EXPANSIONS IN TERMS OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

FIRST PAPER: MULTIPLE FOURIER SERIES EXPANSIONS*

BY

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1. PRELIMINARY DISCUSSIONS

Consider the equation

$$(1) \quad L(u) \equiv \sum_{i=1}^p \frac{\partial u}{\partial x_i} + \lambda u = 0$$

and its adjoint

$$(2) \quad M(v) \equiv - \sum_{i=1}^p \frac{\partial v}{\partial x_i} + \lambda v = 0,$$

which satisfy the identity

$$(3) \quad v L(u) - u M(v) \equiv \sum_{i=1}^p \frac{\partial}{\partial x_i} (u, v) \equiv 0.$$

A particular solution of (1) may be written

$$(4) \quad \bar{u} = e^{-\lambda x_1} g(x_1 - x_2, x_1 - x_3, \dots, x_1 - x_p).$$

To find the general solution we try

$$u = v \bar{u}.$$

Equation (1) then becomes

$$v L(\bar{u}) + \bar{u} \sum_{i=1}^p \frac{\partial v}{\partial x_i} = 0.$$

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A value of λ for which $L(u) = 0$ and $L_i(u) = 0$, $i = 1, 2, \dots, p$, will also be a principal parameter value for the adjoint system. If a particular solution (4) $\neq 0$ is found for which there is a discrete set of principal parameter values, that set so obtained will contain all the principal values for the general solution. To prove this, substitute $u = e^{-\lambda x_1} \varphi(x_1 - x_2, x_1 - x_3, \dots,$

and that the principal parameter values are

$$\lambda = ni \quad (n = 0, \pm 1, \pm 2, \dots, i = \sqrt{-1}).$$

This will be confirmed later by another method.

2. THE FORMAL EXPANSION

If we consider two distinct principal values of λ , say λ_i and λ_j , and the corresponding principal solutions u_i and u_j for a particular q , also the adjoint solutions, v_i and v_j , for the same q , evidently

$$(9) \quad v_j L(u_i) - u_i M(v_j) = \sum_{k=1}^p \frac{\partial}{\partial x_k} (u_i v_j) + (\lambda_i - \lambda_j) u_i v_j.$$

Let us form the p -fold integral of this as we did for equation (3). Reducing as before and using the boundary conditions $L_x(u_i) = 0$, $x = 1, 2, \dots, p$, and the adjoint boundary conditions $L_x(v_j) = 0$, $x = 1, 2, \dots, p$, we obtain

$$(\lambda_i - \lambda_j) \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} u_i v_j dx_1 dx_2 \cdots dx_p = 0.$$

A division by $\lambda_i - \lambda_j$ gives the so called biorthogonality condition

$$(10) \quad \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} u_i v_j dx_1 dx_2 \cdots dx_p = 0.$$

Since $\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} u_i v_i dx_1 \cdots dx_p = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} q^2 dx_1 \cdots dx_p > 0$, we can formally determine the coefficients c_n for the expansion

$$(11) \quad f(x_1, x_2, \dots, x_p) = \sum_{n=-\infty}^{\infty} c_n u_n(x_1, x_2, \dots, x_p),$$

i. e.,

$$(12) \quad c_n = \frac{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(x_1, \dots, x_p) v_i(x_1, \dots, x_p) dx_1 \cdots dx_p}{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} [q(x_1, \dots, x_p)]^2 dx_1 \cdots dx_p}.$$

The important question is whether the series in equation (11) actually converges to the value $f(x_1, x_2, \dots, x_p)$. This is answered by a powerful method due to Birkhoff,* in which he uses contour integrals. Before employing this

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it is convenient to make the problem in the partial differential system depend upon p ordinary differential systems.

3. DEPENDENCE OF THE PARTIAL DIFFERENTIAL SYSTEM ON ORDINARY DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS

By putting

$$(13) \quad \begin{aligned} u &= \prod_{i=1}^p \varrho_i(x_i), \\ v &= \prod_{i=1}^p \gamma_i(x_i), \end{aligned}$$

in equations (1) and (2) and dividing by u and v respectively, we get

$$(14) \quad \begin{aligned} \sum_{i=1}^p \frac{\varrho'_i}{\varrho_i} + \lambda &= 0, \\ -\sum_{i=1}^p \frac{\gamma'_i}{\gamma_i} + \lambda &= 0, \end{aligned}$$

in which a prime indicates the derivative with respect to the one variable involved.

Clearly $\frac{\varrho'_1}{\varrho_1} + \lambda = -\sum_{i=2}^p \frac{\varrho'_i}{\varrho_i} = \text{const.} \equiv \mu_1$, say, and if we put $-\frac{\gamma'_1}{\gamma_1} + \lambda = \sum_{i=2}^p \frac{\gamma'_i}{\gamma_i} = \mu_1$, we shall have a second equation adjoint to the first.

Similarly we write in general

$$(15) \quad \begin{aligned} \frac{\varrho'_x}{\varrho_x} + \mu_{x-1} &= -\sum_{i=x+1}^p \frac{\varrho'_i}{\varrho_i} = \mu_x, \\ -\frac{\gamma'_x}{\gamma_x} + \mu_{x-1} &= \sum_{i=x+1}^p \frac{\gamma'_i}{\gamma_i} = \mu_x \quad (x = 2, 3, \dots, (p-1)) \end{aligned}$$

and finally

$$(16) \quad \frac{\varrho'_p}{\varrho_p} + \mu_{p-1} = 0, \quad -\frac{\gamma'_p}{\gamma_p} + \mu_{p-1} = 0.$$

By this substitution the boundary conditions become, after reduction,

$$(17) \quad \begin{aligned} \varrho_i(-\pi) &= \varrho_i(\pi) \\ \gamma_i(-\pi) &= \gamma_i(\pi) \end{aligned} \quad (i = 1, 2, \dots, p).$$

Solutions of these systems are readily found to be

$$\varrho_1 = e^{(\mu_1 - \lambda)x_1}, \quad \gamma_1 = e^{(\lambda - \mu_1)x_1},$$

$$\varrho_x = e^{(\mu_x - \mu_{x-1})x_x}, \quad \gamma_x = e^{(\mu_{x-1} - \mu_x)x_x}, \quad \text{for } x = 2, 3, \dots, (p-1),$$

and

$$\varrho_p = e^{-\mu_{p-1}x_p}, \quad \gamma_p = e^{\mu_{p-1}x_p}.$$

There is no loss of generality in putting the multiplicative constants equal to unity since the equations are homogeneous and in the expansion (11) they would otherwise cancel out.

The principal parameter values for the ϱ -system are given by

$$\begin{aligned} \mu_1 - \lambda &= \pm n_1 i, & \mu_2 - \mu_1 &= \pm n_2 i, & \dots, & \mu_{p-1} - \mu_{p-2} &= \pm n_{p-1} i, \\ & & \mu_{p-1} &= \pm n_p i, \end{aligned}$$

where the n_j have the values $0, 1, 2, \dots$, and $i = \sqrt{-1}$.

From this it is evident that

$$\varrho_j(x^j) = e^{i n_j x_j} \quad (j = 1, 2, \dots, p; i = \sqrt{-1}; n_j = 0, \pm 1, \pm 2, \dots).$$

Similarly $\gamma^j(x_j) = e^{-i n_j x_j}$ for the same values of j and n_j .

Hence the values for λ found above are confirmed and the form of one solution of (1) and (7) is given explicitly by

$$(18) \quad u_{n_1 n_2 \dots n_p} = e^{i \sum_{x=1}^p n_x x_x} \quad (n_x = 0, \pm 1, \pm 2, \dots).$$

The corresponding solution of the adjoint system is

$$(19) \quad v_{n_1 n_2 \dots n_p} = e^{-i \sum_{x=1}^p n_x x_x} \quad (n_x = 0, \pm 1, \pm 2, \dots).$$

In both solutions the corresponding value of λ is given by

$$(20) \quad \lambda_{n_1 n_2 \dots n_p} = -i \sum_{x=1}^p n_x.$$

It is important to notice that the forms of the solutions given in (18) and (19) are not unique but correspond to only one choice of the φ function. Although any function φ in p variables, which has a period of 2π in each, may be utilized, it is desirable in this paper to restrict ourselves to a consideration of the form given above, since it is this choice of φ which leads always to expansions in multiple Fourier series.

4. CONVERGENCE OF THE EXPANSION IN MULTIPLE FOURIER SERIES

To introduce the proof by the contour method it is thought best to give the results from Birkhoff's article* for the simple case

$$(21) \quad \begin{aligned} \frac{du}{dx} + \lambda u &= 0, & -\frac{dv}{dx} + \lambda v &= 0, \\ u(\pi) - u(-\pi) &= 0, & v(\pi) - v(-\pi) &= 0, \end{aligned}$$

since the present case is made to depend on it.

The Green's function for this system is

$$(22) \quad G(x, s) = \begin{cases} \frac{e^{\lambda(s-\pi-x)}}{2 \sinh \lambda \pi}, & \text{if } s > x, \\ \frac{e^{\lambda(s+\pi-x)}}{2 \sinh \lambda \pi}, & \text{if } s < x. \end{cases}$$

Its residue for the principal value $\lambda = \lambda_i$ is

$$(23) \quad R(x, s; \lambda_i) = \frac{u_i(x) v_i(s)}{\int_{-\pi}^{\pi} u_i(x) v_i(x) dx}.$$

The general term of the formal expansion for $f(x)$ is

$$(24) \quad \frac{u_i(x) \int_{-\pi}^{\pi} f(s) v_i(s) ds}{\int_{-\pi}^{\pi} u_i(s) v_i(s) ds}.$$

This is equal to

$$(25) \quad \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_i} \int_{-\pi}^{\pi} G(x, s; \lambda) f(s) ds d\lambda,$$

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* Loc. cit.

in which Γ_i is a contour in the λ -plane enclosing just the one simple pole λ_i of G . The sum of n terms of the expansion is given by

$$(26) \quad \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma} \int_{-\pi}^{\pi} G(x, s, \lambda) f(s) ds d\lambda,$$

where Γ is a contour containing $\lambda_1, \lambda_2, \dots, \lambda_n$ and no other λ_i .

By assuming a sequence of circles Γ_n with centers at the origin, such that their distance from the nearest poles, ni and $-ni$, where $i = \sqrt{-1}$ and n is an integer, is at least $d > 0$, the value of the series is found as the limit of the contour integrals, as $n \rightarrow \infty$.

By breaking the inner integral into two, i. e., from $-\pi$ to x and from x to π , and integrating as to s by parts, the integral is evaluated. If one uses the more convenient form

$$(27) \quad G(x, s; \lambda) = \frac{e^{\lambda(s-x)}}{2} [\operatorname{sgn}(x-s) + \coth \lambda \pi],$$

the two parts are quite similar. The steps may be sketched as follows:

$$\begin{aligned} I_n &= \frac{1}{2\pi i} \int_{\Gamma_n} \int_{-\pi}^x \frac{e^{\lambda(s-x)}}{2} (1 + \coth \lambda \pi) f(s) ds d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma_n} \left\{ \left[\frac{e^{\lambda(s-x)}}{2\lambda} (1 + \coth \lambda \pi) f(s) \right]_{-\pi}^x + O\left(\frac{1}{\lambda^2}\right) \right\} d\lambda \\ (28) \quad &= \frac{1}{2\pi i} \int_{\Gamma_n} \frac{1 + \coth \lambda \pi}{2\lambda} f(x-0) d\lambda + O\left(\frac{1}{\lambda}\right) \\ &= \frac{f(x-0)}{4\pi i} \int_{\Gamma_n} \frac{d\lambda}{\lambda} + O\left(\frac{1}{\lambda}\right) = \frac{f(x-0)}{2} + O\left(\frac{1}{\lambda}\right) \end{aligned}$$

since

$$\int_{\Gamma_n} \frac{\coth \lambda \pi}{\lambda} d\lambda = O\left(\frac{1}{\lambda}\right).$$

Similarly the other part yields $\frac{f(x+0)}{2} + O\left(\frac{1}{\lambda}\right)$, and the limit of their sum gives $\frac{f(x-0) + f(x+0)}{2}$ uniformly for x , provided $f'(x)$ is continuous.

Let us see how this may be applied to the proof of convergence of the series (11). If we denote

$$\lambda = \mu_1, \quad \mu_1 = \mu_2, \quad \mu_{x-1} = \mu_x, \quad \mu_{p-1} \text{ where } x = 3, 4, \dots, (p-1),$$

by $\nu_1, \nu_2, \nu_x, \nu_p$, respectively, then the Green's functions for the ordinary differential systems may be written

$$(29) \quad G_x(x_x, s_x, \nu_x) = \frac{e^{\nu_x(s_x - x_x)}}{2} [\operatorname{sgn}(x_x - s_x) + \coth \nu_x \pi], \quad x = 1, 2, \dots, p.$$

One may designate the residue of G_x for the principal value $\nu_x = \nu_{xj}$ by

$$(30) \quad R_{xj}(x_x, s_x, \nu_{xj}) = \frac{e_{xj}(x_x) \gamma_{xj}(s_x)}{\int_{-\pi}^{\pi} e_{xj}(x_x) \gamma_{xj}(x_x) dx_x} \quad (x = 1, 2, \dots, p).$$

The general term of our formal expansion (11) will then be

$$\frac{\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} f v_j dx_1 dx_2 \dots dx_p}{\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} u_j v_j dx_1 dx_2 \dots dx_p} u_j,$$

and the series may be represented by

$$\sum_{j=-\infty}^{\infty} \frac{\int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} f \prod_{x=1}^p \gamma_{xj} dx_1 \dots dx_p}{\int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \prod_{x=1}^p e_{xj} \prod_{x=1}^p \gamma_{xj} dx_1 \dots dx_p} \prod_{x=1}^p e_{xj}(x_x) \equiv F(x_1, \dots, x_p).$$

Then, since

$$G_x(x_x, s_x; \nu_{xj}) = \frac{e_{xj}(x_x) \gamma_{xj}(s_x)}{(\nu_x - \nu_{xj}) \int_{-\pi}^{\pi} e_{xj}(x_x) \gamma_{xj}(x_x) dx_x} + \sigma_x(x_x, s_x; \nu_x) \quad (x = 1, 2, \dots, p),$$

where σ_x is analytic in ν_x at $\nu_x = \nu_{xj}$, if we define the residue of p complex variables $z_x, x = 1, 2, \dots, p$, as the coefficient of $\prod_{x=1}^p \frac{1}{(z_x - z_{xj})}$, then the

value of $\prod_{x=1}^p G_x(z_x)$ for $z_x = z_{xj}$ is

$$\frac{1}{(2\pi i)^p} \int_{\Gamma_p} \int_{\Gamma_{p-1}} \cdots \int_{\Gamma_1} \prod_{x=1}^p \frac{G_x(z_x)}{(z_x - z_{xj})} dz_1 dz_2 \cdots dz_p,$$

and that of the term of F for which $\nu_x = \nu_{xj}$, $x = 1, 2, \dots, p$, will be the same as that of

$$(31) \quad \frac{1}{(2\pi i)^p} \int_{\Gamma_p} \cdots \int_{\Gamma_1} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(s_1, \dots, s_p) \prod_{x=1}^p [G_x(x_x, s_x; \nu_x) ds_x d\nu_x]$$

where Γ_x refers to a contour in the z_x -plane, (or ν_x -plane), containing the simple pole z_{xj} , (or ν_{xj}), because terms of the form $\sigma_x(x_x, s_x; \nu_x) / \prod_{\substack{i=1 \\ i \neq x}}^p (\nu_i - \nu_{ij})$ will

contribute nothing.

Hence if F converges, it will be represented by the limit as $\nu_x \rightarrow \infty$, $x = 1, 2, \dots, p$, of the same integral (31) taken around sequences of circular contours in the ν_x -planes, which are drawn in these several planes as that for the simple problem was drawn in the λ -plane.

Let us now break up the integral (31) into 2^p pieces by taking every combination formed by dividing each real interval from $-\pi$ to π into two intervals of the form $-\pi$ to x_x and x_x to π , for $x = 1, 2, \dots, p$.

Consider the piece

$$(32) \quad \frac{1}{(2\pi i)^p} \int_{\Gamma_p} \cdots \int_{\Gamma_1} \int_{-\pi}^{x_p} \cdots \int_{-\pi}^{x_1} f(s_1, \dots, s_p) \prod_{x=1}^p \left[\frac{e^{\nu_x(s_x - x_x)}}{2} (1 + \coth \nu_x \pi) ds_x d\nu_x \right].$$

Integrating by parts as before p times, we obtain

$$\begin{aligned} & \frac{1}{(2\pi i)^p} \int_{\Gamma_p} \cdots \int_{\Gamma_1} \frac{f(x_1 - 0, \dots, x_p - 0)}{2^p} \prod_{x=1}^p \left[(1 + \coth \nu_x \pi) \frac{d\nu_x}{\nu_x} \right] \\ & + O\left(\frac{1}{\nu_p}\right) = \frac{f(x_1 - 0, x_2 - 0, \dots, x_p - 0)}{2^p} + O\left(\frac{1}{\nu_p}\right). \end{aligned}$$

Similarly the piece whose upper limits are π and whose lower limits are $x_p, x_{p-1}, \dots, x_1$, respectively, has the value

$$I_{x_p, x_{p-1}, \dots, x_1}^{\pi, \pi, \dots, \pi} = \frac{f(x_1 + 0, x_2 + 0, \dots, x_p + 0)}{2^p} + O\left(\frac{1}{\nu_p}\right)$$

and it is clear that the limit of F is given by

$$\sum \frac{f(x_1 \pm 0, x_2 \pm 0, \dots, x_p \pm 0)}{2^p},$$

in which the summation includes every possible combination formed by choosing just one sign for each argument of f . We may therefore state the

THEOREM. *Let $f(x_1, x_2, \dots, x_p)$ be made up of a finite number of pieces in the region $-\pi \leq x_x \leq \pi$, $x = 1, 2, \dots, p$, each real, continuous, and possessing continuous partial derivatives. The multiple Fourier expansion connected with the partial differential equation $\sum_{x=1}^p \frac{\partial u}{\partial x_x} + \lambda u = 0$, and the boundary conditions $L_x(u) = 0$ ($x = 1, 2, \dots, p$), namely, the series*

$$F \equiv \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_p=-\infty}^{\infty} \frac{1}{(2\pi)^p} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(s_1, \dots, s_p) \prod_{x=1}^p (e^{-in_x(s_x - x_x)} ds_x),$$

converges to $\frac{1}{2^p} \sum f(x_1 \pm 0, x_2 \pm 0, \dots, x_p \pm 0)$ for each point of the region $-\pi \leq x_x \leq \pi$ ($x = 1, 2, \dots, p$), provided we interpret the argument $(\pm \pi + 0)$ at points of the boundary to mean $(-\pi)$ and the argument $(\pm \pi - 0)$ to mean (π) . In any subregion in which f is continuous and possesses continuous partial derivatives the series converges uniformly to f .

From the fact that the function to be expanded is real and that the solution

$u = e^{\sum_{x=1}^p n_x i x_x}$ may be written

$$u = \prod_{x=1}^p (\cos n_x x_x + i \sin n_x x_x)$$

it is evident that F will contain, besides a possible constant, only terms of a multiple Fourier series. This may be illustrated by the following.

Example. Expand $f(x, y, z) = xyz$ in a multiple Fourier series.

Here

$$f(x, y, z) = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_{lmn} e^{lix + miy + niz}$$

where

$$\begin{aligned} c_{lmn} &= \frac{1}{8\pi^3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} xyz e^{-lix-my-niz} dx dy dz \\ &= \frac{i^3}{lmn} \cos l\pi (\cos m\pi) \cos n\pi. \end{aligned}$$

Hence

$$\begin{aligned} xyz &= \sum_{l=-\infty}^{\infty} \frac{i}{l} \cos l\pi e^{lix} \sum_{m=-\infty}^{\infty} \frac{i}{m} \cos m\pi e^{my} \sum_{n=-\infty}^{\infty} \frac{i}{n} \cos n\pi e^{nz} \\ &= \sum_{l=1}^{\infty} \left(-\frac{2}{l} \cos l\pi \sin lx \right) \sum_{m=1}^{\infty} \left(-\frac{2}{m} \cos m\pi \sin my \right) \sum_{n=1}^{\infty} \left(-\frac{2}{n} \cos n\pi \sin nz \right), \end{aligned}$$

since the terms may be rearranged and the value of any of the c 's with a zero subscript is zero. This series converges uniformly to xyz for any interior point and converges to zero on the boundary.

It is clear that the restrictions on f might be lightened and that its region might easily be extended.

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