ON THE INTEGRALS OF ELEMENTARY FUNCTIONS*

BY

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I. Introduction

This paper contains an extension of Liouville's work on the determination of the circumstances under which the integral of an elementary function is itself elementary.†

The elementary functions are understood here to be those which are obtained by performing algebraic operations and taking exponentials and logarithms. For instance, the function

$$\tan\left[e^{z^2}-\log_z(1+\sqrt{z})\right]+(z^z+\log\arcsin z)^{\frac{1}{2}}$$

is elementary.

Liouville's first result in this field was the theorem that if the integral of an algebraic function of z is elementary, the integral is of the form

$$\alpha_0(z)+c_1\log\alpha_1(z)+c_2\log\alpha_2(z)+\cdots+c_n\log\alpha_n(z),$$

where each $\alpha(z)$ is an algebraic function, and each c a constant.‡ It follows from this theorem that no abelian integral of the first kind is elementary. Liouville also obtained a very broad generalization of the above result, on the basis of which it can be proved, for instance, that the integrals

$$\int e^{z^2}dz, \qquad \int\frac{dz}{\log z},$$

are not elementary functions.§

* Presented to the Society, Feb. 25, 1922.

† No acquaintance with Liouville's work is necessary for the understanding of the present paper.

‡ Journal de l'École polytechnique, vol. 14 (1833), p. 36.

§ Journal für die reine und angewandte Mathematik, vol. 13 (1833), p. 93.

We shall investigate here the circumstances under which the integral of an elementary function can satisfy an elementary transcendental equation. This problem is mentioned by Liouville* and by Hardy,† but nothing seems to have been done towards its solution. Our result is the

THEOREM. *If w, the integral of an elementary function of z, satisfies an equation $F(w,z)=0$, where $F(w,z)$ is an elementary function of w and z, not identically zero, then w is an elementary function of z.*

Thus no abelian integral of the first kind can satisfy an elementary transcendental equation; nor can, to give special cases again, the integrals of e^{z^2} and of $1/\log z$.

If a function satisfies an elementary transcendental equation, its inverse also satisfies such an equation. It follows from the above that no elliptic function can satisfy an elementary transcendental equation. In particular, no elliptic function is elementary.‡

Whereas, notwithstanding their formal character, little need be added to Liouville's papers to make them rigorous, the work in the present paper has to be largely function-theoretic. This is due principally to the fact, seen in § III, that we may not suppose ourselves to be working in a region in which $F(w,z)$ is analytic. On this account, we have had to make, in § II, a careful examination of the elementary functions.

It is natural to inquire as to whether an integral w of an elementary function is itself elementary if it is one of n functions, $w, w_1, \ldots, w_{n-1}$, which satisfy a system of elementary equations

$$F_i(w, w_1, \ldots, w_{n-1}; z) = 0 \qquad (i = 1, 2, \ldots, n).$$

While the formal elements of our proof, given in § IV, can be extended to settle this question affirmatively, we see no way of avoiding certain function-theoretic assumptions, which, though light in almost any other case, would be out of place in a problem of this kind. We shall therefore not write now on this question.

* Journal de l'École polytechnique, vol. 14 (1833), p. 40.

† *The Integration of Functions of a Single Variable,* Cambridge, 1916, p. 41. Appendix I of this tract contains further references to Liouville's work on elementary functions.

‡ This does not follow, of course, from the fact that the integrals of the first kind are not elementary. I do not know where it is proved in the literature that the elliptic functions are not elementary, but this result of the present paper can be established by the same method which Liouville uses for the integrals. It can be shown that if the inverse of an abelian integral is elementary, it is either algebraic, or of the form e^{az+b}, where a and b are constants.

II. The elementary functions

We shall deal with certain analytic functions of w and z, which we shall call *elementary functions.*

An analytic function of w and z will be said to be *analytic almost everywhere,* if, given any element of the function $P(w-w_0,\ z-z_0)$,* any curve

$$w = \varphi(\lambda), \qquad z = \psi(\lambda) \qquad (0 \leqq \lambda \leqq 1),$$

where $\varphi(0) = w_0$ and $\psi(0) = z_0$, and given, finally, any positive ε, there exists a curve

$$w = \varphi_1(\lambda), \qquad z = \psi_1(\lambda) \qquad (0 \leqq \lambda \leqq 1), \tag{1}$$

where $\varphi_1(0) = w_0$ and $\psi_1(0) = z_0$, such that

$$|\varphi_1(\lambda)-\varphi(\lambda)| < \varepsilon, \qquad |\psi_1(\lambda)-\psi(\lambda)| < \varepsilon,$$

for $0 \leqq \lambda \leqq 1$, and such that the element $P(w-w_0,\ z-z_0)$ can be continued along the entire curve (1). Roughly speaking, an element of the function, if it cannot be continued along a given path, can be continued along some path in any neighborhood of the given one.

An algebraic function u, given by an irreducible equation

$$\alpha_0(w,z)\,u^m + \alpha_1(w,z)\,u^{m-1} + \cdots + \alpha_m(w,z) = 0, \tag{2}$$

where each α is a polynomial in w and z with constant coefficients, is analytic almost everywhere, for the space of w and z is four-dimensional, and the singularities of u lie along the two-dimensional manifolds obtained by equating $\alpha_0(w,z)$ and the discriminant of (2) to zero†.

In what follows, the algebraic functions will frequently be called *functions of order zero,* and the variables w and z *monomials of order zero.*

The functions e^v and $\log v$, where v is any non-constant algebraic function, are called by Liouville *monomials of the first order.* It is seen directly that e^v is analytic almost everywhere. If v is analytic, and nowhere zero, along a given curve, $\log v$ is analytic along the curve. If v should vanish for some points of the curve, there is a curve arbitrarily close to the given one on which v is everywhere different from zero‡. Thus $\log v$ is analytic almost everywhere.

* It is to be recalled that an analytic element, $P(w-w_0,\ z-z_0)$, is a series of positive integral powers of $w-w_0$ and $z-z_0$. The point $(w_0,\ z_0)$ is called the center of the element.

† It is not hard to make the proof formal.

‡ A formal proof can be based upon the fact that in continuing v along the given curve, only a finite number of elements of v need be used.

More generally, we shall say, following Liouville, that u is a *function of the first order,* if it is not algebraic, and if it satisfies an equation like (2), in which each α is a rational integral combination of monomials of orders zero and one.

To obtain each α one must be given, for some point (w_0, z_0), an element of each of the monomials on which that α is based. A proper combination of these elements furnishes an element of the α, from which all other elements of the α are found by continuation. Each α is analytic almost everywhere.

Similarly, in stating that u is determined by (2), we mean that for an element of each of the functions u and α_i $(i = 0, 1, \ldots, m)$, the first member of (2) vanishes. If the first member of (2) is reducible in the domain of rationality of the given elements of the α's, we may replace it by that one of its irreducible factors which vanishes for the given element of u. It may thus be assumed that the discriminant of (2), which is analytic wherever every α is analytic and α_0 is not zero, does not vanish identically. We see now that u is analytic almost everywhere, since in the neighborhood of every curve there is a curve along which each α is analytic, and on which α_0 and the discriminant of (2) are everywhere different from zero.

The functions of orders zero and one form together a set which is closed with respect to all algebraic operations. That is, a function determined by an equation like (2), in which each α is a rational integral combination of functions of orders zero and one, is, itself, either algebraic, or a function of the first order. This fact, which will be very important for us, can be proved in exactly the same way as it is shown that algebraic operations performed upon algebraic numbers lead always to algebraic numbers.*

An exponential or a logarithm of a function of order $n - 1$ is called a *monomial of order n,* provided that it is not among the functions of orders $0, 1, \ldots, n - 1$. With the same reservation, any function defined by an equation like (2), in which each α is a rational integral combination of monomials of orders $0, 1, \ldots, n$, is a *function of order n.* As above, we may assume that the discriminant of (2) does not vanish identically.

One sees by a quick induction that a function of any order n is analytic almost everywhere. For every n, the functions of orders $0, 1, \ldots, n$ form a set which is closed with respect to all algebraic operations.

The functions to which orders are assigned by the preceding definitions will be called *elementary functions* of w and z.

We now inquire as to whether an elementary function of w and z becomes, when w is held fast, an elementary function of z; that is, a function which can

* See, for instance, Landau, *Einführung in die elementare und analytische Theorie der algebraischen Zahlen,* Leipzig, 1918, p. 7.

be obtained from z alone, without introducing a second variable, by performing algebraic operations and taking exponentials and logarithms. Perhaps an illustration is necessary to show that there is a real question here. Consider the equation

$$e^{\cot wz}\, u - z\, e^{\csc wz} = 0.$$

It determines u as an entire function of w and z, which reduces to z when $w = 0$. The equation itself is meaningless when $w = 0$. Certainly, then, it is possible for $u(w, z)$ to be an elementary function of w and z and for $u(w_1, z)$ to be analytic in z, but for the operations which produce $u(w, z)$ to be meaningless when $w = w_1$. In the above example, it is easy to show that $u(w_1, z)$ is an elementary function of z, but we cannot be certain that this is possible in all cases.

We shall not settle this question completely, but shall limit ourselves to deriving a result which will suffice for our later purposes. We deal with an elementary function $u(w, z)$, of which some branch is analytic at a point (w_1, z_1). If w is assigned the fixed value w_1, and if z is allowed to range over the neighborhood of z_1, the given branch of u becomes an analytic function of z. It is the monogenic analytic function of z obtained by continuing this branch in which we are interested.

It is of course obvious that if u is an algebraic function of w and z, u becomes an algebraic function of z if w is held fast. Let v be algebraic in w and z. If $\log v$ is analytic at (w_1, z_1), then v must be analytic at (w_1, z_1). Hence $\log v(w_1, z)$ is an elementary function of z. Consider now e^v. If e^v is analytic at (w_1, z_1), there is a point (w_2, z_2), arbitrarily close to (w_1, z_1), at which e^v is not zero, so that v is analytic at (w_2, z_2).* Hence $e^{v(w_2, z)}$ is an elementary function of z. More generally, if w is held fast at a value sufficiently close to w_2, e^v becomes an elementary function of z.

Suppose now that α is a rational integral combination of monomials of orders zero and one, and that α is analytic at (w_1, z_1). We may suppose that the element of α with center at (w_1, z_1) is obtained by continuing, along a path C, the element of α with center at (w_0, z_0) which is found by multiplications and additions from the given elements of the monomials at (w_0, z_0). As each monomial is analytic almost everywhere, we can continue α from (w_0, z_0) along a path C', very close to C, on which every monomial is analytic. We can reach in this way a point (w_2, z_2), arbitrarily close to (w_1, z_1), at which α has an element which is an immediate continuation of the above mentioned element at (w_1, z_1). Also, from what we saw above, there is a point (w_3, z_3),

* Whether e^v can ever be zero, even at a singularity of v, is a matter with which we need not concern ourselves here.

arbitrarily close to (w_2, z_2), such that if w is held fast at any value sufficiently close to w_3, each monomial becomes an elementary function of z. While it might appear at first that (w_3, z_3) is different for different monomials, it becomes plain quickly that a common point may be used for all. For any value of w close to w_3, α is an elementary function of z.

Suppose now that u, analytic at (w_1, z_1), is given by (2), where each α involves monomials of orders zero and one. There exists a point (w_2, z_2), arbitrarily close to (w_1, z_1), at which each α is analytic, and such that if w is given a fixed value sufficiently close to w_2, each α is an elementary function of z. We may suppose also that (w_2, z_2) is so chosen that the coefficients α_i do not all vanish at (w_2, z_2). In that case, if w is held fast at a value close to w_2, u becomes an elementary function of z.

In the general case, it is clear that if u is an elementary function of w and z, analytic at (w_1, z_1), then there is a point (w_2, z_2), arbitrarily close to (w_1, z_1), such that if w is held fast at any value sufficiently close to w_2, u becomes an elementary function of z.

We consider now the differentiation of the elementary functions. Let u be a function of the first order, defined by an equation like (2). The first member of (2) being irreducible, its partial derivative with respect to u cannot vanish for every w and z. Thus, the partial derivative of u with respect to w has an expression rational in u, the α's, and the derivatives of the α's. The derivative of every α is a rational integral combination of the monomials which enter into that α, and of algebraic functions. It follows that u satisfies an equation like (2), in which no monomials of order one appear which do not appear in the equation for u. Proceeding by induction, we can prove that if u is elementary, u_w and u_z are elementary. We prove in the same induction that if, in the equation (2) which defines u, the highest order of the monomials which involve w* is r, and if there are s distinct such monomials of order r, the derivatives of u satisfy equations like (2), in which the monomials which involve w are at most of order r, and if of that order, are among the s monomials mentioned above. This fact will be used in § IV.

III. Formulation of the problem

When, in what follows, we say that $F(w, z)$ assumes the value zero at (w_0, z_0), we shall mean that there exists an analytic element of $F(w, z)$, with center at (w_0, z_0), which assumes there the value zero. There may be other

* In saying that a monomial involves w, we mean not only that w appears in the operations which produce the monomial, but also that the value of the monomial varies when w varies. Furthermore, from what goes before, we know that if a monomial is independent of w, it can be produced by operations which involve only z.

elements with centers at (w_0, z_0) which are not zero there. When we say that $F(w, z)$ has a singularity at (w_0, z_0), we shall mean that there exists a sequence of elements of $F(w, z)$, each an immediate continuation of the preceding one, whose centers approach (w_0, z_0), and the radii of whose restricted domains of convergence approach zero.* $F(w, z)$ may have a singularity at (w_0, z_0), and also have branches which are analytic at (w_0, z_0).

Consider any elementary function of z alone. Let w represent an integral of this function. Our problem is to determine the circumstances under which there exists an equation

$$F(w, z) = 0, \tag{3}$$

where $F(w, z)$ is an elementary function.

Consider any circle in the plane of z, within and on the boundary of which some branch of w is analytic. Pairing the values of z in this circle with the corresponding values of the mentioned analytic branch of w gives a two-dimensional manifold — call it C — in the space of w and z. It will suffice to assume that (3) holds on such a manifold.

It would be unreasonable to require that $F(w, z)$ be analytic on C. It is even possible that every point of C should be a singular point of $F(w, z)$ and that (3) should serve well to determine w. For instance, the equation $(w - z)^{\frac{1}{2}} = 0$ gives $w = z$, but wherever $w = z$, $(w - z)^{\frac{1}{2}}$ has a singularity. In this particular example, the singularities can be removed by squaring, but we have no assurance at present that there do not exist complicated equations which cannot be freed from singularities.

The best way to meet this situation is to extend our problem. We shall undertake *to determine the circumstances under which, for each point of C, F(w, z) either assumes the value zero or else has a singularity.*† *Our result is the*

THEOREM. *If, at each point of an uncountable set of points of C, F(w, z) either assumes the value zero or has a singularity, w is an elementary function of z.*‡

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* The radius of the restricted domain of convergence of an element $P(w - w_0, z - z_0)$ is a number ρ such that the element converges when $|w - w_0| < \rho$, and $|z - z_0| < \rho$, and diverges when $|w - w_0| > \rho$ and $|z - z_0| > \rho$.

† It is understood in this that some of the points of C may be zeros, and the remaining ones singularities.

‡ In using any uncountable set on C in the hypothesis, rather than all of C, we are not seeking generality. Such an uncountable set has to be used in the proof, and it saves time to introduce it immediately.

We need not seek to formulate a notion of $F(w, z)$ assuming the value zero at a singular point. The mere existence of the singular points will permit us to prove that w is elementary.

Let n be the smallest integer such that a function $F(w, z)$ of order n exists which satisfies the hypothesis of our theorem. In what follows, $F(w, z)$ will be assumed to be of order n.

Let us see what happens if $F(w, z)$ is zero for each point of an uncountable set of points of C.

For each such point, $F(w, z)$ has an element with center at the point, which equals zero at its center. There must exist a positive ε such that, for some uncountable subset E of those points, the radii of the restricted domains of convergence of the elements of $F(w, z)$ exceed ε. If no such ε existed, the elements, at the centers of which $F(w, z)$ vanishes, could be denumerated.

Let (a, b) be any point of condensation* of E. Consider the neighborhood of (a, b) given by

$$|w-a| < \frac{\varepsilon}{2}, \qquad |z-b| < \frac{\varepsilon}{2},$$

and the uncountable subset E' of E which lies in this neighborhood.

We now form for (a, b) the immediate continuations of the elements of $F(w, z)$ with centers at the points of E'. The radius of the restricted domain of convergence of each of these elements exceeds $\varepsilon/2$, and each new element equals zero at the center of the element of which it is the immediate continuation.

An infinite number of these immediate continuations must be identical, else there would be an uncountable set of distinct elements of $F(w, z)$ with centers at (a, b), and we would have a contradiction of the well known Poincaré-Volterra theorem.

It follows that one of the elements is zero for an infinite number of points of E'. Such an element is a uniform function of w and z in a four-dimensional region containing part of C. Substituting for w, in this element, its value in terms of z, we have an analytic function of z which vanishes at an infinite number of points of the circle mentioned above, and which is therefore identically zero.

We conclude that $F(w, z)$ is analytic in a four-dimensional region which contains part of C, and vanishes on that part of C.

We consider now the case in which $F(w, z)$ has a singularity at each point of an uncountable set of points of C. Let $u = F(w, z)$ be defined by an

* A point in every neighborhood of which there is an uncountable subset of E.

equation like (2), in which each α is a rational integral combination of monomials of orders not exceeding n. If $F(w, z)$ has a singularity at (a, b), there exists a curve ending at (a, b), along which, except at (a, b), $F(w, z)$ is analytic.* If this curve is subjected to a slight deformation, its end being held fast at (a, b), $F(w, z)$ will also be analytic along the new curve, except at (a, b). As each α in (2) is analytic almost everywhere, we can so choose the new curve that every α is analytic along it, except, perhaps, at (a, b).† If, as the new curve is followed to (a, b), each α should prove to be analytic at (a, b), and if neither α_0 nor the discriminant of (2) should vanish at (a, b), the new curve could not lead to a singularity of $F(w, z)$.

Suppose that there is an α which has an uncountable set of singularities on C. In that case one of the monomials on which that α is based must have an uncountable set of singularities on C. If the monomial is of the form e^v, v must have a singularity wherever the monomial does. If the monomial is of the form $\log v$, v must either assume the value zero, or have a singularity, wherever $\log v$ has a singularity. We find, in any case, a contradiction of the fact that no function of order less than n has an uncountable set of singularities on C, or is zero on such an uncountable set.

Thus, either $\alpha_0(w, z)$ or the discriminant of (2) must be zero on an uncountable set of points of C.

The hypothesis of our theorem has implied the existence of a function of order n, either $F(w, z)$, $\alpha_0(w, z)$, or the discriminant, which is analytic and equal to zero in a neighborhood on C. Let C_1 denote that neighborhood, and let the function — call it u — be defined by an equation like (2).‡ As none of the monomials which appear in the equation (2) for the function can have an uncountable set of singularities on C, there must be a neighborhood C_2, in C_1, for which all of the monomials have analytic branches, which, together with u, satisfy (2).§

As u vanishes in C_2, $\alpha_m(w, z)$, the term independent of u in the equation for u, which, since (2) is irreducible, cannot vanish identically, must vanish in C_2.

IV. Completion of the proof

There is a set — call it A — of rational integral combinations of monomials, each combination of the minimum order n used in § III, which vanish in a

* This statement is equivalent to the definition above of a singular point.

† It should perhaps be emphasized that (a, b) does not correspond to the point (w_0, z_0) used in defining "analytic almost everywhere". It is a point to which we approach by analytic continuation, not from which we start.

‡ If the function is α_0 or the discriminant, the equation will be linear.

§ This statement is easily proved, using the fact that the monomials are analytic almost everywhere.

neighborhood on C, but do not vanish identically;* we have just shown the existence of one such expression. With each expression in A may be associated two numbers; first, r, the maximum of the orders of those monomials in the expression which involve w; second, s, the number of such monomials of order r. Many different expressions will represent the same function; the two numbers will vary with the expression.

Of all the expressions in A, there is a set A_1 for which r has a minimum value r_0. Of the expressions in A_1, there is a set A_2, for which s has a minimum value s_0.

Let u be a particular expression in A_2. Those expressions in A_2 which contain no monomials which do not appear in u constitute a set A_3. All expressions in A_3 contain the same s_0 monomials of order r_0 which involve w. Let θ be one of those monomials.

There is an expression in A_3.

$$\bar{u} = \beta_0 \theta^i + \beta_1 \theta^{i-1} + \cdots + \beta_i,$$

where each β is a polynomial in monomials other than θ, which is of a minimum degree i in θ.

In the neighborhood on C in which $\bar{u}$ vanishes, there is a neighborhood C_1 in which every monomial in the expression for $\bar{u}$ is analytic. There is, in fact, a neighborhood C_1, in which, for every monomial, whether of the form e^v or $\log v$, the function v is analytic.†

The partial derivative

$$\bar{u}_\theta = i\beta_0 \theta^{i-1} + (i-1) \beta_1 \theta^{i-2} + \cdots + \beta_{i-1}$$

cannot vanish for every w and z. If it did, the function $i\bar{u} - \theta\bar{u}_\theta$, which could not vanish identically, since $\bar{u}$ does not, would be in A_3 and would be of degree less than i in θ. Hence $\bar{u}_\theta$ cannot be zero throughout C_1, else it would be a function in A_3 of degree less than i in θ.

Thus there is a neighborhood C_2 in C_1 in which the equation $\bar{u} = 0$ can be turned into the form

$$\theta = f(w, z), \tag{4}$$

where $f(w, z)$ is analytic throughout C_2.

If $r_0 = 0$, (4) states that w is an elementary function of z.

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* The neigborhood may be different for different expressions.

† This follows from the fact that no v can have an uncountable set of singularities on C.

Suppose that $\theta = e^{v(w,z)}$, where $v(w, z)$, of order $r_0 - 1$, is analytic throughout C_1. Let (w_0, z_0) be any point of C_2. Equation (4) shows that $f(w_0, z_0)$ is not zero. Let, then, ϱ be a number such that $v(w, z)$ and $\log f(w, z)$ are analytic for $|w - w_0| < \varrho$ and $|z - z_0| < \varrho$. We have, for a neighborhood C_3 in C_2, which contains (w_0, z_0) and lies within the region just described,

$$v(w, z) = \log f(w, z). \tag{5}$$

Let $\varphi(z)$ be the elementary function of which w is the integral. Differentiating in (5) with respect to z, we have, throughout C_3,

$$v_w(w, z)\,\varphi(z) + v_z(w, z) = \frac{f_w(w,z)\,\varphi(z) + f_z(w, z)}{f(w, z)}. \tag{6}$$

Having regard to the fact that the equation (2) for $f(w, z)$ is the equation $\bar{u} = 0$ with θ replaced by u, and to the remarks in the last paragraph of § II, we see that the second member of (6) satisfies an equation (2) in which no monomial which involves w is of order greater than r_0, and in which there are at most $s_0 - 1$ such monomials of order r_0. The first member of (6) satisfies an equation (2) in which every monomial involving w is of order less than r_0.

The two members of (6), considered as functions of w and z, must be identical. If they were not, the integral w would satisfy an elementary equation (6), from which we could derive an expression in A_2 with $r \leqq r_0$, and with $s < s_0$ if $r = r_0$.

The two functions

$$v(w + \mu, z), \qquad \log f(w + \mu, z) \tag{7}$$

are analytic in w, μ and z for $|w + \mu - w_0| < \varrho$, $|z - z_0| < \varrho$. Let δ be any positive number less than ϱ, and let C_4 be any neighborhood in C_3 for which $|w - w_0| < \varrho - \delta$, $|z - z_0| < \varrho$. If μ is assigned a fixed value, less in modulus than δ, and if (w, z) is kept in C_4, the functions (7) become functions of z whose derivatives are given by the members of (6), with $w + \mu$ substituted for w. Since (6), being an identity, will hold after this substitution, the difference of the functions in (7) must stay constant as (w, z) varies over C_4. We have thus, on C_4,

$$v(w + \mu, z) = \beta(\mu) + \log f(w + \mu, z), \tag{8}$$

where $\beta(\mu)$, being the difference of two analytic functions of μ, is analytic for $|\mu| < \delta$. Differentiating in (8) with respect to μ, and putting $\mu = 0$ in the result, we have, on C_4,

$$v_w(w, z) = \beta'(0) + \frac{f_w(w, z)}{f(w, z)}. \tag{9}$$

As above, (9) must be an identity, so that, integrating, we have for $|w - w_0| < \varrho$, $|z - z_0| < \varrho$,

$$v(w, z) = \beta'(0)\, w + \log f(w, z) + \gamma(z), \tag{10}$$

where $\gamma(z)$, analytic for $|z - z_0| < \varrho$, is still to be determined.

As (5) is not an identity, there is a point (w_1, z_1), with $|w_1 - w_0| < \varrho$, $|z - z_0| < \varrho$, which does not satisfy (5). By § II, (w_1, z_1) may be so chosen that $f(w_1, z)$ is an elementary function of z alone. Substituting w_1 for w in (10), and comparing the resulting equation with (10), we obtain, for $|w - w_0| < \varrho$, $|z - z_0| < \varrho$,

$$v(w, z) - \beta'(0)\, w - \log f(w, z) = v(w_1, z) - \beta'(0)\, w_1 - \log f(w_1, z). \tag{11}$$

Because of (5), we must have, in C_3,

$$\beta'(0)\, w = \beta'(0)\, w_1 + \log f(w_1, z) - v(w_1, z). \tag{12}$$

Now $\beta'(0)$ cannot be zero. If it were, $\log f(w_1, z) - v(w_1, z)$ would vanish in an area in the z-plane, and therefore for every z, whereas, by the choice of (w_1, z_1), it cannot vanish at z_1.

Thus (12) gives w as an elementary function of z.

Consider now the case of $\theta = \log v(w, z)$. Equation (4) becomes

$$f(w, z) = \log v(w, z),$$

and with almost no modification in the treatment of (5) other than an interchange of the letters f and v, we find again that w is an elementary function of z.

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