ASYMPTOTIC PLANETOIDS*

BY

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1. **Introduction.** Near the vertices of the equilateral triangles described in the plane of Jupiter's orbit and on the line joining the Sun and Jupiter as base are to be found six of the planetoids. Four of these,† Achilles (588), Patrocles (617), Hector (624), and Nestor (689), oscillate about one vertex in much the same way as the mythological luminaries of the same names circulated about the walls of Troy. Two planetoids are found at the other vertex, but their names are unknown to the author.‡ These six planetoids are in the vicinity of two of the five well known points of libration in the problem of three bodies.§ The three other points lie on the line joining the Sun and Jupiter, one point lying between the Sun and Jupiter, another on the side of the sun remote from Jupiter, and the third on the side of Jupiter remote from the Sun. One of these straight-line points in the case of the Sun and Earth, viz., the one on the side of the Earth remote from the Sun, has a physical significance in that it may account for the "Gegenshein."|| The equilateral triangle points of libration were considered by Lagrange in his celebrated prize memoir of 1772 as "pure curiosities," but recent astronomical discoveries show that these points likewise have some physical significance attached to them.

The object of this article is to determine orbits for these planetoids of the Sun and Jupiter which will approach the equilateral triangle points of libration as the time approaches infinity. As these orbits are asymptotic to the above mentioned points of libration, we have designated the planetoids which move in such orbits as "asymptotic planetoids." These planetoids are considered to be of appreciable mass, and their perturbations upon the Sun and Jupiter are determined. They are assumed to move in the plane of Jupiter's orbit.

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* Presented to the Society, December 31, 1919.

† Marcolongo, *Il problema dei tre corpi*, Scientia, vol. 1 (1919), No. 8.

‡ I am indebted to Professor E. W. Brown for this information, as he communicated it to me when this paper was presented to the Society.

§ Lagrange, *Collected Works*, vol. 6, pp. 229–324; Tisserand, *Mécanique Céleste*, vol. 1, chap. 8; Moulton, *Celestial Mechanics*, pp. 309–318.

|| Gyldèn, *Sur un Cas Particulier du Problème Astronomique*, vol. 1; Moulton, *Celestial Mechanics*, p. 305.

In plain mathematical language, divorced from every astronomical application, the paper treats of two-dimensional asymptotic orbits near the equilateral triangle equilibrium points in the problem of three finite bodies. The paper concludes with numerical examples of orbits in which the ratios of the masses are not those of the Sun, Jupiter and the planetoids, but are chosen so that the orbits near each point can be drawn to the same scale.

Several classes of asymptotic orbits have already been obtained, but with one exception, number (6) below, they belong to the particular case of the problem of three bodies in which one body is infinitesimal and the finite bodies move in circles. The following is a list of the solutions which have been obtained:

(1) Warren determined two-dimensional orbits which are asymptotic to the points of libration lying on the straight line joining the two finite bodies. (American Journal of Mathematics, vol. 38, No. 3, pp. 221–248.)

The remaining cases were found by the author of the present paper.

(2) Two- and three-dimensional orbits which are respectively asymptotic to the two- and three-dimensional periodic orbits near the straight line equilibrium points determined by Moulton* in the chapter on *Oscillating satellites*. (American Journal of Mathematics, vol. 41, No. 2, pp. 79–110.)

(3) Two-dimensional orbits which approach the equilateral triangle points of libration. (Transactions of the Cambridge Philosophical Society, vol. 22, No. 15, pp. 309–340.)

(4) Three-dimensional orbits which are asymptotic to the three-dimensional periodic oscillations near the equilateral triangle equilibrium points which were determined by Buck.† This paper forms part of the paper mentioned in (3).

(5) Three-dimensional orbits which are asymptotic to the isosceles triangle solutions‡ of the problem of three bodies. (Proceedings of the London Mathematical Society, ser. 2, vol. 17, No. 1, pp. 54–74.)

(6) Two-dimensional orbits which are asymptotic to the straight line equilibrium points when the three masses are finite.§

2. **The differential equations.** Let m_1, m_2, and m_3 represent the masses of the three bodies and let M denote their sum. Let a system of rectangular axes be chosen having the origin at the center of mass of the three bodies and the plane of their motion as the plane of reference. Let the axes rotate about the origin with the uniform angular velocity n and let the coördinates of the three bodies m_i be denoted by (x_i, y_i), $i = 1, 2, 3$. If the bodies are subject to their mutual

* *Periodic Orbits*, chap. 5.

† Buck, chap. 9 of Moulton's *Periodic Orbits*.

‡ Buchanan, chap. 10 of Moulton's *Periodic Orbits*.

§ Rendiconti del Circolo Matematico di Palermo, vol. 45, pp. 1–25.

attractions according to the Newtonian law, then the differential equations which define their motion are

$$
(1)\qquad
\begin{aligned}
&x_i'' - 2ny_i' - n^2x_i = \frac{1}{m_i}\frac{\partial U}{\partial x_i},\\
&y_i'' + 2nx_i' - n^2y_i = \frac{1}{m_i}\frac{\partial U}{\partial y_i} \qquad\qquad (i = 1, 2, 3),\\
&U = k^2\left[\frac{m_1m_2}{r_{12}} + \frac{m_2m_3}{r_{23}} + \frac{m_3m_1}{r_{31}}\right],\\
&r_{ij} = [(x_i - x_j)^2 + (y_i - y_j)^2]^{1/2} \qquad (i, j = 1, 2, 3;\ i \neq j),
\end{aligned}
$$

where k^2 is the factor of proportionality and the accents denote derivation with respect to t.

These equations admit the vis viva integral

$$
(2)\qquad \sum_{i=1}^{3} m_i(x_i'^2 + y_i'^2 - x_i^2 - y_i^2) = 2U + C,
$$

where C is the constant of integration.

It is shown in treatises on celestial mechanics* that the equilateral triangle configuration with proper initial components of velocity is a particular solution of the differential equations of motion (1). If the units of distance and time are so chosen that the mutual distances and k^2 are unity, then the angular velocity of rotation n must satisfy the condition

$$
n^2 = M.
$$

Since the unit of mass is so far arbitrary, it is possible, without loss of generality, to put $M = 1$. Hence n^2 is also unity.

If the axes are initially orientated so that the x-axis is parallel to the side of the equilateral triangle joining m_2 and m_3, then the values of the coördinates for the particular solutions are either

Configuration I (Fig. I),

$$
(3)\qquad
\begin{aligned}
x_1 &= \frac{1}{2}(m_3 - m_2), & y_1 &= \frac{1}{2}\sqrt{3}\,(m_2 + m_3),\\
x_2 &= \frac{1}{2}m_1 + m_3, & y_2 &= -\frac{1}{2}\sqrt{3}\,m_1,\\
x_3 &= -\left(\frac{1}{2}m_1 + m_2\right), & y_3 &= -\frac{1}{2}\sqrt{3}\,m_1,
\end{aligned}
$$

* See Moulton's *Introduction to Celestial Mechanics*, pp. 309–11.

or

Configuration II (Fig. 2),

$$
(4) \qquad
\begin{aligned}
x_1 &= \frac{1}{2}(m_3 - m_2), & y_1 &= -\frac{1}{2}\sqrt{3}\,(m_2 + m_3),\\
x_2 &= \frac{1}{2}m_1 + m_3, & y_2 &= \frac{1}{2}\sqrt{3}\,m_1,\\
x_3 &= -\left(\frac{1}{2}m_1 + m_2\right), & y_3 &= \frac{1}{2}\sqrt{3}\,m_1.
\end{aligned}
$$

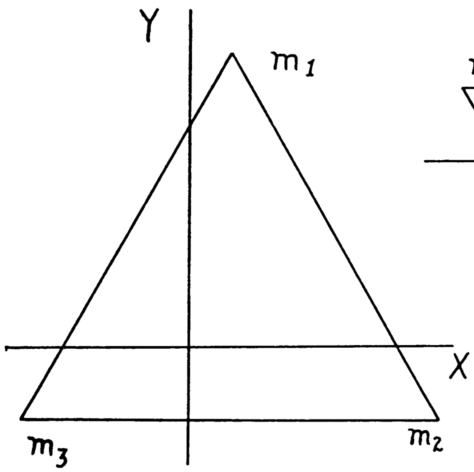

FIG. 1.—Configuration I.

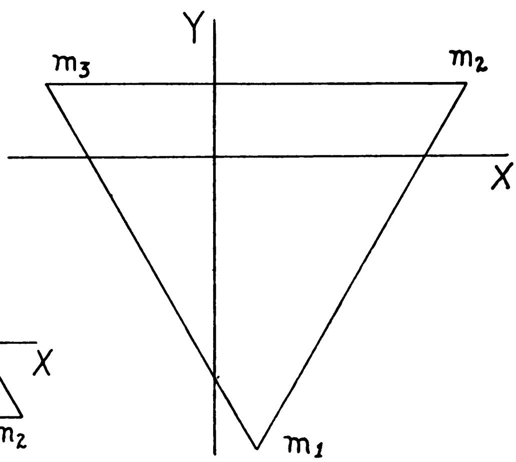

FIG. 2.—Configuration II.

Since the origin of coördinates is at the center of mass, it follows that

$$m_1x_1 + m_2x_2 + m_3x_3 = 0, \qquad m_1y_1 + m_2y_2 + m_3y_3 = 0,$$

and therefore the coördinates of one of the bodies may be eliminated from the differential equations. Let us suppose x_3 and y_3 are thus eliminated by

$$(5) \qquad x_3 = -\frac{1}{m_3}(m_1x_1 + m_2x_2), \qquad y_3 = -\frac{1}{m_3}(m_1y_1 + m_2y_2).$$

When these substitutions are made in (1), and k^2 and n are both put equal to unity, the differential equations of motion become

$$
(6)\qquad
\begin{aligned}
x_1'' - 2y_1' - x_1 &= -\frac{m_2(x_1 - x_2)}{\rho_{12}^3} - \frac{(m_1 + m_3)x_1 + m_2x_2}{\rho_{13}^3},\\
y_1'' + 2x_1' - y_1 &= -\frac{m_2(y_1 - y_2)}{\rho_{12}^3} - \frac{(m_1 + m_3)y_1 + m_2y_2}{\rho_{13}^3},\\
x_2'' - 2y_2' - x_2 &= -\frac{m_1(x_2 - x_1)}{\rho_{12}^3} - \frac{m_1x_1 + (m_2 + m_3)x_2}{\rho_{23}^3},\\
y_2'' + 2x_2' - y_2 &= -\frac{m_1(y_2 - y_1)}{\rho_{12}^3} - \frac{m_1y_1 + (m_2 + m_3)y_2}{\rho_{23}^3},\\
\rho_{12} &= [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2},\\
\rho_{13} &= \frac{1}{m_3}\,[\{(1 - m_2)x_1 + m_2x_2\}^2 + \{(1 - m_2)y_1 + m_2y_2\}^2]^{1/2},\\
\rho_{23} &= \frac{1}{m_3}\,[\{m_1x_1 + (1 - m_1)x_2\}^2 + \{m_1y_1 + (1 - m_1)y_2\}^2]^{1/2}
\end{aligned}
$$

3. **The equations of variations.** Let

$$
(7)\qquad
\begin{aligned}
x_1 &= \frac{1}{2}(m_3 - m_2) + \epsilon u_1, & y_1 &= \frac{1}{2}\sqrt{3}\,(m_2 + m_3) + \epsilon v_1,\\
x_2 &= \frac{1}{2}m_1 + m_3 + \epsilon u_2, & y_2 &= -\frac{1}{2}\sqrt{3}\,m_1 + \epsilon v_2,
\end{aligned}
$$

where $u_1, \ldots, v_2$ are new dependent variables, and ϵ is an arbitrary parameter. The additive quantities $\epsilon u_1, \ldots, \epsilon v_2$ denote the components of displacement of m_1 and m_2 from the vertices of the equilateral triangle in configuration I (Fig. 1). We shall not consider configuration II (Fig. 2) in detail as the coördinates in Fig. 2 differ from the corresponding ones in Fig. 1 only in the sign of $\sqrt{3}$. Since $\sqrt{3}$ does not occur explicitly in the differential equations, we may change the sign of $\sqrt{3}$ in any solutions which we obtain for configuration I and thereby obtain the corresponding solutions for configuration II.

When equations (7) are substituted in (6) the right members of the differential equations can be expanded as power series in ϵ which will converge for ϵ sufficiently small. Since the expressions in (7) that are independent of ϵ are particular solutions of equations (6), there will be no terms independent of ϵ after the substitutions (7) have been made. Hence the factor ϵ can be divided out of the equations, and when m_3 is replaced by $1 - m_1 - m_2$ the differential equations (6) become

$$
(8)\quad
\begin{aligned}
&\left(D_2 - \frac{3}{4}\right) u_1 - \left\{ 2D + \frac{3\sqrt{3}}{4}(1 - 2m_2)\right\} v_1 + 0u_2 - \frac{3\sqrt{3}}{2} m_2 v_2 \\
&\qquad = \epsilon U_1^{(2)} + \epsilon^2 U_1^{(3)} + \cdots + \epsilon^{n-1} U_1^{(n)} + \cdots, \\
&\left\{ 2D - \frac{3\sqrt{3}}{4}(1 - 2m_2)\right\} u_1 + \left(D^2 - \frac{9}{4}\right) v_1 - \frac{3\sqrt{3}}{2} m_2 u_2 + 0v_2 \\
&\qquad = \sqrt{3}\,[\epsilon V_1^{(2)} + \epsilon^2 V_1^{(3)} + \cdots + \epsilon^{n-1} V_1^{(n)} + \cdots], \\
&-\frac{9}{4} m_1 u_1 - \frac{3\sqrt{3}}{4} m_1 v_1 + \left(D^2 - 3 + \frac{9}{4} m_1\right) u_2 - \left(2D - \frac{3\sqrt{3}}{4} m_1\right) v_2 \\
&\qquad = \epsilon U_2^{(2)} + \epsilon^2 U_2^{(3)} + \cdots + \epsilon^{n-1} U_2^{(n)} + \cdots, \\
&-\frac{3\sqrt{3}}{4} m_1 u_1 + \frac{9}{4} m_1 v_1 + \left(2D + \frac{3\sqrt{3}}{4} m_1\right) u_2 + \left(D^2 - \frac{9}{4} m_1\right) v_2 \\
&\qquad = \sqrt{3}\,[\epsilon V_2^{(2)} + \epsilon^2 V_2^{(3)} + \cdots + \epsilon^{n-1} V_2^{(n)} + \cdots],
\end{aligned}
$$

where D denotes the operator d/dt, and $U_1^{(k)}, \ldots, V_2^{(k)}$ $(k = 2, \ldots, n, \ldots)$ are homogeneous polynomials in u_1, u_2, v_1, v_2 of degree k. In all these polynomials, $\sqrt{3}$ occurs only as a factor of the odd powers of v_1 and v_2, considered together.

If we consider only the linear terms in (8) we obtain the equations of variation. They are

$$
(9)\quad
\begin{aligned}
&\left(D^2 - \frac{3}{4}\right) u_1 - \left\{ 2D + \frac{3\sqrt{3}}{4}(1 - 2m_2)\right\} v_1 + 0u_2 - \frac{3\sqrt{3}}{2} v_2 = 0, \\
&\left\{ 2D - \frac{3\sqrt{3}}{4}(1 - 2m_2)\right\} u_1 + \left(D^2 - \frac{9}{4}\right) v_1 - \frac{3\sqrt{3}}{2} m_2 u_2 + 0v_2 = 0, \\
&-\frac{9}{4} m_1 u_1 - \frac{3\sqrt{3}}{4} m_1 v_1 + \left(D^2 - 3 + \frac{9}{4} m_1\right) u_2 - \left(2D - \frac{3\sqrt{3}}{4} m_1\right) v_2 = 0, \\
&-\frac{3\sqrt{3}}{4} m_1 u_1 + \frac{9}{4} m_1 v_1 + \left(2D + \frac{3\sqrt{3}}{4} m_1\right) u_2 + \left(D^2 - \frac{9}{4} m_1\right) v_2 = 0.
\end{aligned}
$$

4. **The solutions of the equations of variations.** The solutions of the equations of variation can usually be obtained by differentiating the generating solutions, equations (3), with respect to the arbitrary constants which denote the initial time and the scale factor.* This is not possible in the case under consideration since the generating solutions are constants.

Equations (9) are linear differential equations with constant coefficients, and in order to obtain their solutions we consider the operator D as an alge-

* Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. 1, chap. 4.

braic quantity and equate to zero the determinant formed from the coefficients of u_1, u_2, v_1, v_2 in (9). This gives

$$(10)\quad \Delta = \begin{vmatrix} D^2 - \frac{3}{4}, & -2D - \frac{3\sqrt{3}}{4}(1-2m_2), & 0, & -\frac{3\sqrt{3}}{2}m_2 \\ 2D - \frac{3\sqrt{3}}{4}(1-2m_2), & D^2 - \frac{9}{4}, & -\frac{3\sqrt{3}}{2}m_2, & 0 \\ -\frac{9}{4}m_1, & -\frac{3\sqrt{3}}{4}m_1, & D^2 - 3 + \frac{9}{4}m_1, & -2D + \frac{3\sqrt{3}}{4}m_1 \\ -\frac{3\sqrt{3}}{4}m_1, & \frac{9}{4}m_1, & 2D + \frac{3\sqrt{3}}{4}m_1, & D^2 - \frac{9}{4}m_1 \end{vmatrix} = 0.$$

This determinant reduces to

$$(11)\quad \Delta = D^2\Big[D^6 + 2D^4 + \Big\{1 + \frac{27}{4}(m_1m_2 + m_2m_3 + m_3m_1)\Big\}D^2 + \frac{27}{4}(m_1m_2 + m_2m_3 + m_3m_1)\Big] = 0.$$

If we neglect the factor D^2, the preceding equation is a cubic in D^2 and its discriminant is

$$\frac{27}{16}(m_1m_2 + m_2m_3 + m_3m_1)^2\,[27(m_1m_2 + m_2m_3 + m_3m_1) - 1].$$

Since $m_1 + m_2 + m_3 = 1$ and since m_1, m_2, and m_3 are all positive, this discriminant is positive, and therefore the cubic has one negative root and two conjugate complex roots. Let these roots be λ_1^2, λ_2^2, and λ_3^2 where

$$(12)\quad \lambda_1 = \sqrt{-1}\,\lambda, \qquad \lambda_2 = \mu + \sqrt{-1}\,\nu, \qquad \lambda_3 = \mu - \sqrt{-1}\,\nu.$$

Then the solutions of (11) are

$$(13)\quad D = 0, \quad 0, \quad \pm\lambda_1, \quad \pm\lambda_2, \quad \pm\lambda_3,$$

and the solutions of the differential equations (9) are therefore

$$(14)\quad \begin{aligned} u_1 &= \alpha_1 + \alpha_2 t + \alpha_3 e^{\lambda_1 t} + \alpha_4 e^{-\lambda_1 t} + \alpha_5 e^{\lambda_2 t} + \alpha_6 e^{-\lambda_2 t} + \alpha_7 e^{\lambda_3 t} + \alpha_8 e^{-\lambda_3 t}, \\ v_1 &= \beta_1^{(0)}\alpha_1 + \beta_2^{(0)}\alpha_2 t + \beta_1^{(1)}\alpha_3 e^{\lambda_1 t} - \beta_2^{(1)}\alpha_4 e^{-\lambda_1 t} + \beta_1^{(2)}\alpha_5 e^{\lambda_2 t} - \beta_2^{(2)}\alpha_6 e^{-\lambda_2 t} \\ &\quad + \beta_1^{(3)}\alpha_7 e^{\lambda_3 t} - \beta_2^{(3)}\alpha_8 e^{-\lambda_3 t}, \\ u_2 &= \gamma_1^{(0)}\alpha_1 + \gamma_2^{(0)}\alpha_2 t + \gamma_1^{(1)}\alpha_3 e^{\lambda_1 t} + \gamma_2^{(1)}\alpha_4 e^{-\lambda_1 t} + \gamma_1^{(2)}\alpha_5 e^{\lambda_2 t} + \gamma_2^{(2)}\alpha_6 e^{-\lambda_2 t} \\ &\quad + \gamma_1^{(3)}\alpha_7 e^{\lambda_3 t} + \gamma_2^{(3)}\alpha_8 e^{-\lambda_3 t}, \\ v_2 &= \delta_1^{(0)}\alpha_1 + \delta_2^{(0)}\alpha_2 t + \delta_1^{(1)}\alpha_3 e^{\lambda_1 t} - \delta_2^{(1)}\alpha_4 e^{-\lambda_1 t} + \delta_1^{(2)}\alpha_5 e^{\lambda_2 t} - \delta_2^{(2)}\alpha_6 e^{-\lambda_2 t} \\ &\quad + \delta_1^{(3)}\alpha_7 e^{\lambda_3 t} - \delta_2^{(3)}\alpha_8 e^{-\lambda_3 t}, \end{aligned}$$

where $\alpha_1, \ldots, \alpha_8$ are the constants of integration. The remaining symbols $\beta_1^{(0)}$, $\ldots, \delta_2^{(3)}$ denote constants which are defined by the following equations and properties:

$$\beta_1^{(0)} = \frac{1}{1-m_1}\left[\frac{8}{9} - \frac{\sqrt{3}}{3}(1-m_1) - \frac{2}{3}(m - \sqrt{3}\,m_2) + \frac{2}{9(m_1-1)}\{m_1(m_2+m_3) + 4m_2m_3\}\right],$$

$$\gamma_1^{(0)} = \frac{(m_1-1)(\sqrt{3}\,m_1 + 9m_1m_2) + \sqrt{3}\,\{m_1(m_2+m_3) + 4m_2m_3\}}{9m_2(m_1-1)^2},$$

$$\delta_1^{(0)} = -\frac{1}{2m_2}\left[(1-2m_2)\,\beta_1^{(0)} + \frac{8(m_3-m_2)}{9(m_1-1)} + \frac{\sqrt{3}}{3}\right],$$

$$\beta_2^{(0)} = \frac{m_3-m_2}{\sqrt{3}(m_1-1)}, \qquad \gamma_2^{(0)} = \frac{m_1}{m_1-1}, \qquad \delta_2^{(0)} = \frac{m_1+2m_3}{\sqrt{3}(m_1-1)},$$

$$\beta_1^{(i)} = \frac{3\sqrt{3}\,[(m_3-m_2)\lambda_i^2 + 2\sqrt{3}(1-m_1)\lambda_i + 3(m_2-m_3)]}{4\lambda_i^4 - \lambda_i^2(5-9m_1) + 6\sqrt{3}(m_3-m_2)\lambda_i + 27(1-m_1)},$$

$$\gamma_1^{(i)} = \frac{2\sqrt{3}}{9m_2}\left(\lambda_i^2 - \frac{9}{4}\right)\beta_1^{(i)} + \frac{4\sqrt{3}}{9m_2}\lambda_i - \frac{1-2m_2}{2m_2},$$

$$\delta_1^{(i)} = \frac{2\sqrt{3}}{9m_2}\left(\lambda_i^2 - \frac{3}{4}\right) - \left\{\frac{4\sqrt{3}}{9m_2}\lambda_i + \frac{1-2m_2}{2m_2}\right\}\beta_1^{(i)} \qquad (i = 1, 2, 3).$$

The following pairs of constants differ only in the sign of $\sqrt{3}$:

$$\beta_1^{(i)}, \beta_2^{(i)}; \quad \gamma_1^{(i)}, \gamma_2^{(i)}; \quad \delta_1^{(i)}, \delta_2^{(i)}; \qquad (i = 1, 2, 3).$$

The following pairs are conjugate complex:

$$\beta_1^{(1)}, -\beta_2^{(1)}; \quad \gamma_1^{(1)}, \gamma_2^{(1)}; \quad \delta_1^{(1)}, -\delta_2^{(1)}; \quad \beta_k^{(2)}, \beta_k^{(3)}; \quad \gamma_k^{(2)}, \gamma_k^{(3)}; \quad \delta_k^{(2)}, \delta_k^{(3)}; \quad (k = 1, 2)$$

The solutions for D in equations (13) are called the *characteristic exponents*. Since the original differential equations (1) admit the integral (2), then two of the characteristic exponents will be zero.* It was to be expected, therefore, that two roots of (11) should be zero. Poincaré† has also shown that if equa-

* Poincaré, loc. cit., p. 188.

† *Loc. cit.*, p. 69.

tions (1) do not contain t explicitly, which is always the case in all problems of mechanics in which there is a conservative system, then the characteristic exponents are always equal in pairs but opposite in sign. It is this property of the solutions that makes the determinant (11) even in D.

5. **Construction of Asymptotic Solutions in $e^{-\lambda_2 t}$ and $e^{-\lambda_3 t}$.** We shall now construct solutions of the differential equations (6) which are asymptotic in the sense of Poincaré,* that is, *each* term of the solution must have the form

$$e^{ct}\,P(t),$$

where c is a constant having its real part different from zero, and $P(t)$ is a periodic function of t or, in particular, a constant. Such solutions will therefore approach zero as t approaches $+\infty$ or $-\infty$ according as the real part of c is negative or positive, respectively.

The only terms of the solutions of the equations of variation which are asymptotic in the sense just defined are those in $e^{\pm\lambda_2 t}$ and $e^{\pm\lambda_3 t}$. These terms approach zero as t approaches $-\infty$ or $+\infty$ according as the $+$ or $-$ signs are taken with λ_2 and λ_3.

In this section we shall construct the solutions which approach zero as t approaches $+\infty$. These solutions will obviously involve powers of $e^{-\lambda_2 t}$ and $e^{-\lambda_3 t}$. In the next section we shall show how the solutions in $e^{+\lambda_2 t}$ and $e^{+\lambda_3 t}$ can be obtained directly from the solutions in $e^{-\lambda_2 t}$ and $e^{-\lambda_3 t}$.

Only the formal construction of the solutions is considered in §§ 5 and 6, but the convergence of the solutions obtained is established in § 7.

We propose to integrate equations (8) as power series in ϵ and this is the reason why ϵ was introduced in equations (7). Accordingly we substitute

$$(15) \qquad u_i = \sum_{j=0}^{\infty} u_{ij}\,\epsilon^j, \qquad v_i = \sum_{j=0}^{\infty} v_{ij}\,\epsilon^j \qquad (i = 1, 2),$$

in equations (8), and as we shall have frequent occasion to refer to the resulting equations we shall cite them as (8′). These equations (8′) are to be satisfied identically in ϵ and we may therefore equate the coefficients of the same powers of ϵ. In this way we obtain sequences of differential equations in u_{ij} and v_{ij} which can be integrated step by step, as we shall show, subject to suitable initial conditions and restrictions.

The first two steps of the integration will be considered in detail, and then an induction to the general term will be made to show that the process of integration can be carried on indefinitely.

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* Loc. cit., p. 340.

Step 1. *Terms in* (8′) *independent of* ϵ. The terms in (8′) which are independent of ϵ are obviously the same as the equations of variation (9) if the subscripts on the dependent variables in (9) are altered so as to read u_{10}, v_{10}, u_{20} and v_{20} instead of u_1, u_2, v_1, v_2. Hence the solutions of these equations are the same as (14) with the corresponding changes of subscripts. Now only two of the exponentials, viz., $e^{-\lambda_2 t}$ and $e^{-\lambda_3 t}$, approach zero as t approaches $+\infty$ and we therefore put equal to zero the arbitrary constants associated with the other exponentials. The solutions at this step are, then,

$$(16)\quad \begin{aligned} u_{10} &= \alpha_6^{(0)} e^{-\lambda_2 t} + \alpha_8^{(0)} e^{-\lambda_3 t}, & v_{10} &= -\beta_2^{(2)} \alpha_6^{(0)} e^{-\lambda_2 t} - \beta_2^{(3)} \alpha_8^{(0)} e^{-\lambda_3 t}, \\ u_{20} &= \gamma_2^{(2)} \alpha_6^{(0)} e^{-\lambda_2 t} + \gamma_2^{(3)} \alpha_8^{(0)} e^{-\lambda_3 t}, & v_{20} &= -\delta_2^{(2)} \alpha_6^{(0)} e^{-\lambda_2 t} - \delta_2^{(3)} \alpha_8^{(0)} e^{-\lambda_2 t}, \end{aligned}$$

where $\alpha_6^{(0)}$ and $\alpha_8^{(0)}$ are arbitrary, and $\beta_2^{(2)}, \ldots, \delta_2^{(3)}$ are defined as in equations (14).

It is evident that the complementary functions at all the succeeding steps of the integrations will be the same as (14) and that after the exponentials are rejected which do not have the proper form, two constants of integration will remain undetermined. Hence, besides rejecting all the exponentials except $e^{-\lambda_2 t}$ and $e^{-\lambda_3 t}$, it is necessary to impose two initial conditions upon these solutions so as to determine the arbitrary constants arising at each step of the integration.

Let us suppose that

$$(17)\qquad u_1(0) = \epsilon_1, \qquad u_2(0) = \gamma,$$

where ϵ_1 and γ are arbitrary. Since u_1 and u_2 are multiplied by the arbitrary parameter ϵ in (7) we may put ϵ_1 or γ equal to unity. Let us suppose $\epsilon_1 = 1$. When the conditions (17) are imposed on (15) it follows that

$$(18)\qquad \begin{aligned} u_{10}(0) &= 1, & u_{1j}(0) &= 0, \\ u_{20}(0) &= \gamma, & u_{2j}(0) &= 0 \qquad (j = 1, \ldots, \infty). \end{aligned}$$

Now applying these conditions to the solutions (16) we obtain

$$\alpha_6^{(0)} = \frac{\gamma_2^{(3)} - \gamma}{\gamma_2^{(3)} - \gamma_2^{(2)}}, \qquad \alpha_8^{(0)} = \frac{\gamma - \gamma_2^{(2)}}{\gamma_2^{(3)} - \gamma_2^{(2)}}.$$

Since $\gamma_2^{(2)}$ and $\gamma_2^{(3)}$ are conjugate imaginaries, it follows that $\alpha_6^{(0)}$ and $\alpha_8^{(0)}$ are likewise conjugates.

In order to unify the notation, we put

$$(19)\quad \begin{aligned} \alpha_6^{(0)} &= \alpha_{10}^{(10)}, & -\beta_2^{(2)} \alpha_6^{(0)} &= \beta_{10}^{(10)}, & \gamma_2^{(2)} \alpha_6^{(0)} &= \alpha_{20}^{(10)}, & -\delta_2^{(2)} \alpha_6^{(0)} &= \beta_{20}^{(10)}, \\ \alpha_8^{(0)} &= \alpha_{10}^{(01)}, & -\beta_2^{(3)} \alpha_8^{(0)} &= \beta_{10}^{(01)}, & \gamma_2^{(3)} \alpha_8^{(0)} &= \alpha_{20}^{(01)}, & -\delta_2^{(3)} \alpha_8^{(0)} &= \beta_{20}^{(01)}. \end{aligned}$$

Since $\alpha_6^{(0)}$, $\alpha_8^{(0)}$; $\beta_2^{(2)}$, $\beta_2^{(3)}$; $\gamma_2^{(2)}$, $\gamma_2^{(3)}$; $\delta_2^{(2)}$, $\delta_2^{(3)}$ are conjugate pairs, it follows that the symbols in (19) which differ only in a permutation of superscripts are likewise conjugate pairs. This notation will be adopted in the sequel to denote conjugate complexes. Thus $\alpha_{i1}^{(jk)}$ and $\alpha_{i1}^{(jk)}$ $(j \neq k)$, are conjugate imaginaries.

On employing the above notation, we obtain for the desired solutions at this step

$$(20)\quad u_{i0} = \alpha_{i0}^{(10)} e^{-\lambda_2 t} + \alpha_{i0}^{(01)} e^{-\lambda_3 t}, \quad v_{i0} = \beta_{i0}^{(10)} e^{-\lambda_2 t} + \beta_{i0}^{(01)} e^{-\lambda_3 t} \qquad (i = 1, 2),$$

where $\alpha_{i0}^{(10)}, \ldots, \beta_{i0}^{(01)}$ are linear in γ.

If we put

$$\lambda_2 = \mu + \sqrt{-1}\,\nu, \qquad \lambda_3 = \mu - \sqrt{-1}\,\nu,$$

and suppose that

$$\alpha_{i0}^{(10)} = \tfrac{1}{2}\,[a_{i0}^{(1)} + \sqrt{-1}\,b_{i0}^{(1)}], \qquad \beta_{i0}^{(10)} = \tfrac{1}{2}\,[c_{i0}^{(1)} + \sqrt{-1}\,d_{i0}^{(1)}],$$

then the solutions (20) become

$$(21)\quad \begin{aligned} u_{i0} &= e^{-\mu t}\,[a_{i0}^{(1)} \cos \nu t + b_{i0}^{(1)} \sin \nu t], \\ v_{i0} &= e^{-\mu t}\,[c_{i0}^{(1)} \cos \nu t + d_{i0}^{(1)} \sin \nu t]. \end{aligned}$$

This second form of the solutions is more convenient than (20) for numerical computation but is more cumbersome in obtaining the solutions at the succeeding steps.

Step 2. *Coefficients of* ϵ *to the first degree.* When the solutions (20) have been substituted in (8′), the differential equations obtained by equating the coefficients of ϵ to the first degree are

$$\left(D^2 - \frac{3}{4}\right) u_{11} - \left\{2D + \frac{3\sqrt{3}}{4}(1 - 2m_2)\right\} v_{11} + 0\, u_{21} - \frac{3\sqrt{3}}{2} m_2 v_{21} = U_{11},$$

$$\left\{2D - \frac{3\sqrt{3}}{4}(1 - 2m_2)\right\} u_{11} + \left(D^2 - \frac{9}{4}\right) v_{11} - \frac{3\sqrt{3}}{2} m_2 u_{21} + 0\, v_{21} = V_{11},$$

$$(22)\quad -\frac{9}{4} m_1 u_{11} - \frac{3\sqrt{3}}{4} m_1 v_{11} + \left(D^2 - 3 + \frac{9}{4} m_1\right) u_{21} - \left(2D - \frac{3\sqrt{3}}{4} m_1\right) v_{21} = U_{21},$$

$$-\frac{3\sqrt{3}}{4} m_1 u_{11} + \frac{9}{4} m_1 v_{11} + \left(2D + \frac{3\sqrt{3}}{4} m_1\right) u_{21} + \left(D^2 - \frac{9}{4} m_1\right) v_{21} = V_{21};$$

$$
(23)\qquad
\begin{aligned}
U_{11} &= A_{11}^{(20)}\, e^{-2\lambda_2 t} + A_{11}^{(11)}\, e^{-(\lambda_2+\lambda_3)t} + A_{11}^{(02)}\, e^{-2\lambda_3 t},\\
V_{11} &= B_{11}^{(20)}\, e^{-2\lambda_2 t} + B_{11}^{(11)}\, e^{-(\lambda_2+\lambda_3)t} + B_{11}^{(02)}\, e^{-2\lambda_3 t},\\
U_{21} &= A_{21}^{(20)}\, e^{-2\lambda_2 t} + A_{21}^{(11)}\, e^{-(\lambda_2+\lambda_3)t} + A_{21}^{(02)}\, e^{-2\lambda_3 t},\\
V_{21} &= B_{21}^{(20)}\, e^{-2\lambda_2 t} + B_{21}^{(11)}\, e^{-(\lambda_2+\lambda_3)t} + B_{21}^{(02)}\, e^{-2\lambda_3 t},
\end{aligned}
$$

where $A_{11}^{(20)}, \ldots, B_{21}^{(02)}$ are quadratic expressions in γ having constant coefficients The constants like $A_{11}^{(20)}$ and $A_{11}^{(02)}$ which have the same subscripts but which have their superscripts reversed are conjugate imaginaries. The other constants $A_{11}^{(11)}, \ldots, B_{21}^{(11)}$ are real.

The complementary functions of equations (22) are the same as (14), and the particular integrals can be found by the method of the variation of parameters as at the previous step. These particular integrals are

$$
(24)\qquad
\begin{aligned}
u_{i1} &= \sum_{j,\,k=0}^{2} \alpha_{i1}^{(jk)}\, e^{-(j\lambda_2+k\lambda_3)t} \qquad (j+k=2, \quad i=1,2),\\
v_{i1} &= \sum_{j,\,k=0}^{2} \beta_{i1}^{(jk)}\, e^{-(j\lambda_2+k\lambda_3)t},
\end{aligned}
$$

where

$$
(25)\qquad
\begin{aligned}
\alpha_{11}^{(jk)} &= \frac{\Delta_1^{(jk)}}{\Delta[-(j\lambda_2+k\lambda_3)]}, &\qquad \beta_{11}^{(jk)} &= \frac{\Delta_2^{(jk)}}{\Delta[-(j\lambda_2+k\lambda_3)]},\\
\alpha_{21}^{(jk)} &= \frac{\Delta_3^{(jk)}}{\Delta[-(j\lambda_2+k\lambda_3)]}, &\qquad \beta_{21}^{(jk)} &= \frac{\Delta_4^{(jk)}}{\Delta[-(j\lambda_2+k\lambda_3)]}.
\end{aligned}
$$

The preceding symbols are defined as follows:

$\Delta[-(j\lambda_2+k\lambda_3)]$ denotes the determinant Δ in (10) when D has been replaced by $-(j\lambda_2+k\lambda_3)$;

$\Delta_i^{(jk)}$, $i=1, 2, 3, 4$, denotes the preceding determinant $\Delta[-(j\lambda_2+k\lambda_3)]$ when the elements of the ith row, reading from top to bottom, have been replaced by $A_{11}^{(jk)}, B_{11}^{(jk)}, A_{21}^{(jk)}, B_{21}^{(jk)}$, respectively.

The constants in (25), viz., $\alpha_{11}^{(jk)}, \ldots, \beta_{21}^{(jk)}$ are found to have the same form as the constants in (23). Thus $\alpha_{11}^{(jk)}$ and $\alpha_{11}^{(kj)}$ are conjugate complex if $j \neq k$, but real if $j=k$.

If the right members in (22) contained any constant terms, or terms in t or in $e^{\pm\lambda_i t}$, $i=1, 2, 3$, that is, terms which are the same functions of t as the terms in the complementary functions, then the denominators in (25) would vanish and the particular integrals would not take the form (24), but would contain terms in t, t^2 or $t\, e^{\pm\lambda_i t}$. Hence, such terms would appear in the particular integrals if and only if the right members contained constants, terms in t or $e^{\pm\lambda_i t}$, respectively. In order, then, to show that there are no terms in the particular integrals which do not satisfy Poincaré's definition of asymptotic solutions. it is sufficient to show that the right members do not contain any terms which are the same functions of t as any part of the complementary functions.

To obtain the complete solutions of (22), we combine the complementary functions, equations similar to (14), and the particular integrals (24). The particular integrals have the desired form for asymptotic solutions, but we must reject all the terms of the complementary functions except those in $e^{-\lambda_2 t}$ and $e^{-\lambda_3 t}$. This is possible, of course, by equating to zero all the arbitrary constants of integration except those associated with the exponentials $e^{-\lambda_2 t}$ and $e^{-\lambda_3 t}$. The desired solutions are therefore

$$
\begin{aligned}
u_{11} &= \alpha_6^{(1)} e^{-\lambda_2 t} + \alpha_8^{(1)} e^{-\lambda_3 t} + \alpha_{11}^{(20)} e^{-2\lambda_2 t} + \alpha_{11}^{(11)} e^{-(\lambda_2+\lambda_3)t} + \alpha_{11}^{(02)} e^{-2\lambda_3 t},\\
v_{11} &= -\beta_2^{(2)} \alpha_6^{(1)} e^{-\lambda_2 t} - \beta_2^{(3)} \alpha_8^{(1)} e^{-\lambda_3 t} + \beta_{11}^{(20)} e^{-2\lambda_2 t} + \beta_{11}^{(11)} e^{-(\lambda_2+\lambda_3)t}\\
&\qquad + \beta_{11}^{(02)} e^{-2\lambda_3 t},\\
u_{21} &= \gamma_2^{(2)} \alpha_6^{(1)} e^{-\lambda_2 t} + \gamma_2^{(3)} \alpha_8^{(1)} e^{-\lambda_3 t} + \alpha_{21}^{(20)} e^{-2\lambda_2 t} + \alpha_{21}^{(11)} e^{-(\lambda_2+\lambda_3)t} + \alpha_{21}^{(02)} e^{-2\lambda_3 t},\\
v_{21} &= -\delta_2^{(2)} \alpha_6^{(1)} e^{-\lambda_2 t} - \delta_2^{(3)} \alpha_8^{(1)} e^{-\lambda_3 t} + \beta_{21}^{(20)} e^{-2\lambda_2 t} + \beta_{21}^{(11)} e^{-(\lambda_2+\lambda_3)t}\\
&\qquad + \beta_{21}^{(02)} e^{2\lambda_3 t},
\end{aligned}
$$

where $\alpha_6^{(1)}$ and $\alpha_8^{(1)}$ are the constants of integration. By virtue of the initial conditions (18), these constants must satisfy the equations

$$
\begin{aligned}
\alpha_6^{(1)} + \alpha_8^{(1)} &= -[\alpha_{11}^{(20)} + \alpha_{11}^{(11)} + \alpha_{11}^{(02)}] \equiv R_1^{(1)},\\
\gamma_2^{(2)} \alpha_6^{(1)} + \gamma_2^{(3)} \alpha_8^{(1)} &= -[\alpha_{21}^{(20)} + \alpha_{21}^{(11)} + \alpha_{21}^{(02)}] \equiv R_2^{(1)},
\end{aligned}
$$

or

$$
\alpha_6^{(1)} = \frac{R_1^{(1)} \gamma_2^{(3)} - R_2^{(1)}}{\gamma_2^{(3)} - \gamma_2^{(2)}}, \qquad \alpha_8^{(1)} = \frac{R_2^{(1)} - R_1^{(1)} \gamma_2^{(2)}}{\gamma_2^{(3)} - \gamma_2^{(2)}}.
$$

Since $\alpha_{11}^{(20)}$, $\alpha_{11}^{(02)}$ and $\alpha_{21}^{(20)}$, $\alpha_{21}^{(02)}$ are conjugate pairs and $\alpha_{11}^{(11)}$ and $\alpha_{21}^{(11)}$ are real, $R_1^{(1)}$ and $R_2^{(1)}$ are likewise real. But as $\gamma_2^{(2)}$ and $\gamma_2^{(3)}$ are conjugates, it follows that $\alpha_6^{(1)}$ and $\alpha_8^{(1)}$ are also conjugates.

In order to unify the notation, as at step 1, we put

$$
\begin{aligned}
&\alpha_6^{(1)} = \alpha_{11}^{(10)}, \quad -\beta_2^{(2)} \alpha_6^{(1)} = \beta_{11}^{(10)}, \quad \gamma_2^{(2)} \alpha_6^{(1)} = \alpha_{21}^{(10)}, \quad -\delta_2^{(2)} \alpha_6^{(1)} = \beta_{21}^{(10)},\\
&\alpha_8^{(1)} = \alpha_{11}^{(01)} \quad -\beta_2^{(3)} \alpha_8^{(1)} = \beta_{11}^{(01)}, \quad \gamma_2^{(3)} \alpha_8^{(1)} = \alpha_{21}^{(01)}, \quad -\delta_2^{(2)} \alpha_8^{(1)} = \beta_{21}^{(01)},
\end{aligned}
$$

and the desired solutions at this step take the form

$$
(26) \qquad
\begin{aligned}
u_{i1} &= \sum_{j,k=0}^{2} \alpha_{i1}^{(jk)} e^{-(j\lambda_2 + k\lambda_3)t},\\
v_{i1} &= \sum_{j,k=0}^{2} \beta_{i1}^{(jk)} e^{-(j\lambda_2 + k\lambda_3)t} \qquad (j+k = 1 \text{ or } 2, \quad i = 1, 2).
\end{aligned}
$$

If we put

$$\lambda_2 = \mu + \sqrt{-1}\,\nu, \qquad \lambda_3 = \mu - \sqrt{-1}\,\nu,$$

and if we suppose that

$$(27)\qquad \begin{aligned} \alpha_{i1}^{(jk)} &= \frac{1}{2}\,[a_{i1}^{(jk)} + \sqrt{-1}\,b_{i1}^{(jk)}], \quad \alpha_{i1}^{(kj)} = \frac{1}{2}\,[a_{i1}^{(jk)} - \sqrt{-1}\,b_{i1}^{(jk)}],\\ \beta_{i1}^{(jk)} &= \frac{1}{2}\,[c_{i1}^{(jk)} + \sqrt{-1}\,d_{i1}^{(jk)}], \quad \beta_{i1}^{(kj)} = \frac{1}{2}\,[c_{i1}^{(jk)} - \sqrt{-1}\,d_{i1}^{(jk)}] \quad (i = 1, 2), \end{aligned}$$

where $b_{i1}^{(jk)}$ and $d_{i1}^{(jk)}$ are different from zero when $j \neq k$, but equal to zero when $j = k$, then the solutions (26) become

$$u_{i1} = e^{-\mu t}\,[a_{i1}^{(10)} \cos \nu t + b_{i1}^{(10)} \sin \nu t] + e^{-2\mu t}\left[\frac{1}{2}\,a_{i1}^{(11)} + a_{i1}^{(20)} \cos 2\nu t + b_{i1}^{(20)} \sin 2\nu t\right] \quad (i = 1, 2),$$

$$v_{i1} = e^{-\mu t}\,[c_{i1}^{(10)} \cos \nu t + d_{i1}^{(10)} \sin \nu t] + e^{-2\mu t}\left[\frac{1}{2}\,c_{i1}^{(11)} + c_{i1}^{(20)} \cos 2\nu t + d_{i1}^{(20)} \sin 2\nu t\right].$$

The succeeding steps of the integration are similar to the preceding step, and an induction to the general term will now be made to show that the process of integration may be carried on for any desired number of steps.

Let us suppose that u_{il}, v_{il}, $i = 1, 2$, have been computed for $l = 0, \ldots, n - 1$, and that

$$(28)\qquad \begin{aligned} u_{il} &= \sum_{j,k=0}^{l+1} \alpha_{il}^{(jk)}\, e^{-(j\lambda_2 + k\lambda_3)t},\\ v_{il} &= \sum_{j,k=0}^{l+1} \beta_{il}^{(jk)}\, e^{-(j\lambda_2 + k\lambda_3)t} \qquad (j + k = 1, 2, \ldots, l + 1). \end{aligned}$$

where $\alpha_{il}^{(jk)}$ and $\alpha_{il}^{(kj)}$; $\beta_{il}^{(jk)}$ and $\beta_{il}^{(kj)}$ are conjugate complex if $j \neq k$ but real if $j = k$. In order to make the induction it will be necessary to show from the differential equations in u_{in} and v_{in} obtained from (8′) that the solutions for these variable are the same as (28) if $l = n$.

Step $n + 1$: Coefficients of ϵ^n in (8′). Let us consider the differential equations obtained by equating the coefficients of ϵ^n in (8′) after the various solutions in (28) have been substituted. They are found to have the same form as (22) if the second subscript on the variables and the right members is changed to n. The right members, however, have the form

$$
(29)\qquad
\begin{aligned}
U_{in} &= \sum_{j,k=0}^{n+1} A_{in}^{(jk)}\, e^{-(j\lambda_2 + k\lambda_3)t},\\
V_{in} &= \sum_{j,k=0}^{n+1} B_{in}^{(jk)}\, e^{-(j\lambda_2 + k\lambda_3)t} \qquad (j + k = 2, 3, \ldots, n + 1;\ \ i = 1, 2),
\end{aligned}
$$

where $A_{in}^{(jk)}$ and $B_{in}^{(jk)}$ are complex quantities similar to those in (28). The complementary functions of these differential equations which have the desired exponentials are the same as (16) if the second subscript on the variables and the superscript on the constants of integration are replaced by n. The particular integrals of these differential equations can be found as at the previous step by the method of the variation of parameters. They are

$$
(30)\qquad
\begin{aligned}
u_{1n} &= \sum_{j,k=0}^{n+1} \frac{\Delta_{1n}^{(j+k)}}{\Delta[-(j\lambda_2 + k\lambda_3)]}\, e^{-(j\lambda_2 + k\lambda_3)t} \equiv \sum_{j,k=0}^{n+1} \alpha_{1n}^{(jk)}\, e^{-(j\lambda_2 + k\lambda_3)t},\\
v_{1n} &= \sum_{j,k=0}^{n+1} \frac{\Delta_{2n}^{(j+k)}}{\Delta[-(j\lambda_2 + k\lambda_3)]}\, e^{-(j\lambda_2 + k\lambda_3)t} \equiv \sum_{j,k=0}^{n+1} \beta_{1n}^{(jk)}\, e^{-(j\lambda_2 + k\lambda_3)t},\\
u_{2n} &= \sum_{j,k=0}^{n+1} \frac{\Delta_{3n}^{(j+k)}}{\Delta[-(j\lambda_2 + k\lambda_3)]}\, e^{-(j\lambda_2 + k\lambda_3)t} \equiv \sum_{j,k=0}^{n+1} \alpha_{2n}^{(jk)}\, e^{-(j\lambda_2 + k\lambda_3)t},\\
v_{2n} &= \sum_{j,k=0}^{n+1} \frac{\Delta_{4n}^{(j+k)}}{\Delta[-(j\lambda_2 + k\lambda_3)]}\, e^{-(j\lambda_2 + k\lambda_3)t} \equiv \sum_{j,k=0}^{n+1} \beta_{2n}^{(jk)}\, e^{-(j\lambda_2 + k\lambda_3)t}
\end{aligned}
$$

$$(j + k = 2, 3, \ldots, n + 1).$$

The various Δ's in the preceding equations are defined as follows:

$\Delta[-(j\lambda_2 + k\lambda_3)]$ denotes the determinant Δ in (10) if D is replaced by $-(j\lambda_2 + k\lambda_3)$. Since $j + k$ is not less than 2, these denominators do not become $\Delta(-\lambda_2)$ or $\Delta(-\lambda_3)$ and are therefore different from zero.

$\Delta_{in}^{(j+k)}$, $i = 1, 2, 3, 4$, denote the preceding determinant if the elements of the ith row, reading from top to bottom, are replaced by $A_{1n}^{(jk)}$, $B_{1n}^{(jk)}$, $A_{2n}^{(jk)}$ and $B_{2n}^{(jk)}$, respectively.

On determining the constants of integration $\alpha_6^{(n)}$ and $\alpha_8^{(n)}$ by the initial conditions (18), and then unifying the notation as at step 2, the complete solutions which have the desired form are found to be the same as (28) if $l = n$. This completes the induction.

The solutions, therefore, at the general step n are

$$
(31)\qquad
\begin{aligned}
u_{in} &= \sum_{j,k=0}^{n+1} \alpha_{in}^{(jk)} e^{-(j\lambda_2 + k\lambda_3)t}, \\
v_{in} &= \sum_{j,k=0}^{n+1} \beta_{in}^{(jk)} e^{-(j\lambda_2 + k\lambda_3)t}
\end{aligned}
$$

$$
(i = 1, 2; \quad j + k = 1, 2, \ldots, n + 1; \quad n = 0, 1, 2, \ldots).
$$

When equations (31) are substituted in (15), the asymptotic solutions of the differential equations (8) are then found to be

$$
(32)\qquad
\begin{aligned}
\epsilon u_i &= \sum_{n=1}^{\infty} \sum_{j,k=0}^{n} \alpha_{in}^{(jk)} e^{-(j\lambda_2 + k\lambda_3)t} \epsilon^n, \\
\epsilon v_i &= \sum_{n=1}^{\infty} \sum_{j,k=0}^{n} \beta_{in}^{(jk)} e^{-(j\lambda_2 + k\lambda_3)t} \epsilon^n \qquad (i = 1, 2; \quad j + k = 1, 2, \ldots, n),
\end{aligned}
$$

where the second subscript on the constants $\alpha_{in}^{(jk)}$ and $\beta_{in}^{(jk)}$ has been made to conform with the powers of ϵ.

If these solutions are expressed in trigonometric form by the substitution used in (27) we obtain

$$
(33)\qquad
\begin{aligned}
\epsilon u_i = f_i(+\sqrt{3}, -t) \equiv \sum_{n=1}^{\infty} \sum_{j=1}^{\frac{n}{2}, \frac{n+1}{2}}{}' \sum_{k=0 \text{ or } 1}^{j}{}^{*} &[\{a_{in}^{(2j-1, 2k-1)} \cos(2k - 1) \nu t \\
+ b_{in}^{(2j-1, 2k-1)} \sin(2k - 1) \nu t\} e^{-(2j-1)\mu t} &+ \{a_{in}^{(2j, 2k)} \cos 2k \nu t \\
&+ b_{in}^{(2j, 2k)} \sin 2k \nu t\} e^{-2j\mu t}] \epsilon^n, \\
\epsilon v_i = g_i(+\sqrt{3}, -t) \qquad\qquad &(i, = 1, 2),
\end{aligned}
$$

where Σ' denotes that the highest value of $2l - 1$ or $2l$ is n, and Σ^* denotes that the lowest values of $2k - 1$ and $2k$ are 1 and 0, respectively. The function $g_i(+\sqrt{3}, -t)$ is the same as $f_i(+\sqrt{3}, -t)$ if the constants a and b with their subscripts and superscripts are replaced by c and d, respectively, with the same subscripts and superscripts. When these equations are substituted in (7) we obtain the parametric equations of the orbits which approach the vertices of the equilateral triangles in configuration I, viz.,

$$
(34) \qquad
\begin{aligned}
x_1 &= \frac{1}{2}(m_3 - m_2) + f_1(+\sqrt{3}, -t),\\
y_1 &= \frac{\sqrt{3}}{2}(m_2 + m_3) + g_1(+\sqrt{3}, -t),\\
x_2 &= \frac{1}{2}m_1 + m_3 + f_2(+\sqrt{3}, -t),\\
y_2 &= -\frac{\sqrt{3}}{2}m_1 + g_2(+\sqrt{3}, -t),\\
x_3 &= -\frac{1}{m_3}(m_1 x_1 + m_2 x_2),\\
y_3 &= -\frac{1}{m_3}(m_1 y_1 + m_2 y_2).
\end{aligned}
$$

The orbits which approach the configuration II are obtained by changing the sign of $\sqrt{3}$ in (34). They have the equations

$$
(35) \qquad
\begin{aligned}
x_1 &= \frac{1}{2}(m_3 - m_2) + f_1(-\sqrt{3}, -t),\\
y_1 &= -\frac{\sqrt{3}}{2}(m_2 + m_3) + g_1(-\sqrt{3}, -t),\\
x_2 &= \frac{1}{2}m_1 + m_3 + f_2(-\sqrt{3}, -t),\\
y_2 &= \frac{\sqrt{3}}{2}m_1 + g_2(-\sqrt{3}, -t),\\
x_3 &= -\frac{1}{m_3}(m_1 x_1 + m_2 x_2),\\
y_3 &= -\frac{1}{m_3}(m_1 y_1 + m_2 y_2).
\end{aligned}
$$

Equations (34) and (35) therefore represent the orbits which approach the vertices of the equilateral triangles in configurations I and II, respectively, as the time approaches plus infinity. These orbits deal with the *future* of the system and, at the risk of being censored, we shall cite them as *future* I and *future* II, respectively.

6. **The asymptotic solutions in $e^{\lambda_2 t}$ and $e^{\lambda_3 t}$.** This section deals with the asymptotic orbits which approach the vertices of the equilateral triangles as the time approaches $-\infty$. Such orbits deal with the *past* while the previous orbits forecast the *future*. So then the mathematical astronomer can say with more than *poetic license*

"Backward, turn backward, O time in your flight."

It is obvious that these orbits can be constructed by making the same use of the exponentials $e^{\lambda_2 t}$ and $e^{\lambda_3 t}$ as was made of $e^{-\lambda_2 t}$ and $e^{-\lambda_3 t}$ in constructing the previous orbits. This construction will not be considered in detail as we shall show that the *past* orbits can be obtained from the *future* orbits in a very simple way. This method, it will be observed, is the converse of the adage of "history repeating itself."

Let us consider the differential equations (8) and let their solutions (33) be denoted by $f_i(+\sqrt{3}, -t)$, $g_i(+\sqrt{3}, -t)$, $i = 1, 2$. Next consider the effect of changing the signs of t, $\sqrt{3}$, v_i in (8). The left members of the first and third equations of (8) remain unchanged while those of the second and fourth change signs. Since $u_1^{(k)}$ and $v_1^{(k)}$ are even in $\sqrt{3}$, v_1, and v_2, considered together, the above changes of signs will leave these expressions unaltered. But as the right members of the second and fourth equations in (8) contain the factor $\sqrt{3}$, these changes of signs will produce a change of sign not only in the left members but also in the right members of these two equations and the minus sign can be cancelled off in both equations. Hence the differential equations (8) are unchanged if the signs of t, $\sqrt{3}$, v_1 and v_2 are changed. These changes of signs have no effect upon the initial conditions (18) and consequently if we make the same changes of signs in the solutions $f_i(+\sqrt{3}, -t)$, $g_i(+\sqrt{3}, -t)$, it will still leave them solutions of (8). Thus $f_i(-\sqrt{3}, +t)$ and $-g_i(-\sqrt{3}, +t)$ are solutions of (8) and the corresponding solutions of (1) are

$$(36)\quad \begin{aligned} x_1 &= \frac{1}{2}(m_3 - m_2) + f_1(-\sqrt{3}, +t), \\ y_1 &= \frac{\sqrt{3}}{2}(m_2 + m_3) - g_1(-\sqrt{3}, +t), \\ x_2 &= \frac{1}{2}m_1 + m_3 + f_2(-\sqrt{3}, +t), \\ y_2 &= -\frac{\sqrt{3}}{2}m_1 - g_2(-\sqrt{3}, +t), \\ x_3 &= -\frac{1}{m_3}(m_1 x_1 + m_2 x_2), \\ y_3 &= -\frac{1}{m_3}(m_1 y_1 + m_2 y_2). \end{aligned}$$

These are the orbits which approach the vertices of the equilateral triangle in configuration I as the time approaches $-\infty$, and will be cited as *past* I. The corresponding orbits for configuration II, *past* II, are obtained from (36) by changing the sign of $\sqrt{3}$. They are

$$
(37)\qquad
\begin{aligned}
x_1 &= \frac{1}{2}(m_3 - m_2) + f_1(+\sqrt{3}, +t),\\
y_1 &= -\frac{\sqrt{3}}{2}(m_2 + m_3) - g_1(+\sqrt{3}, +t),\\
x_2 &= \frac{1}{2}m_1 + m_3 + f_2(+\sqrt{3}, +t),\\
y_2 &= +\frac{\sqrt{3}}{2}m_1 - g_2(+\sqrt{3}, +t),\\
x_3 &= -\frac{1}{m_3}(m_1 x_1 + m_2 x_2),\\
y_3 &= -\frac{1}{m_3}(m_1 y_1 + m_2 y_2).
\end{aligned}
$$

From the form of the equations of these four orbits it is evident that the *past* II orbits are obtained from the *future* I orbits by changing the sign of t in the latter and reflecting in the x-axis. The same relation exists between *past* I and *future* II orbits.

7. **The convergence of solutions.** Only the formal construction of the solutions (8) has been made, and we shall now consider their convergence.

The convergence of the solutions (32) of the differential equations (8) depends upon the form of the characteristic exponents in terms of which the solutions are expanded. Now it has been shown by Poincaré* that such solutions as (32), (34) and (35) will converge as t approaches $+\infty$ provided that the real parts of the characteristic exponents in terms of which the solutions have been expanded are different from zero and positive, and likewise solutions such as (36) and (37) will converge as t approaches $-\infty$ provided that the corresponding exponents are different from zero and negative. The characteristic exponents of this problem are $0, 0, \pm\lambda_1, \pm\lambda_2, \pm\lambda_3$, equations (13), but only λ_2 and λ_3 have their real parts different from zero, and it is in terms of only these exponents that the solutions have been expanded. Hence the solutions (34) and (35) converge as t approaches $+\infty$, and the solutions (36) and (37) converge as t approaches $-\infty$.

8. **Illustrative examples.** We shall conclude this paper with illustrative numerical examples. The values of the masses chosen are not the ratios of the masses of the Sun, Jupiter and the planetoids mentioned in § 1, but more simple values to illustrate the nature of the orbits.

Let $m_1 = 0.2$, $m_2 = 0.3$ and $m_3 = 0.5$. Then M, the sum of the masses, is unity, and n, the mean angular motion, is $+1$ or -1. We have chosen the

* *Loc. cit.*, vol. 1, pp. 338–343.

counter-clockwise direction of rotation and put $n = +1$. The vertices of the equilateral triangle for configuration I are $(0.1, 0.4\sqrt{3})$, $(0.6, -0.1\sqrt{3})$ and $(-0.4, -0.1\sqrt{3})$, and for configuration II $(0.1, -0.4\sqrt{3})$, $(0.6, 0.1\sqrt{3})$ and $(-0.4, 0.1\sqrt{3})$ for m_1, m_2 and m_3, respectively.

The computation has been carried out for only the linear terms in ϵ for the orbits (32). The values of the various constants together with the equations which define them are listed in the following table. The conjugates of the various terms are omitted.

TABLE I

Constant	Equation	Value
Δ	(11)	$D^2(D^6+2D^4+3.0925D^2+2.0925)$
λ_1	(12)	$\sqrt{-1}$
$\lambda_2 = \mu + \sqrt{-1}\,\nu$	(12)	$0.688+0.987\sqrt{-1}$
$\beta_2^{(2)}$	(14) et seq.	$0.152+0.948\sqrt{-1}$
$\gamma_2^{(2)}$	(14) et seq.	$-0.246+0.547\sqrt{-1}$
$\delta_2^{(2)}$	(14) et seq.	$-0.630-0.315\sqrt{-1}$
$\alpha_6^{(0)} = \alpha_{10}^{(10)}$	(16), (19)	$0.500-(0.225+0.914\gamma)\sqrt{-1}$
$-\beta_2^{(2)}\alpha_6^{(0)} = \beta_{10}^{(10)}$	(16), (19)	$-(0.289+0.866\gamma)-(0.440-0.139\gamma)\sqrt{-1}$
$\gamma_2^{(2)}\alpha_6^{(0)} = \alpha_{20}^{(10)}$	(16), (19)	$0.500\gamma+(0.328+0.225\gamma)\sqrt{-1}$
$-\delta_2^{(2)}\alpha_6^{(0)} = \beta_{20}^{(10)}$	(16), (19)	$(0.386+0.288\gamma)+(0.015-0.578\gamma)\sqrt{-1}$

The solutions for ϵu_1, ϵv_1, ϵu_2, and ϵv_2 are

$$\epsilon u_1 = e^{-0.688t}\,[\cos 0.987t - (0.450 + 1.828\gamma)\sin 0.987t]\,\epsilon + \cdots,$$
$$\epsilon v_1 = e^{-0.688t}\,[(0.578 + 1.732\gamma)\cos 0.987t + (0.880 - 0.278\gamma)\sin 0.987t]\,\epsilon + \cdots,$$
$$\epsilon u_2 = e^{-0.688t}\,[\gamma\cos 0.987t + (0.656 + 0.450\gamma)\sin 0.987t]\,\epsilon + \cdots,$$
$$\epsilon v_2 = e^{-0.688t}\,[(0.772 + 0.576\gamma)\cos 0.987t + (0.030 - 1.152\gamma)\sin 0.987t]\,\epsilon + \cdots.$$

Example I.

If we put $\epsilon = \gamma = 0.1$ and consider only the linear terms in ϵ, these solutions become

$$\epsilon u_1 = e^{-0.688t}\,[0.1000\cos 0.987t - 0.0633\sin 0.987t],$$
$$\epsilon v_1 = -e^{-0.688t}\,[0.0751\cos 0.987t + 0.0852\sin 0.987t],$$
$$\epsilon u_2 = e^{-0.688t}\,[0.0100\cos 0.987t + 0.0701\sin 0.987t],$$
$$\epsilon v_2 = e^{-0.688t}\,[0.0830\cos 0.987t - 0.0085\sin 0.987t].$$

The values of the above quantities for various values of t are listed in Table II. The corresponding values for ϵu_3 and ϵv_3 have been computed from the center of gravity equations. The coördinates $(\epsilon\, u_1, \epsilon\, v_1)$, $(\epsilon\, u_2, \epsilon\, v_2)$ and $(\epsilon\, u_3, \epsilon\, v_3)$ denote the x- and y-displacements of m_1, m_2 and m_3, respectively, from the vertices of the equilateral triangle in configuration I.

TABLE II

$\epsilon = 0.1 \quad \gamma = 0.1$

t	ϵu_1	ϵv_1	ϵu_2	ϵv_2	ϵu_3	ϵv_3
0	+0.100	−0.075	+0.010	+0.083	−0.046	−0.020
0.1	+0.087	−0.078	+0.016	+0.076	−0.044	−0.015
0.2	+0.075	−0.079	+0.021	+0.070	−0.042	−0.010
0.3	+0.063	−0.079	+0.024	+0.063	−0.040	−0.006
0.4	+0.052	−0.078	+0.028	+0.056	−0.037	−0.002
0.5	+0.041	−0.076	+0.030	+0.049	−0.034	+0.001
0.6	+0.032	−0.073	+0.031	+0.043	−0.032	+0.004
0.7	+0.023	−0.069	+0.032	+0.036	−0.029	+0.006
0.8	+0.015	−0.065	+0.033	+0.030	−0.026	+0.008
0.9	+0.008	−0.061	+0.033	+0.025	−0.023	+0.010
1	+0.001	−0.057	+0.032	+0.019	−0.020	+0.011
1.2	−0.009	−0.047	+0.030	+0.010	−0.014	+0.013
1.4	−0.017	−0.037	+0.027	+0.003	−0.010	+0.013
1.6	−0.021	−0.028	+0.023	−0.003	−0.005	+0.013
1.8	−0.024	−0.020	+0.019	−0.007	−0.002	+0.012
2	−0.025	−0.012	+0.015	−0.010	+0.001	+0.011
2.4	−0.022	−0.001	+0.008	−0.013	+0.004	+0.008
2.8	−0.017	+0.006	+0.002	−0.012	+0.005	+0.005
3.2	−0.011	+0.008	−0.001	−0.009	+0.005	+0.002
3.6	−0.006	+0.009	−0.003	−0.006	+0.004	+0.0002
4	−0.0015	+0.0072	−0.0037	−0.0033	+0.0028	−0.0009
4.5	+0.0015	+0.0046	−0.0032	−0.0006	+0.0011	−0.0015
5	+0.0027	+0.0021	−0.0021	+0.0008	+0.0002	−0.0014
5.5	+0.0026	+0.0003	−0.0011	+0.0014	−0.0004	−0.00097
6	+0.0019	+0.0006	−0.0003	+0.0013	−0.0006	−0.00052
7	+0.0004	+0.0009	+0.0004	+0.0005	−0.0004	+0.00005

The diagram of the orbit of m_1 for Example I is shown in Figure 3. This orbit is, of course, with respect to the rotating axes. The arrow indicates the directions of motion.

Example II.

We have also computed the orbits when $\epsilon = 0.1$ and $\gamma = 1$. The equations determining the displacements, in so far as the linear terms in ϵ are concerned, are

$$\begin{aligned}
\epsilon u_1 &= e^{-0.688t}\,(0.100 \cos 0.987t - 0.228 \sin 0.987t),\\
\epsilon v_1 &= -e^{-0.688t}\,(0.231 \cos 0.987t + 0.060 \sin 0.987t),\\
\epsilon u_2 &= e^{-0.688t}\,(0.100 \cos 0.987t + 0.111 \sin 0.987t),\\
\epsilon v_2 &= e^{-0.688t}\,(0.135 \cos 0.987t - 0.112 \sin 0.987t).
\end{aligned}$$

On substituting in these equations the various values of t as in Table II and making use of the center of gravity equations to determine $\epsilon\, u_3$ and $\epsilon\, v_3$, the following displacements are found:

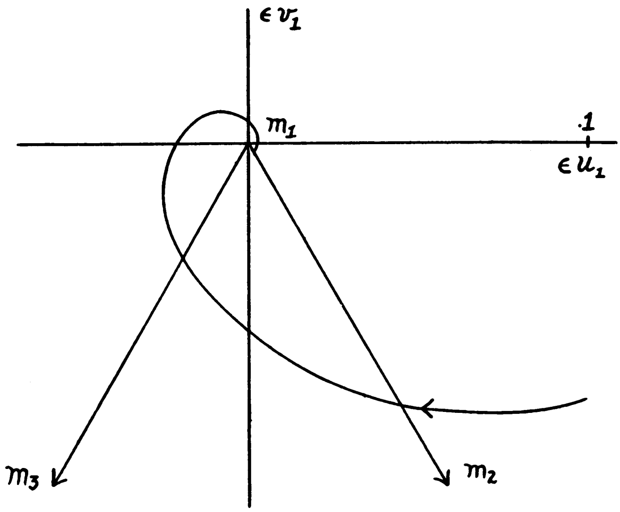

FIG. 3.

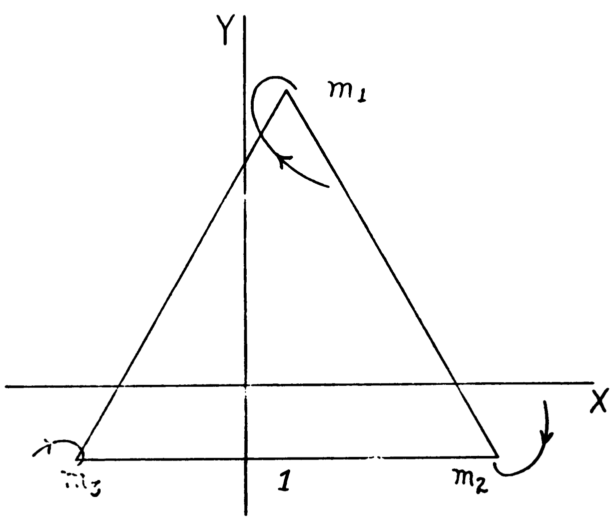

FIG. 4.

TABLE III

$\epsilon = 0.1 \quad \nu = 1$

t	ϵu_1	ϵv_1	ϵu_2	ϵv_2	ϵu_3	ϵv_3
0	+0.100	−0.231	+0.100	+0.135	−0.100	+0.011
0.1	+0.072	−0.221	+0.103	+0.115	−0.091	+0.019
0.2	+0.046	−0.208	+0.105	+0.096	−0.081	+0.026
0.3	+0.024	−0.195	+0.104	+0.079	−0.072	+0.031
0.4	+0.004	−0.179	+0.103	+0.062	−0.063	+0.034
0.5	−0.014	−0.164	+0.100	+0.047	−0.054	+0.038
0.6	−0.029	−0.149	+0.096	+0.033	−0.046	+0.040
0.7	−0.042	−0.133	+0.092	+0.020	−0.038	+0.041
0.8	−0.053	−0.119	+0.086	+0.008	−0.031	+0.042
0.9	−0.061	−0.104	+0.080	−0.001	−0.024	+0.042
1.0	−0.068	−0.090	+0.074	−0.010	−0.018	+0.041
1.2	−0.076	−0.063	+0.062	−0.023	−0.007	+0.039
1.4	−0.078	−0.039	+0.049	−0.032	+0.002	+0.035
1.6	−0.076	−0.019	+0.037	−0.038	+0.009	+0.030
1.8	−0.071	−0.003	+0.026	−0.040	+0.013	+0.025
2	−0.063	+0.009	+0.016	−0.040	+0.016	+0.020
2.4	−0.044	+0.024	+0.001	−0.034	+0.017	+0.011
2.8	−0.026	+0.028	−0.008	−0.024	+0.015	+0.003
3.2	−0.011	+0.026	−0.011	−0.015	+0.011	−0.001
3.6	−0.0001	+0.0198	−0.0114	−0.0066	+0.0069	−0.0039
4	+0.0061	+0.0130	−0.0095	−0.0008	+0.0033	−0.0047
4.5	+0.0087	+0.0054	−0.0061	+0.0032	+0.0001	−0.0041
5	+0.0078	+0.0003	−0.0028	+0.0045	−0.0015	−0.0028
5.5	+0.0054	−0.0024	−0.0004	+0.0039	−0.0019	−0.0014
6	+0.0028	−0.0031	+0.0009	+0.0027	−0.0017	−0.0004
7	−0.0004	−0.0018	+0.0012	+0.0004	−0.0005	−0.0005

The diagram of the orbits of the three bodies in this example is found in Figure 4. The length of the side of the equilateral triangle is taken as the unit length.

In conclusion, the author wishes to express his thanks to his former colleague, F. M. Wood, M.A., B.Sc., for verifying the algebraic and numerical computations in this paper.

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Page 51. J. F. RITT, *Prime and composite polynominals.*

Page 63. Line 7, for $\psi^{-1}(v)$ read $\psi^{-1}(w)$;

line 25, for $\varphi^{-1}(v)$ read $\psi^{-1}(w)$.