ON THE COMPLETE INDEPENDENCE OF THE POSTULATES FOR BETWEENNESS*

BY

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The paper on betweenness published by Huntington and Kline in 1917† contained eleven sets of independent postulates, selected from a basic list of twelve postulates, as follows:

(1) A, B, C, D, 1,2;	(7) A, B, C, D, 2,5;
(2) A, B, C, D, 1,5;	(8) A, B, C, D, 3,5;
(3) A, B, C, D, 1,6;	(9) A, B, C, D, 3,4,6;
(4) A, B, C, D, 1,7;	(10) A, B, C, D, 3,4,7;
(5) A, B, C, D, 1,8;	(11) A, B, C, D, 3,4,8.
(6) A, B, C, D, 2,4;	

The purpose of the present paper is to exhibit the “complete existential theory” (in the sense of E. H. Moore‡) of each of these sets. This requires the discussion, in the usual way, of $2^6 = 64$ examples for each of the sets (1)–(8), and $2^7 = 128$ examples for each of the sets (9)–(11).

The results show that sets (1)–(10) are completely independent while set (11) is not.

In the case of sets (1), (2), (3), (5), (6), (7), which happen to be the sets which do not contain either postulate 3 or postulate 7, the necessary examples are given in terms of a class K containing only four elements.

In the case of sets (4), (8), (9), (10), and (11), some of the examples require the use of a class K containing five elements. These five-element

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† E. V. Huntington and J. R. Kline, *Sets of independent postulates for betweenness*, these Transactions, vol. 18 (1917), pp. 301–325. For a twelfth set of postulates, which need not here be considered, see E. V. Huntington, *A new set of postulates for betweenness with proof of complete independence*, in the present number of these Transactions.

‡ For a discussion of the significance of “complete independence”, with bibliographical references, see the paper by E. V. Huntington, in the present number of these Transactions.

examples are used, however, only in cases where an exhaustive examination has shown that no four-element example with the same record exists.

The failure of set (11) to be completely independent is due to the non-existence of only two examples,* namely an example satisfying postulates A, B, C, D and 8, and violating postulates 3 and 4; and a corresponding example violating D.

Table I defines 294 systems (K, R) by listing explicitly the triads which are supposed to be true in each case.

Tables II and III show how these examples are used in establishing complete independence, a plus sign indicating that a postulate holds, a minus sign, that it fails. For example, in connection with Set (6) we need to exhibit a system (K, R) having the record

D	A	B	C	2	4
+	+	—	+	—	—

Turning to Table II, record No. 12, we see that Example 72 is the system required.

To obtain a similar record with the D + changed to D —, we have only to change Example 72 to Example 72d.

It will be noted that postulate D (which demands that if ABC is a true triad, then A , B , and C shall be distinct) plays a peculiar rôle, since, though it is strictly independent and cannot be omitted, yet it is never used in proving any of the “theorems of deducibility”, and its holding or failing does not effect the holding or failing of any of the other postulates.

* The proof of the non-existence of these examples was communicated to the writer by Professor Huntington, who showed that the simpler example, satisfying A, B, C, D, 8 and violating 3, can be found only when $n = 4$ or $n = 5$, and does not exist when $n = 6$ or $n > 6$ (where n is the number of elements in the class K). This is an altogether unexpected state of affairs, since in previous discussions of complete independence, an increase in the number of elements has always increased (instead of diminishing) the likelihood of finding an example of any desired type. It also suggests a wider inquiry into the validity of the “Lemmas on non-deducibility” (pp. 272–74 of the second paper cited) when n is greater than four. For example, although postulate 3 is in general not deducible from postulates A, B, C, D, 8 (see Lemma 3.1), yet if we add the further condition that the class K shall contain six or more elements, then postulate 3 can be so deduced. Again Mr. C. H. Langford has shown that postulate 4, though not deducible from postulates A, B, C, D, 8, alone (see Lemma 4.1), can be deduced from these postulates with the added condition that the class K shall contain at least five elements. How many other similar instances may exist has not yet been investigated.

TABLE ONE

Ex.	In Examples 1-129, the class K consists of four elements, 1, 2, 3, 4.												
	In Examples 501-518, the class K consists of five elements, 1, 2, 3, 4, 5.												
1	123	124	134	234	321	421	431	432					
2	123	124	134	321	324	421	423	431					
3	123	143	214	234	321	341	412	432					
4	123	124	134	243	321	342	421	431					
5	123	124	132	134	231	234	321	324	421	423	431	432	
6	123	124	132	134	231	234	321	421	431	432			
7	123	132	214	231	234	314	321	412	413	432			
8	123	124	132	134	231	243	321	342	421	431			
9	123	124	321	421									
10	123	142	241	321									
11	123	214	321	412									
12	123	142	241	314	321	413							
13	123	132	231	321									
14	123	124	132	231	321	421							
15	123	132	231	234	321	432							
16	123	132	214	231	321	412							
17	123	124	134	234									
18	123	124	134	324									
19	123	124	134	342									
20	123	234	341	421									
21	123	124	132	134	234								
22	123	124	132	342	431								
23	123	124	132	243	431								
24	123	124	132	324	413								
25	123	124											
26	123	243											
27	123	234											
28	123	241	413										
29	123	124	132										
30	123	132	243										
31	123	132	234										
32	123	132	412										
33	123	124	234	314	321	413	421	432					
34	123	132	142	143	231	241	243	321	341	342			
35	123	124	132	143	231	234	321	341	421	432			
36	132	142	231	241									
37	123	143	214	321	341	412							

Ex.													
79	123	124	234	413									
80	123	124	143	423									
81	123	124	132	341	342								
82	123	124	143	213	243	312	321	341	342	421			
83	123	132	142	143	231	241	321	341					
84	123	124	234	431									
85	123	124	134	243									
86	123	124	132	143	432								
87	123	124	132	134	243								
88	123	413											
89	123	243	321	341									
90	123	142	143	213	241	243	312	321	341	342			
91	123	124	132	143	231	321	324	341	421	423			
92	123	124	314	321	413	421							
93	123	132	231	314	321	413							
94	123	132	142	234	241	321	324	423					
95	123	143	243	321	421								
96	123	124	324	413									
97	123	243	421	431									
98	123	124	243	321	341	423							
99	123	132	143	421	432								
100	123	124	132	413	432								
101	123	132	243	421	431								
102	123	132	142	231	243	431							
103	123	143	231	431									
104	241	321	421										
105	123	213	243										
106	123	213	243	423									
107	123	132	143	432									
108	123	132	213	243	432								
109	123	124	132	134	142	143	231	234	241	243	321	341	
	342	421	431	432									
110	123	124	132	134	142	143	231	241	321	324	341	421	
	423	431											
111	132	142	143	231	241	341							
112	123	124	132	134	231	321	421	431					
113	123	132	142	231	241	243	321	342					
114	123	143	214	243									
115	123	143	243	421									
116	123	124	213	243	413								
117	123	124	132	134	423								

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Ex.												
118	123	124	132	314	432							
119	123	132	142	241	243	341						
120	123	243	423									
121	123	132	142	342								
122	123	132	243	423								
123	123	132	213	243	423							
124	123	132	412	423								
125	123	132	142	143	213	231	241	243	312	321	341	342
126	123	143	213	214	243	312	321	341	342	412		
127	123	132	214	231	234	321	412	432				
128	123	124	132	143	231	321	341	421				
129	123	132	142	432								

1d-129d. Same as 1-129, with the addition of the triad 111.

Ex.												
501	123	125	134	135	142	145	234	241	245	321	325	345
	431	432	521	523	531	541	542	543				
502	123	125	234	321	432	521						
503	123	153	243	321	342	351						
504	123	125	143	145	214	234	254	315	321	325	341	345
	412	432	452	513	521	523	541	543				
505	123	124	145	153	215	235	243	245	314	321	342	345
	351	413	421	512	532	541	542	543				
506	123	125	143	145	213	214	215	243	245	312	315	321
	325	341	342	345	412	512	513	521	523	541	542	543
507	123	143	321	325	341	523						
508	123	153	243	321	342	351						
509	123	124	153	243	321	342	351	421				
510	123	132	231	245	321	542						
511	123	143	235	321	325	341	523	532				
512	123	125	135	143	145	214	234	254	321	325	341	345
	412	432	452	521	523	531	541	543				
513	123	125	134	142	145	153	241	243	245	321	325	342
	345	351	431	521	523	541	542	543				
514	123	124	134	153	154	215	235	243	254	321	342	351
	354	421	431	451	452	453	512	532				
515	123	153	243	321	325	342	351	523				
516	123	132	142	143	231	234	241	321	341	415	432	514
517	123	125	143	145	153	214	234	254	321	325	341	345
	351	412	432	452	521	523	541	543				
518	123	153	243	321	325	342	351	523				

501d-518d. Same as 501-518, with the addition of the triad 111.

TABLE THREE

Rec.	Postulates							Independent Sets		
	D A B C 3 4 6							(9)		
	D A B C 3 4 7								(10)	
	D A B C 3 4 8									(11)
1	+	+	+	+	+	+	+	1	1	1
2	+	+	+	+	+	+	—	2	2	2
3	+	+	+	+	+	—	+	3	3	3
4	+	+	+	+	+	—	—	504	512	517
5	+	+	+	+	—	+	+	65	65	65
6	+	+	+	+	—	—	+	33	513	33
7	+	+	+	+	—	—	+	4	4	—
8	+	+	+	+	—	—	—	505	514	4
9	+	+	+	—	+	+	+	90	109	125
10	+	+	+	—	+	+	—	6	6	6
11	+	+	+	—	+	—	+	82	82	82
12	+	+	+	—	+	—	—	506	110	110
13	+	+	+	—	—	+	+	48	48	48
14	+	+	+	—	—	+	—	91	91	91
15	+	+	+	—	—	—	+	8	8	126
16	+	+	+	—	—	—	—	35	35	8
17	+	+	—	+	+	+	+	11	11	11
18	+	+	—	+	+	+	—	9	9	9
19	+	+	—	+	+	—	+	36	36	36
20	+	+	—	+	+	—	—	507	111	72
21	+	+	—	+	—	+	+	10	10	10
22	+	+	—	+	—	+	—	92	92	92
23	+	+	—	+	—	—	+	508	508	508
24	+	+	—	+	—	—	—	509	515	518
25	+	+	—	—	+	+	+	510	510	510
26	+	+	—	—	+	+	—	15	15	15
27	+	+	—	—	+	—	+	83	516	83
28	+	+	—	—	—	+	—	511	112	127
29	+	+	—	—	—	+	+	93	93	93
30	+	+	—	—	—	+	—	14	14	14
31	+	+	—	—	—	—	+	38	38	38
32	+	+	—	—	—	—	—	94	113	128
33	+	—	+	+	+	+	+	17	17	17
34	+	—	+	+	+	+	—	53	53	53
35	+	—	+	+	+	—	+	75	114	74
36	+	—	+	+	+	—	—	95	115	75
37	+	—	+	+	—	+	+	85	85	85
38	+	—	+	+	—	+	—	96	96	96
39	+	—	+	+	—	—	+	97	97	97
40	+	—	+	+	—	—	—	98	116	116
41	+	—	+	—	+	+	+	21	21	21
42	+	—	+	—	+	+	—	55	117	55
43	+	—	+	—	+	—	+	62	77	76
44	+	—	+	—	+	—	—	99	62	62
45	+	—	+	—	—	+	+	87	87	87
46	+	—	+	—	—	+	—	100	118	100
47	+	—	+	—	—	—	+	101	101	101
48	+	—	+	—	—	—	—	102	119	102
49	+	—	—	+	+	+	+	25	25	25
50	+	—	—	+	+	+	—	57	57	57
51	+	—	—	+	+	—	+	44	44	44
52	+	—	—	+	+	—	—	103	78	78
53	+	—	—	+	—	+	+	89	88	88
54	+	—	—	+	—	+	—	104	120	120
55	+	—	—	+	—	—	+	105	105	105
56	+	—	—	+	—	—	—	106	106	106
57	+	—	—	—	+	+	+	29	29	29
58	+	—	—	—	+	+	—	59	59	59
59	+	—	—	—	+	—	+	46	46	46
60	+	—	—	—	+	—	—	107	121	129
61	+	—	—	—	—	+	+	32	32	32
62	+	—	—	—	—	+	—	124	122	124
63	+	—	—	—	—	—	+	47	47	47
64	+	—	—	—	—	—	—	108	123	123

Records No. 65-128 are the same as records No. 1-64 with the D+ changed to D—, and the letter "d" added to each example-number.

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