_1 and the regions T_2' and T_3' similarly formed from S_2 and S_3 as the loci of z_2 and z_3, the locus of z_4 is that circular region T_4' bounded by τ_1 which contains τ_2, ω_1, ω_2. Since T_1' entirely contains T_1, T_2' entirely contains T_2, and T_3' entirely contains T_3, and since every point of τ_1 is a point of the locus of z_4 corresponding to the regions T_1, T_2, T_3, it follows that the entire circle τ_1 is a boundary of T_4; there are points z_4 corresponding to T_1, T_2, T_3 lying on one side of and all along τ_1, but there are no such points z_4 lying on the opposite side of τ_1.

The circle τ_1 is a part of the boundary of the annular region corresponding to the three circles S_1, S_2, S_3 as in Theorem VI; otherwise it is surely not part of the boundary for the regions T_1, T_2, T_3. Similarly, if we fix a point P_1 on S_1 we know that σ_1 must be a part of the boundary of the locus for P_1, S_2, S_3.

If we fix P_1 on S_1 and P_2 on S_2, we have points z_3 of T_3 on one side of and all along S_3, and hence we have points z_4 of T_4 on one side of and all along the circle S_4' of the proof of Theorem VI. Transform P_1 to infinity. When P_2 traces S_2, if S_4' always lies between σ_1 and σ_2 but surrounds neither, it passes through every point of the annular region between the circles. Moreover, if the points z_4 of T_4 just mentioned lie outside of S_4', no point of either σ_1 or σ_2 can be a point of the boundary of T_4, and hence no arc of τ_1 can be a part of the boundary of T_4. Then the points z_4 which correspond to T_4' must all lie interior to S_4' and as P_2 varies on S_2 we have no new points added to the locus of z_4 because T_4 was replaced by T_4'.

When P_2 moves from S_2 into T_2, for example along the line L through P_2 cutting S_2 and S_3 at the same angle, S_4' moves so as to continue to cut L at the same angle, but is no longer tangent to σ_1 or σ_2. In fact, the intersections of S_4' with L are the points z_4 determined by the cross ratio of points P_1 (fixed), P_2 (variable on L), and the fixed intersections of L with S_3. Motion of P_2 causes these two intersections to vary. When P_2 is at either intersection of L with S_2, S_4' is tangent to σ_1 and σ_2; when P_2 moves on L from one of these intersections to the other in T_2', S_4' varies, but always varies so as to cut L at the same angle, which is the angle cut on L by σ_1. If we assume for definiteness that S_4' is interior to σ_1, we see that this motion of S_4' must always keep S_4' interior to σ_1, and σ_2 is not a part of the boundary of T_4 nor of T_4', and that the entire interior of σ_1 belongs to T_4'.

Precisely similar reasoning obtains if S_4' lies between σ_1 and σ_2 but encloses one circle and not the other, but we shall not give the details. We further omit the detailed treatment if S_2 or S_3 is a straight line (P_1 being at infinity). In every case we add no new boundary nor take away a part of the boundary of the locus of z_4 by replacing T_2 and T_3 by T_2' and T_3', so far as concerns the circle τ_1. Moreover the locus of z_4 in the latter case is not the whole plane unless the locus in the former case is the whole plane.

8. Theorem VII, circular boundaries lead to circular boundaries; general case. We shall now extend this reasoning by considering the locus of the points of the circle σ_1 as P_1 traces S_1. Suppose for definiteness that τ_1 is exterior to τ_2 and that σ_1 lies between the two circles but τ_2 is exterior to σ_1. We have assumed a part of τ_1 a part of the boundary of T_4, so the points of T_4 and hence of T_4' previously determined lie interior to σ_1. Similarly, a small motion of P_1 on S and into the interior of T_1 either moves σ_1 always interior to τ_1, or always exterior to τ_1; this follows from the continuity of the motion of σ_1 due to the continuous motion of P_1, and from the properties of the cross ratio determining the intersection of σ_1 with S which are shortly to be considered in detail. Since we are assuming at least an arc of τ_1 to be part of the boundary of T_4, it follows that such small motion of P_1 always moves σ_1 interior to τ_1.

Continuous motion of P_1 in one sense along S and in T_1' from one intersection of S with S_1 to the other intersection causes σ_1 to move. Each intersection of S with σ_1 is a point z_4 determined by its cross ratio with P_1, an intersection of S with S_2, and an intersection of S with S_3. The angle of intersection of S and σ_1 does not change, and this is the same as the angle of intersection of S and τ_1.

In its initial and final positions σ_1 is interior to τ_1; it therefore follows (in fact becomes evident if S is transformed into a straight line) that every point of σ_1 remains always interior to τ_1.

Determination of the locus of z_4 in this manner gives us the entire region T_4'; we have shown every point of T_4' to lie interior to or on τ_1. Every point interior to or on τ_1 is on a circle S and is interior to or on the circle σ_1 for some choice of P_1 in T_1' and hence T_4' is the interior (boundary included) of τ_1.

There is a possibility which could conceivably arise in this proof, namely that for some positions of S that circle does not cut S_1 and that for some point P_1 interior to T_1' but not on a circle S cutting S_1*, the circle σ_1 should be completely or partially exterior to τ_1; our reasoning as given does not permit this possibility. If there is one of the regions T_1', T_2', T_3' for which the circle S passes through every point, we change our notation if necessary so that that region shall be T_1', and our proof is valid as given. If there is no region T_1', T_2', T_3' through every point of which passes a circle S cutting S_1, S_2, S_3, it follows from the proof of Lemma IV (I, p. 105) that a point P_1 on S_1 but on neither S_2 nor S_3 can be chosen which lies in both T_2' and T_3'. It follows from the development of §7 that no point of either σ_1 or σ_2 can be a boundary point of T_4', so the possibility suggested need not be considered further.

The case where σ_1 lies between τ_1 and τ_2 but separates those two circles needs to be considered in detail, but only slight modifications in the reasoning are necessary and these are left to the reader. This completes the proof that τ_1 is the boundary of T_4' and hence part of the boundary of T_4. There are points z_4 of T_4 all along one side of and adjacent to τ_1, so no other bounding circle of T_4 (the entire circle necessarily part of the boundary of T_4) can intersect τ_1.

We have made the assumption that no two bounding circles of any one of the regions T_i are tangent, that no three circles, boundaries respectively of the three

* A similar possibility should have been pointed out in I, pp. 111–112 in proving that (notation of I) either the entire exterior or the entire interior of each of the circles C_4', C_4'', C_4''', C_4'''' belongs to the region C_4. For definiteness suppose this possibility to arise in connection with Lemma IV (I, p. 105), Case I; we consider the configuration simplified by transformation as in I, p. 106. The statement desired is evident if the regions C_1, C_2, C_3 are all interior to their bounding circles, so we may suppose one of these regions, say C_3, exterior to its bounding circle C_3. We wish to prove a point z_4 exterior to C_4' a point of the final envelope C_4 and need consider only points z_4 on circles C which do not cut C_1, C_2, C_3. Then there is a circle through z_4 which cuts C_1 and C_2 and which lies entirely in the region C_3. We may choose z_1 and z_2 on this circle and lastly z_3, so that these three points are in their proper envelopes and have the proper cross ratio with z_4.

The entire reasoning given in I, pp. 110–112, to prove that arcs of but one of the circles $C_4^{(i)}$ can be a part of the boundary of C_4 is no longer strictly necessary, for the proof of Theorem VII of the present paper contains a proof of Theorem I. Moreover, the reasoning of I, pp. 110–112, can be replaced by the more simple and elegant argument used in §6 (or even §15) of a paper by the writer shortly to appear in these Transactions.

given regions, are coaxial, and that none of the given regions is a point or a circle. To extend our result to include these special cases, we consider a sequence of sets of three regions which satisfy our restrictions but which approach the more special given regions; such an auxiliary sequence can always be constructed. Then the limit of the locus of z_4 for the variable regions is the locus of z_4 for the limit regions, from which it follows that for the limit case the locus T_4 is bounded by a finite number of entire circles.

9. Theorem VII, number of circular boundaries. It remains for us to prove that the number of circles bounding T_4 is not greater than the greatest number of circles bounding any of the regions T_1, T_2, T_3. In the detailed proof we restrict the boundaries of the original regions as before; this restriction is raised by the limiting process previously used and need not be mentioned further.

We keep P_1 (z_1 in any position) fixed for the moment, and suppose that the point at infinity is not a point of the locus T_4'' corresponding to the points z_2 of T_2 and z_3 of T_3. Consider the mechanism used in the proof of Theorem VI; first suppose S_4' to lie between σ_1 and σ_2 and σ_2 to be interior to σ_1 but not interior to S_4'. Under the assumption that σ_1 is part of the boundary of T_4'', when P_2 traces S_2, S_4' moves tangent to σ_1 and σ_2. If Σ_4' is a circle which is obtained from the proper cross ratio from P_1, P_2, and the points of a bounding circle Σ_3 of T_3 other than S_3, then Σ_4' is interior to S_4'. During the motion of P_2 on S_2, Σ_4' cannot trace a boundary of T_4''. Moreover, if P_2 is moved continuously, Σ_4' moves continuously; Σ_4' can never be made to surround a lacuna in T_4'', from which it follows that Σ_3 in combination with any bounding circle of T_2 cannot lead to a boundary of T_4''. Bounding circles of T_4'' can proceed only from combination of S_3 with bounding circles of T_2, so the number of bounding circles of T_4'' is not greater than the number of bounding circles of T_2.

Second, suppose S_4' to lie between σ_1 and σ_2 and σ_2 to be interior to both σ_1 and S_4'. When σ_1 is supposed as before part of the boundary of T_4'', the circle Σ_4' must as before be interior to S_4'. There is a circle Σ_4' interior to S_4' corresponding to each of the m bounding circles Σ_3 of T_3 (other than S_3), and in the region traced out by these circles as P_2 moves around S_2 there are at most m lacunae, each constantly surrounded by a circle Σ_4'. When P_2 is made to trace the whole of T_2, one or more of these lacunae may disappear, but none of them can divide to make two or more lacunae in T_4''. It follows that the number of bounding circles of T_4'' is not greater than the number of bounding circles of T_3.

The case that σ_2 is a point leads simply to one bounding circle of T_4'', so in every case T_4'' is bounded by a number of circles not greater than the greater number of bounding circles of T_2 and T_3.

If the reasoning just used is again applied regarding the locus of T_4'' as P_1 varies over T_1, it is seen that T_4 is bounded by a number of circles not greater than the greater number of bounding circles of T_1 and T_4'', which is not greater than the greatest number of bounding circles of T_1, T_2, T_3, so the proof of Theorem VII is complete. It is also true that if T_1, T_2, T_3 are each bounded by at most a countable infinity of non-intersecting circles then T_4 is also bounded by at most a countable infinity of non-intersecting circles.

10. Theorem VII, ruler-and-compass construction for boundaries. As in Theorem I, the circles which bound the region T_4 of Theorem VII can be constructed by ruler and compass whenever λ is rational or is given geometrically; indeed this follows from the fact that any circle C_4 of Lemma IV (I, p. 105) can be so constructed; see I, p. 105, footnote. We also have a test for determining whether or not a given circle S_4 (in the notation of §7) is actually a part of the boundary of T_4. If none of the regions T_1, T_2, T_3 reduces to the points of a circle, we need merely suppose the circles S_1, S_2, S_3 to bound entire circular regions which lie on the same side of those circles as the given regions; then S_4 is a boundary of T_4 when and only when the locus of z_4 in the simplified situation is a circular region bounded by S_4. A test for this latter fact has already been determined in I, p. 112. If any of the regions T_1, T_2, T_3 reduces to the points of a circle, we may determine in what sense to consider the corresponding circular regions to lie by an investigation such as that of §7.*

To be sure, the test for determining whether S_4 can be a boundary of T_4 was proved only under certain restrictions on the bounding circles of T_1, T_2, T_3, but the test can be applied whether those restrictions are satisfied or not. Thus, if the circular regions which lie on the same side of S_1, S_2, S_3 as do the given regions T_1, T_2, T_3 lead to a region T_4' bounded by a circle τ_1 and which is not the whole plane, the original regions lead to a region T_4 bounded by the entire circle τ_1 and which is not the whole plane. On the other hand, if a circle S_4 is actually a boundary of T_4, there must be a circle which bounds the locus of z_4 of the approximating sequence and which approaches S_4. There are therefore entire variable circular regions T_1', T_2', T_3' corresponding which lead to a locus of z_4 not the whole plane; these approach entire circular regions formed in the manner described from certain boundaries of T_1, T_2, T_3, and which lead to a locus of z_4 not the whole plane. It follows that the test to determine whether S_4 is a boundary of T_4 is valid in every case. If any of the given regions is a circle, the results in the given case may also be indicated by approximating non-special cases.

* One particular case of three circles S_1, S_2, S_3 can be treated directly, namely, where one of those circles separates the intersections of the other two; compare I, p. 111. The locus of z_4 is always the entire plane. For any given point P of the plane there can be determined points z_1, z_2, z_3 on S_1, S_2, S_3 such that their cross ratio with P has a value very small and positive; these points can be moved continuously on their respective circles so that this cross ratio increases in value and becomes less than but nearly equal to unity. Then these points can be chosen so that λ takes any value between zero and unity; similarly, so that it takes any real value.

Great care should be used in determining loci and limits of loci of this nature if boundaries of the three given regions are tangent at a single point, for it may be merely by virtue of the fact that when z_1, z_2, z_3 coincide z_4 is undetermined that the locus of z_4 is the entire plane. Thus, if two finite points z_1 and z_2 have as their common locus a half-plane, the locus of the mid point z_4 of their segment is also that half-plane. When we consider not the problem of the ordinary ratio but that of the *cross ratio* of the two given points and the mid point of their segment with the point at infinity we are compelled to admit the point at infinity as a possibility for z_1 and z_2 and that the entire plane is the locus of z_4 (compare I; p. 111). For there can be found two variable points z_1 and z_2 such that z_4 is the mid point of their segment and which approach the point at infinity. When these two points coincide at infinity we ought therefore to consider that the four limit points have the proper cross ratio. But for an arbitrary point z_4 of the plane there cannot be found two variable finite points z_1 and z_2 *in their proper loci*, such that z_4 is the mid point of their segment and which approach the point at infinity. This phenomenon can occur only if the three regions have but a single common point and if the respective boundaries are tangent at that point.

This difference of behavior of ratios and cross ratios corresponds indeed to a difference of behavior in our original problem of the roots of the jacobian, as compared with the problem of the roots of the derivative of a polynomial. For most purposes these problems are equivalent if one of the ground forms has all its roots at infinity and the other has as its roots the roots of the polynomial considered. If all the roots of both ground forms coincide, the jacobian vanishes identically. Hence if in Theorem III (I, p. 112) the loci of m roots of f_1 and the remaining p_1-m roots of f_1 are two coincident half-planes, and if f_2 has all its roots coincident at infinity, the locus of the roots of the jacobian is the entire plane. But if these two coincident half-planes are the loci of m roots of a polynomial and the remaining p_1-m roots of that polynomial, and if the polynomial is not allowed to have infinite roots, the locus of the roots of the derivative of this polynomial is this same half-plane instead of the entire plane.

We have assumed in Theorem VII not only that the boundaries of T_1, T_2, T_3 are entire circles but also that no two of the bounding circles of one of these regions cut each other. If this latter part of the hypothesis is omitted, it is not true that T_4 is bounded by entire circles. Consider for example T_1 the point at infinity, T_2 the two finite crescents bounded by two intersecting circles S_2' and S_2'', and T_3 the interior and boundary of a circle S_3 whose center is A and whose radius is small in comparison with the radii of S_2' and S_2''. Choose λ so that when z_1 is at infinity z_4 lies midway between the finite points z_2 and z_3.

The locus of z_4 may be determined by fixing z_2, determining the locus of z_4 while z_3 varies over T_3, and then allowing z_2 to vary over T_2. That is, we shrink T_2 toward A as center of similitude in the ratio $1:2$, and determine the locus of the points of a small circle whose radius is half the radius of S_3 and whose center varies all over this new configuration. The locus T_4 is therefore bounded by arcs of four different circles; no entire circle is a part of its boundary.

11. Successive application of Theorem I in determination of loci. An evident way of generalizing Theorem I is by successive application of that theorem. Thus, suppose we have n sets of the circular regions of Theorem I, $C_k^{(i)}$ $(k=1, 2, 3, 4; i=1, 2, \ldots, n)$. Any point z_4 which corresponds to points $z_k (k=1, 2, 3)$ in all the regions $C_k^{(i)}$ $(i=1, 2, \ldots, n)$ is located in all the regions $C_4^{(i)}$ $(i=1, 2, \ldots, n)$, so the region T_4 which is the locus of points z_4 determined by points z_k in the regions T_k common to all the $C_k^{(i)}$ $(i=1, 2, \ldots, n; k=1, 2, 3)$ is contained in all the regions $C_4^{(i)}$. But ordinarily T_4 will not be the entire region common to this last set of regions. A simple example is the case, $\lambda=\frac{1}{2}$:

$$\begin{array}{llllll}
C_1^{(1)} : & \infty & C_1^{(2)} : & \infty & T_1 : & \infty \\
C_2^{(1)} : & |z_2+2| \leqq 1 & C_2^{(2)} : & -2 & T_2 : & -2 \\
C_3^{(1)} : & 2 & C_3^{(2)} : & |z_3-2| \leqq 1 & T_3 : & 2 \\
C_4^{(1)} : & |z_4| \leqq \frac{1}{2} & C_4^{(2)} : & |z_4| \leqq \frac{1}{2} & T_4 : & 0.
\end{array}$$

A particularly interesting case such that T_4 is the entire region common to the $C_4^{(i)}$ occurs when all the sets of circles $C_1^{(i)}, C_2^{(i)}, C_3^{(i)}, C_4^{(i)}$ come under a single case (e. g., Case I) of Lemma IV (I, p. 105), where these circles have a single coaxial system of circles C cutting them all at the same angle or a definite set or sets at angles supplementary to the angles cut on the other sets, and where all the circular regions $C_k^{(i)}$ for one value of i have the same disposition with respect to their bounding circles as the corresponding regions for every other value of i. It follows from the reasoning of I, pp. 111–112, as supplemented in a footnote to §8 of the present paper, that every point of the region common to all the $C_4^{(i)}$ is a point of the locus of z_4, so that common region is precisely T_4. The circle C then cuts the boundaries of T_1, T_2, T_3, T_4 all at the same angle or one or two of those regions at an angle supplementary to the angle cut on the others. Our notation here has supposed the sets of circles $C_k^{(i)}$ finite in number, but of course that is unnecessary.

A special case of this last result will be considered in some detail, where a set of regions $C_k^{(i)}$ is one and the same point for all values of i. We first state explicitly a special case of Theorem I, which corresponds to a theorem given in I, p. 115, and which is proved explicitly in S.

THEOREM VIII. *If the loci of the points z_1 and z_2 are the interiors and boundaries of the circles C_1 and C_2 whose centers are α_1 and α_2 and radii r_1 and r_2, respectively, then the locus of the point*

$$z=\frac{m_2 z_1+m_1 z_2}{m_1+m_2}$$

which divides the segment (z_1, z_2) in the real constant ratio $m_1:m_2$ $(m_1 m_2 > 0)$ is the interior and boundary of the circle C whose center is

$$\frac{m_2 a_1 + m_1 a_2}{m_1 + m_2}$$

and radius

$$\frac{m_2 r_1 + m_1 r_2}{m_1 + m_2}.$$

The point

$$\frac{r_2 a_1 - r_1 a_2}{r_2 - r_1}$$

(which may be the point at infinity) is an external center of similitude for any pair of the circles, C_1 C_2, C.

We next prove a generalization of Theorem VIII:

THEOREM IX. *If two finite or infinite convex regions C_1 and C_2 having an external center of similitude P (which may be the point at infinity) are the loci of two points z_1 and z_2, respectively, then the locus of the point z which divides the segment (z_1, z_2) in the real constant ratio $m_1:m_2$ $(m_1 m_2 > 0)$ is a region C such that P is an external center of similitude for any pair of the regions C_1, C_2, C.*

In Theorem IX, we tacitly assume C_1 and C_2 to be closed regions bounded by regular curves. We formulate the proof of the theorem for the case that P is finite, but a very slight change in phraseology will give the proof when P is infinite.

Consider a line L through P cutting the boundary of C_1 in points A_1 and A_1' and the boundary of C_2 in corresponding points A_2 and A_2'. If C_1 and C_2 are infinite, A_1' and A_2' are to be considered the point at infinity. When L rotates about P, the points A and A' which divide the segments (A_1, A_2) and (A_1', A_2'), respectively, in the ratio $m_1 : m_2$ trace the boundary of a convex region C such that P is an external center of similitude for any pair of the regions C_1, C_2, C. We shall prove C to be the locus of z. Any point z of C is a point of the locus, for we need merely choose z_1 and z_2 on the line Pz and having the same relative situation in C_1 and C_2 as z in C. In order to prove that every point z is in C we shall use a preliminary theorem which is a special case of Theorem IX and a limiting case of Theorem VIII:

THEOREM X. *If two half-planes π_1 and π_2 bounded by two parallel lines λ_1 and λ_2 and lying on the same side of those lines are the loci of two points z_1 and z_2, respectively, then the locus of the point z which divides the segment (z_1, z_2) in the real constant ratio $m_1:m_2$ $(m_1 m_2 > 0)$ is a half-plane π bounded by a line λ parallel to λ_1 and λ_2; π lies on the same side of λ as π_1 and π_2 of λ_1 and λ_2. Any line which* *cuts* $\lambda_1, \lambda_2, \lambda$ *cuts them in points* z_1, z_2, z *such that* z *divides the segment* (z_1, z_2) *in the ratio* $m_1 : m_2$.

We prove Theorem X in the same spirit as Theorem VIII was proved in S. We use rectangular coördinates $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $z = x + iy$, and suppose the regions π_1 and π_2 to be $x_1 \geqq a_1$ and $x_2 \geqq a_2$, respectively. The region π is

$$x \geqq \frac{m_2 a_1 + m_1 a_2}{m_1 + m_2}.$$

For given z_1 and z_2 in their proper loci, the formula

$$x = \frac{m_2 x_1 + m_1 x_2}{m_1 + m_2}$$

shows at once that z is in π. If z is given in π we have only to choose

$$y_1 = y_2 = y, \quad x_1 = a_1 - \frac{m_2 a_1 + m_1 a_2}{m_1 + m_2} + x, \quad x_2 = a_2 - \frac{m_2 a_1 + m_1 a_2}{m_1 + m_2} + x,$$

and we shall have z_1 and z_2 in π_1 and π_2 and such that z divides their segment in the ratio $m_1 : m_2$. The axis of reals cuts λ_1, λ_2, λ in points z_1, z_2, z such that z divides the segment (z_1, z_2) in the ratio $m_1 : m_2$, and hence that is true of any other line which cuts λ_1, λ_2, λ.

Theorem X is to be used in proving that every point z of Theorem IX lies in C. Through the point A_1 draw a line λ_1 which does not cut C_1 and through A_2 a parallel line λ_2, where A_1 and A_2 have the same significance as before. There is a half-plane π_1 bounded by λ_1 which contains all of C_1 and a corresponding half-plane π_2 bounded by λ_2 which contains all of C_2. The corresponding half-plane π bounded by the line λ through A contains all of C, from which it follows that any point z of Theorem IX lies in π. This is always true as L rotates about P and as λ_1, λ_2, λ envelop C_1, C_2, C_3, so z lies in C and Theorem IX is completely proved.

Our proof depends essentially on the fact that C_1 and hence also C_2 and C are convex, since we have passed a line λ_1 through an arbitrary point A_1 of the boundary of C_1 and supposed C_1 to lie entirely on one side of that line. If C_1 is not convex not only does the proof break down, but the theorem is false and we can even show that *the locus of* z *cannot be the region* C found as previously described. There are two points U_1 and V_1 on the boundary of C_1 such that no point of the segment $U_1 V_1$ belongs to C_1. The analogous segment $U_2 V_2$ has no point in C_2. Denote by U and V the points on the boundary of C which divide the segments $U_1 U_2$ and $V_1 V_2$ in the ratio $m_1 : m_2$; the segment UV has no point in C. The lines $U_1 V_1$, $U_2 V_2$, UV are all parallel and the point z which divides the segment $U_2 V_1$ in the ratio $m_1 : m_2$ lies on the segment UV and hence is exterior to the region C.

We terminate our study of the loci connected with cross ratios by stating without further proof the theorem analogous to Theorem IX if $m_1m_2 < 0$:

THEOREM XI. *If two finite or infinite convex regions C_1 and C_2 having a finite internal center of similitude P are the loci of two points z_1 and z_2, respectively, then the locus of the point z which divides the segment (z_1, z_2) in the real constant ratio $m_1 : m_2$ $(m_1m_2 < 0)$ is a region C such that P is an external center of similitude for C_1 and C (or C_2 and C) and an internal center of similitude for C_2 and C (or C_1 and C).*

If C_1 and C_2 are not convex but satisfy the other conditions of the theorem, the corresponding region C cannot be the locus of z.

12. Application to the location of the roots of the jacobian of two binary forms. Theorem I was originally proved in I not for its own sake but for application to the study of the location of the roots of the jacobian of two binary forms. Thus there can be proved (I, p. 114):

THEOREM XII. *Let f_1 and f_2 be binary forms of degrees p_1 and p_2, respectively, and let the circular regions C_1, C_2, C_3 be the respective loci of m roots of f_1, the remaining p_1-m roots of f_1, and all the roots of f_2. Denote by C_4 the circular region which is the locus of points z_4 such that*

$$(z_1, z_2, z_3, z_4) = \frac{p_1}{m}$$

when z_1, z_2, z_3 have the respective loci C_1, C_2, C_3. Then the locus of the roots of the jacobian of f_1 and f_2 is composed of the region C_4 together with the regions C_1, C_2, C_3, except that among the latter the corresponding region is to be omitted if any of the numbers m, p_1-m, p_2 is unity. If a region C_i $(i=1, 2, 3, 4)$ has no point in common with any other of those regions which is a part of the locus of the roots of the jacobian, it contains precisely $m-1$, p_1-m-1, p_2-1, or 1 of those roots according as $i=1, 2, 3$, or 4.

Theorem XII can be generalized so as to give applications for some of the results of the present paper. Thus we have directly:

THEOREM XIII. *Let f_1 and f_2 be binary forms of degrees p_1 and p_2, respectively, and let regions T_1, T_2, T_3 be the loci of m roots of f_1, the remaining p_1-m roots of f_1, and all the roots of f_2, these three sets of roots always to be contained in three variable circular regions all points of which are points of T_1, T_2, T_3, respectively. Denote by T_4 the locus of points z_4 such that*

$$(z_1, z_2, z_3, z_4) = \frac{p_1}{m}$$

when z_1, z_2, z_3 have the respective loci T_1, T_2, T_3. Then the locus of the roots of the jacobian of f_1 and f_2 is composed of the region T_4 together with the regions T_1, T_2, T_3, except that among the latter the corresponding region is to be omitted if any of the *numbers* m, $p_1 - m$, p_2 *is unity. If a region* T_i $(i = 1, 2, 3, 4)$ *has no point in common with any other of those regions which is a part of the locus of the roots of the jacobian, it contains precisely* $m-1$, $p_1 - m - 1$, $p_2 - 1$, *or* 1 *of those roots according as* $i = 1, 2, 3,$ *or* 4.

It is in connection with Theorem XIII that most of the results of the present paper are to be considered so far as concerns their application to the location to the roots of the jacobian. It should be noticed, however, that Theorem XIII gives no better indication of the location of the roots of the jacobian of particular fixed forms than does Theorem XII, but may give a better indication than Theorem XII of the location of the roots of the jacobian of two forms whose roots vary in particular ways. The results of §11 can also be used independently of Theorem XIII in establishing slight extensions of Theorem XII which do apply to the jacobian of two fixed forms; this application will be made in a later paper.

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