A TYPE OF DIFFERENTIAL SYSTEM CONTAINING A PARAMETER*

BY

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Poincaré† and others have discussed the continuity with respect to a parameter μ of the solutions of a system of equations

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \dots, x_n, \mu, t),$$

in which the function X_i is analytic in μ if $|\mu| < c$, and satisfies certain other conditions with respect to $x_1, x_2, \dots, x_n, t$ in a domain

$$|x_i - x_i^0| < b, \quad 0 \leq t \leq T \quad (i = 1, 2, \dots, n).$$

For certain problems in mechanics it is convenient to have a similar discussion of equations of the form

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \dots, x_n, \cos \nu t, \sin \nu t) \quad (i = 1, 2, \dots, n)$$

for very large values of the parameter ν ; the present paper is devoted to this type of equations, and an application is made to a problem related to the restricted problem of three bodies.

1. INTEGRATION BY THE METHOD OF SUCCESSIVE APPROXIMATIONS

In the equations

$$(1) \quad \frac{dx_i}{dt} = X_i(x_1, x_2, \dots, x_n, \cos \nu t, \sin \nu t)$$

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† *Les Méthodes nouvelles de la Mécanique céleste*, vol. I, § 27.

assume that the functions X_i are continuous in all their arguments, and satisfy the conditions

$$\begin{aligned} & |X_i(x_1'', \dots, x_n'', \cos \nu t, \sin \nu t) - X_i(x_1', \dots, x_n', \cos \nu t, \sin \nu t)| \\ (2) \quad & < A_1 |x_1'' - x_1'| + \dots + A_n |x_n'' - x_n'|, \\ & |X_i| < M \quad (i = 1, 2, \dots, n), \end{aligned}$$

if the arguments lie in the domain

$$(D) \quad |x_i - x_i^0| < b, \quad -\infty < t < \infty \quad (i = 1, 2, \dots, n).$$

Then if $K = \sum_{i=1}^n A_i$, equations (1) can be integrated by the method of successive approximations,* and the solutions are defined if

$$(3) \quad 0 \leq t - t_0 < \frac{1}{K} \log \left(1 + \frac{bK}{M} \right).$$

In the discussion of the successive approximations we shall require the following

LEMMA. Assume that $x_1(t), \dots, x_n(t)$ are continuous, possess continuous derivatives, and that $|x_i - x_i^0| < b$ for $0 \leq t \leq T$, and suppose $y_i(t), Y_i(t)$ defined by the equations

$$\begin{aligned} Y_i(t) &= \int_0^t X_i(x_1, \dots, x_n, \cos \nu t, \sin \nu t) dt, \\ (4) \quad y_i(t) &= \int_0^t \frac{1}{2\pi} \int_0^{2\pi} X_i(x_1, \dots, x_n, \cos \omega, \sin \omega) d\omega dt. \end{aligned}$$

* Picard, *Traité d'Analyse*, vol. 2, 1905, Chap. XI.

Then if ϵ is an arbitrarily small positive quantity, N can be found such that if $\nu > N$,

$$|Y_i(t) - y_i(t)| < \epsilon, \quad 0 \leq t \leq T.$$

If $\Delta t = \frac{2\pi}{\nu}$,

$$\cos \nu(t + \Delta t) = \cos \nu t, \quad \sin \nu(t + \Delta t) = \sin \nu t.$$

Suppose

$$m\Delta t \leq t' < (m+1)\Delta t, \quad m\Delta t = \bar{t}.$$

Then

$$(5) \quad Y_i(t') = \sum_{k=1}^m \int_{(k-1)\Delta t}^{k\Delta t} X_i(x_1, x_2, \dots, x_n, \cos \nu t, \sin \nu t) dt \\ + \int_{\bar{t}}^{t'} X_i(x_1, x_2, \dots, x_n, \cos \nu t, \sin \nu t) dt.$$

Since $\nu(k-1)\Delta t = 2(k-1)\pi$, $\nu k\Delta t = 2k\pi$,

$$I_k = \int_{(k-1)\Delta t}^{k\Delta t} X_i(x_1, \dots, x_n, \cos \nu t, \sin \nu t) dt \\ = \frac{1}{\nu} \int_0^{2\pi} X_i(x_1, \dots, x_n, \cos \omega, \sin \omega) d\omega$$

if

$$\omega = \nu[t - (k-1)\Delta t],$$

or finally

$$(6) \quad I_k = \frac{\Delta t}{2\pi} \int_0^{2\pi} X_i(x_1^{(k-1)}, \dots, x_n^{(k-1)}, \cos \omega, \sin \omega) d\omega,$$

where $x_i^{(k-1)}$ denotes the function $x_i(t)$ in the interval $(k-1)\Delta t \leq t \leq k\Delta t$, or the corresponding function of ω . If $\bar{x}_i^{(k-1)} = x_i[(k-1)\Delta t]$, we obtain from (2)

$$|X_i(x_1^{(k-1)}, \dots, x_n^{(k-1)}, \cos \omega, \sin \omega) - X_i(\bar{x}_1^{(k-1)}, \dots, \bar{x}_n^{(k-1)}, \cos \omega, \sin \omega)|$$

$$< \sum_{r=1}^n A_r |x_r^{(k-1)} - \bar{x}_r^{(k-1)}|.$$

Since $x'_i(t)$ exists and is continuous for $0 \leq t \leq T$, we have, for some constant B ,

$$|x'_i(t)| < B, \quad 0 \leq t \leq T.$$

Then

$$|x_i^{(k-1)} - \bar{x}_i^{(k-1)}| < B\Delta t,$$

$$(7) \quad \sum_{i=1}^n A_i |x_i^{(k-1)} - \bar{x}_i^{(k-1)}| < BK\Delta t.$$

If

$$I'_k = \frac{\Delta t}{2\pi} \int_0^{2\pi} X_i(\bar{x}_1^{(k-1)}, \dots, \bar{x}_n^{(k-1)}, \cos \omega, \sin \omega) d\omega,$$

$$|I_k - I'_k| \leq \frac{\Delta t}{2\pi} \int_0^{2\pi} BK\Delta t d\omega$$

$$(8) \quad \leq BK(\Delta t)^2.$$

Also

$$(9) \quad \left| \int_{\bar{t}}^t X_i(x_1, \dots, x_n, \cos \nu t, \sin \nu t) dt \right| < M\Delta t.$$

From (5), (8), (9),

$$\begin{aligned} \left| Y_i(t') - \sum_{k=1}^m I'_k \right| &\leq M \Delta t + B K m(\Delta t)^2 \\ (10) \qquad \qquad \qquad &\leq \Delta t (M + B K \bar{t}). \end{aligned}$$

From the definition of $y_i(t)$,

$$\begin{aligned} y_i(\bar{t}) &= \int_0^{\bar{t}} \frac{1}{2\pi} \int_0^{2\pi} X_i(x_1, x_2, \dots, x_n, \cos \omega, \sin \omega) d\omega dt \\ &= \sum_{k=1}^m \frac{\Delta t}{2\pi} \int_0^{2\pi} X_i[x_1^{(k-1)}(\xi), x_2^{(k-1)}(\xi), \dots, x_n^{(k-1)}(\xi), \cos \omega, \sin \omega] d\omega, \\ &\qquad \qquad \qquad (k-1) \Delta t \leq \xi \leq k \Delta t, \end{aligned}$$

from the theorem of the mean. Also,

$$|y_i(\bar{t}) - u_i(t')| < M \Delta t.$$

Employing inequalities (2),

$$\left| y_i(\bar{t}) - \sum_{k=1}^m I'_k \right| \leq B K \bar{t} \Delta t.$$

Hence, finally,

$$(11) \qquad \left| y_i(t') - \sum_{k=1}^m I'_k \right| \leq \Delta t [M + B K \bar{t}].$$

Combining (10) and (11),

$$(12) \quad |Y_i(t') - y_i(t')| \leq \frac{4\pi}{\nu} [M + BKT], \quad 0 \leq t' \leq T.$$

From (12) the lemma follows immediately.

Equations (1) can be integrated by the construction of the functions

$$(13) \quad u_i^{(k)} = x_i^0 + \int_0^t X_i(u_1^{(k-1)}, u_2^{(k-1)}, \dots, u_n^{(k-1)}, \cos \nu t, \sin \nu t) dt, \quad u_i^0 = x_i^0 \\ (i = 1, 2, \dots, n; k = 1, 2, \dots).$$

Consider the differential equations

$$(14) \quad \frac{dz_i}{dt} = \frac{1}{2\pi} \int_0^{2\pi} X_i(z_1, z_2, \dots, z_n, \cos \omega, \sin \omega) d\omega \quad (i = 1, 2, \dots, n),$$

and suppose these integrated by the same method;

$$(15) \quad v_i^{(k)} = x_i^0 + \int_0^t \frac{1}{2\pi} \int_0^{2\pi} X_i(v_1^{(k-1)}, v_2^{(k-1)}, \dots, v_n^{(k-1)}, \cos \omega, \sin \omega) d\omega dt, \\ v_i^0 = x_i^0, \quad (i = 1, 2, \dots, n; k = 1, 2, \dots).$$

It is seen immediately that the same constants K, M can be employed for equations (14). Also the functions $u_i^{(k)}, v_i^{(k)}$ lie in the domain D .

From (13)

$$(16) \quad u_i^{(k)} = x_i^0 + \int_0^t X_i(v_1^{(k-1)}, v_2^{(k-1)}, \dots, v_n^{(k-1)}, \cos \nu t, \sin \nu t) dt \\ + \int_0^t [X_i(u_1^{(k-1)}, \dots, u_n^{(k-1)}, \cos \nu t, \sin \nu t) \\ - X_i(v_1^{(k-1)}, \dots, v_n^{(k-1)}, \cos \nu t, \sin \nu t)] dt.$$

Suppose $|u_i^{(k-1)} - v_i^{(k-1)}| < \varrho$ ($i = 1, 2, \dots, n$); then if $w_i^{(k)}$ is defined by the equation

$$w_i^{(k)} = x_i^0 + \int_0^t X_i(v_1^{(k-1)}, \dots, v_n^{(k-1)}, \cos \nu t, \sin \nu t) dt,$$

$$|u_i^{(k)} - w_i^{(k)}| < K\varrho t \quad (i = 1, 2, \dots, n).$$

Also, from the lemma,

$$|w_i^{(k)} - v_i^{(k)}| \leq \frac{4\pi}{\nu} [M + BKT].$$

Hence

$$|u_i^{(k)} - v_i^{(k)}| \leq K\varrho t + \frac{4\pi}{\nu} [M + BKT].$$

From the definition of the functions u , v , w it follows that B can be replaced by M .

If $k = 1$, we obtain

$$|u_i' - v_i'| \leq \frac{4\pi M}{\nu} [1 + KT] = \underline{C_1}.$$

Similarly,

$$|u_i'' - v_i''| \leq \frac{KTC_1}{\nu} + \frac{C_1}{\nu} = \frac{C_2}{\nu},$$

and in general

$$(17) \quad |u_i^{(k)} - v_i^{(k)}| \leq \frac{C_k}{\nu} \quad (k = 1, 2, \dots),$$

where

$$C_k = 4\pi M(1 + KT) \sum_{r=0}^{k-1} (KT)^r.$$

Now if $T < \frac{1}{K} \log \left(1 + \frac{bK}{M}\right)$, the integer ν can be chosen so large that if $\sigma > \nu$, $0 < t \leq T$,

$$|x_i(t) - u_i^{(\sigma)}(t)| < \frac{\varepsilon}{3},$$

$$|z_i(t) - v_i^{(\sigma)}(t)| < \frac{\varepsilon}{3},$$

if ε is any previously assigned positive quantity.

Now suppose σ fixed; from (17)

$$|u_i^{(\sigma)} - v_i^{(\sigma)}| \leq \frac{C_\sigma}{\nu}.$$

Hence if ν is chosen sufficiently large $\frac{C_\sigma}{\nu} < \frac{\varepsilon}{3}$, and we obtain

$$|x_i(t) - z_i(t)| < \varepsilon \quad (i = 1, 2, \dots, n).$$

Hence the theorem: *If $x_1(t), x_2(t), \dots, x_n(t)$ is a solution of (1), and $z_1(t), z_2(t), \dots, z_n(t)$ the solution of (14) satisfying the same initial conditions, and if $0 < T < \frac{1}{K} \log \left(1 + \frac{bK}{M}\right)$, then given any positive quantity ε , a number N can be found such that if $\nu > N$,*

$$|x_i(t) - z_i(t)| < \varepsilon \quad (0 \leq t \leq T; i = 1, 2, \dots, n).$$

If $KT < 1$, then $C_\sigma < C$,

$$C = \frac{4\pi M(1 + KT)}{1 - KT}.$$

Hence for any ν , ϵ can be so chosen that if $\sigma > \epsilon$,

$$|x_i(t) - u_i^{(\sigma)}(t)| < \frac{1}{2\nu}, \quad |z_i(t) - v_i^{(\sigma)}(t)| < \frac{1}{2\nu}, \quad |u_i^{(\sigma)} - v_i^{(\sigma)}| < \frac{C}{\nu}.$$

Consequently

$$|x_i(t) - z_i(t)| < \frac{C+1}{\nu} \quad (0 \leq t \leq T; i = 1, \dots, n),$$

an inequality independent of ϵ .

2. EXAMPLE

Suppose μ a parameter on the interval $0 < \mu < 1$, and assume masses μ and $1 - \mu$ connected by a rigid weightless bar of unit length. Assume this system to rotate about its center of gravity in the (x, y) plane, with an angular velocity n , the origin coinciding with the center of gravity. Then if a particle of unit mass moves in space under the newtonian attraction of the first two masses, its coordinates satisfy the equations

$$\begin{aligned} \frac{dx}{dt} &= x', & \frac{dx'}{dt} &= \frac{\partial U}{\partial x}, \\ \frac{dy}{dt} &= y', & \frac{dy'}{dt} &= \frac{\partial U}{\partial y}, \\ \frac{dz}{dt} &= z', & \frac{dz'}{dt} &= \frac{\partial U}{\partial z}, \end{aligned}$$

(18)

$$U = \frac{1 - \mu}{r_1} + \frac{\mu}{r_2},$$

$$x_1 = \mu \cos nt, \quad y_1 = \mu \sin nt,$$

$$x_2 = (1 - \mu) \cos nt, \quad y_2 = -(1 - \mu) \sin nt,$$

$$r_1^2 = (x - x_1)^2 + (y - y_1)^2 + z^2,$$

$$r_2^2 = (x - x_2)^2 + (y - y_2)^2 + z^2.$$

If $P_0(x_0, y_0, z_0)$ is such that $r_1^0 > 1$, $r_2^0 > 1$, or if $z_0 \neq 0$, then a certain neighborhood of P_0 can be found within which the first and second partial derivatives of U , with respect to x, y, z , are continuous and their absolute values have upper bounds independent of n . Within this neighborhood equations (18) are of the form (1), and the theorem of § 1 can be applied. The motion approaches that defined by the equations

$$\begin{aligned}
 \frac{d^2 \bar{x}}{dt^2} &= \frac{\partial \bar{U}}{\partial \bar{x}}, \\
 \frac{d^2 \bar{y}}{dt^2} &= \frac{\partial \bar{U}}{\partial \bar{y}}, \quad \bar{U} = \frac{1-\mu}{2\pi} \int_0^{2\pi} \frac{d\omega}{r_1} + \frac{\mu}{2\pi} \int_0^{2\pi} \frac{d\omega}{r_2}, \\
 \frac{d^2 \bar{z}}{dt^2} &= \frac{\partial \bar{U}}{\partial \bar{z}}, \\
 \bar{x}_1 &= \mu \cos \omega, & \bar{y}_1 &= \mu \sin \omega, \\
 \bar{x}_2 &= (1-\mu) \cos \omega, & \bar{y}_2 &= -(1-\mu) \sin \omega.
 \end{aligned}
 \tag{19}$$

The limiting motion is that of a particle moving in space under the attraction of two concentric rings, each of uniform density; the equations (19) admit the area integral

$$xy' - x'y = C,$$

in addition to the energy integral. Consequently the plane problem is integrable.

The interest of this result lies in the fact that while in the restricted problem of three bodies $n = 1$, yet the analytic discussion in many cases* is precisely the same as for n arbitrary ($\neq 0$)

* For instance, Birkhoff, *The restricted problem of three bodies*, Rendiconti del Circolo Matematico di Palermo, vol. 39 (1915).