ON THE OSCILLATION OF A CONTINUUM AT A POINT*

BY

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1. The concept of oscillation of a continuum at a given point was introduced by S. Mazurkiewicz in his paper on Jordan curves† and the idea has since been used frequently by other writers‡. Less attention has been given to the general concept, however, than to the special case where the oscillation is zero, a property which H. Hahn has shown to be equivalent to the property of connectedness im kleinen.

In this article several new properties of the oscillatory function are developed. In particular it is shown that the oscillation is an upper semi-continuous function and therefore has the well known characteristics of pointwise discontinuous functions. Furthermore, the behavior of the function in the neighborhood of points where its value is greater than zero is investigated. (See §§ 11–15.)

The results are obtained by the introduction of an auxiliary function $\tau(a)$ (§ 3) and oscillatory sub-sets (§§ 7–10) of a continuum, both of which have interesting properties in themselves. The work is confined to continua, although several of the theorems can be readily extended to non-closed connected sets.

2. **Notation.** The ordinary notation of the theory of aggregates is employed, with the following modifications.

If A is the common part of a system of aggregates $\{C\}$, we write $A = Dv[C]$.

If A is a real part of B, we write $A \subset B$.

If A is a part of B and may be identical with B, we write $A \subseteq B$.

3. **Definitions.** Let A be any continuum and $a \varepsilon A$. Let $V_\delta\,(a)$ denote the set of points of A whose distance from a is less than δ, $\delta > 0$. Let C_δ denote a subcontinuum of A which contains all points of $V_\delta(a)$. The lower bound of the diameters of all such sets C_δ, for all $\delta > 0$, is denoted by $\tau_A(a)$, or simply $\tau(a)$.

For convenience the previous sentence may be written

$$\tau(a) = \text{Min Diam } [C_\delta].$$

* Presented to the Society, September 10, 1925.

† *Sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 1, pp. 166 ff.

‡ E. g., Z. Janiszewski, C. Kuratowski, and B. Knaster.

If a set is unlimited, we call its diameter ∞. If for a point a every subcontinuum C_δ is unlimited, we say that $\tau(a) = \infty$.

It is a comparatively simple matter to show the existence of the auxiliary function just defined for each point of any continuum. It is also evident that the above definition might well be employed as a definition of the oscillation of a continuum at a point. Without entering into details, it may be stated that $\tau(a)$ has all the properties of S. Mazurkiewicz' oscillatory function $\sigma(a)$ as given in the article referred to above, pp. 170–178. It is not, however, precisely the same function for continua in general, as the next section will show.

4. The function $\sigma(a)$ is defined as follows (loc. cit., p. 170). If x and y are any two points of a continuum A, let $C(x, y)$ be any subcontinuum of A containing x and y. The number $\varrho_A(x, y) =$ Min Diam $[C(x, y)]$ for all possible sets $C(x, y)$ is called the relative distance between x and y with respect to A. Then $\sigma(a) = \overline{\lim}\, \varrho_A(x, y)$ as x and y approach a.

It is easy to construct continua for which $\tau(a) \neq \sigma(a)$ at certain points. However, as the following theorem shows, $\tau(a)$ and $\sigma(a)$ vanish at the same points. Hence the genre of a point is the same for both definitions of oscillation.

THEOREM. *If A is a continuum and $a \varepsilon A$, then $\sigma(a) \leqq \tau(a) \leqq 2\sigma(a)$.*

Proof. For any $\varepsilon > 0$ there exists a $\delta > 0$ and a sub-continuum C_δ of A for which

$$V_\delta(a) \subseteq C_\delta \subseteq A, \tag{1}$$

and

$$\text{Diam } C_\delta \leqq \tau(a) + \varepsilon. \tag{2}$$

From (2) we have at once that $\varrho_A(x, y) \leqq \tau(a) + \varepsilon$ for any pair of points x and y in $V_\delta(a)$. Therefore

$$\sigma(a) \leqq \tau(a). \tag{3}$$

On the other hand, for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\varrho_A(x, y) \leqq \sigma(a) + \frac{\varepsilon}{2} \tag{4}$$

for any pair of points x and y in $V_\delta(a)$. By definition of $\varrho_A(x, y)$ there exists a sub-continuum $C(x, y)$ of A such that

$$\text{Diam } C(x, y) \leqq \varrho_A(x, y) + \frac{\varepsilon}{2} \leqq \sigma(a) + \varepsilon. \tag{5}$$

Now relations (4) and (5) hold for the particular case that $x = a$. Hence the union of all the sets $C(a, y)$ is a connected sub-set C of A which contains $V_\delta(a)$ and $\bar{C}$ is a sub-continuum of A containing $V_\delta(a)$. From (5) we have

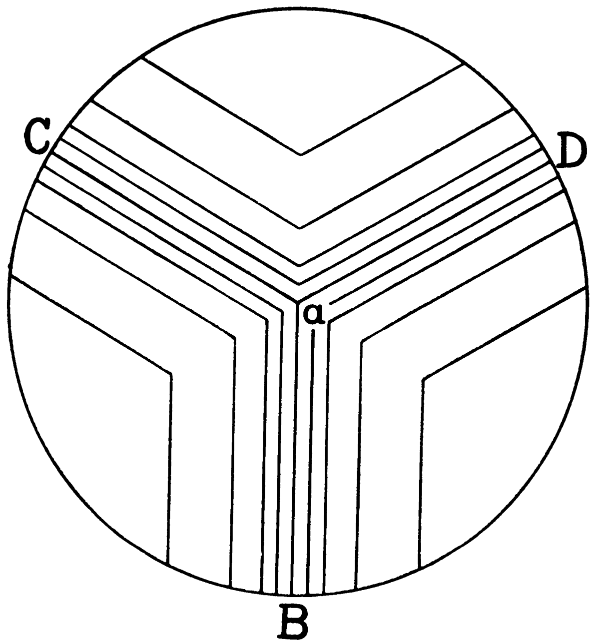
$$\text{Diam}\,\bar{C} = \text{Diam}\,C \leqq 2\sigma(a) + 2\varepsilon,$$

whence

$$\tau(a) \leqq 2\sigma(a) + 2\varepsilon. \tag{6}$$

Since (6) holds for any $\varepsilon > 0$, (3) and (6) give the theorem.

Example. The following is a case where $\sigma(a) \neq \tau(a)$. Draw a circle of center a and radius r and three radii trisecting the circle. On the radii bisecting each of the sectors thus formed take a sequence of points whose



distance from a is $r/2^n$, $n = 1, 2, 3, \cdots$, and through each of these points draw two lines parallel to the radii bounding the sector and terminated by the arc of the sector. Let A be the continuum consisting of the circumference, the three radii, and the series of broken lines.

Two points lying in any $V_\delta(a)$ and not lying on the radii aB, aC, or aD will lie in the same sector or in adjacent sectors, say those whose arcs are BC and CD. They can be joined by a sub-continuum passing through C and of diameter not greater than $r + 4\delta$ and not less than r. Thus $\sigma(a) = r$.

But $V_\delta(a)$ contains points in all three sectors. Hence any sub-continuum containing $V_\delta(a)$ must contain two of the points B, C, D and therefore has a diameter at least as great as the distance $BC = r\sqrt{3}$. Thus $\tau(a) \geqq r\sqrt{3}$ and it is easy to see that $\tau(a) = r\sqrt{3}$. In this example, then, $\tau(a) = \sqrt{3}\,\sigma(a)$.

5. THEOREM. *Let $A = \{x\}$ be a continuum. Then $\tau(x)$ and $\sigma(x)$ are upper semi-continuous at all points of A.*

Proof. At points where $\tau(x)$ is infinite the theorem obviously holds. Let a be a point of A at which $\tau(a) = k \neq \infty$. Then for $\varepsilon > 0$, there is a $\delta > 0$ and a sub-continuum C_δ of A for which

(1) $$V_\delta(a) \subseteq C_\delta \subseteq A,$$

and

(2) $$\text{Diam}\, C_\delta \leqq k + \varepsilon.$$

Now for any x in $V_\delta(a)$ there is an $\eta > 0$ such that $V_\eta(x) \subseteq V_\delta(a)$. Hence

(3) $$V_\eta(x) \subseteq C_\delta.$$

Relations (1), (2), and (3) show that

$$\tau(x) \leqq k + \varepsilon = \tau(a) + \varepsilon, \quad \text{for } x \text{ in } V_\delta(a),$$

which is the definition of upper semi-continuity.

Likewise for any $\varepsilon > 0$ there is a $\delta > 0$ such that any two points x and y in $V_\delta(a)$ can be joined by a sub-continuum $C(x, y)$ of A whose diameter is not greater than $\sigma(a) + \varepsilon$. For any point x' in $V_\delta(a)$ there is an $\eta > 0$ such that $V_\eta(x') \subseteq V_\delta(a)$. Hence any two points of $V_\eta(x')$ can be joined by a sub-continuum of A of diameter not greater than $\sigma(a) + \varepsilon$.

Then we have $\sigma(x') \leqq \sigma(a) + \varepsilon$ for all x' in $V_\delta(a)$, which is the requirement for upper semi-continuity.

The above theorem shows that the oscillation of a continuum, if everywhere finite, is continuous or pointwise discontinuous, like the oscillation of a one-valued function of a real variable. Among the properties deducible from this fact may be mentioned the following.

COROLLARY 1. *Let $A = \{x\}$ be a continuum. Then $\tau(x)$ and $\sigma(x)$ are continuous at each point of A for which their value is zero.*

COROLLARY 2. *Let $A = \{x\}$ be a continuum. Then the set of points for which $\tau(x) \geqq k$, or $\sigma(x) \geqq k$, any constant, is a closed set; the set of points for which either function is discontinuous is of the first category; and the set for which either function is continuous is of the second category with respect to A.*

COROLLARY 3. *Let $A = \{x\}$ be a continuum and B be a closed limited part of A. If $\tau(x)$ or $\sigma(x)$ is finite for each point of B, it is limited in B and there is at least one point of B at which it takes on its maximum value.*

6. **Generalized irreducible continua.** We now proceed to extend the notion of a continuum irreducible between two points. Let C be a continuum and let A be any sub-set of C. The continuum C is called *irreducible about* A if there is no continuum C' satisfying the relation $A \subseteq C' \subset C$.

THEOREM. *Let C be a limited continuum and $A \subseteq C$. Then there exists at least one sub-continuum D of C which is irreducible about A.*

Proof. In order to prove this theorem it is only necessary to show, as C. Kuratowski has demonstrated,* that if C_i is a sequence of continua such that

$$C \supseteq C_1 \supseteq C_2 \supseteq \cdots \supseteq C_i \supseteq C_{i+1} \supseteq \cdots \supseteq A \tag{1}$$

and $C_\omega = Dv\,[C_i]$, then C_ω is a continuum and contains A.

Since each C_i is closed and limited, C_ω exists. By (1) $A \subseteq C_\omega$. Since all the aggregates C_i form a part of the limited aggregate C and $C_\omega \neq 0$, it follows† that C_ω is a continuum. Our theorem is therefore proved.

Without going into the properties of irreducible continua, it is perhaps well to call attention to the fact that under the conditions of the above theorem there may be many sub-continua of C irreducible about A. If C is unlimited, there may be no irreducible sub-continuum.

7. **Oscillatory sets.** Let A be a continuum and $a \varepsilon A$. Let $\delta_1 > \delta_2 > \cdots$, $\delta_i \to 0$, and let C_i denote a sub-continuum of A irreducible about $V_{\delta_i}(a)$. If the sequence $\{C_i\}$ is monotone decreasing, i. e., $C_1 \supseteq C_2 \supseteq \cdots$, we call $C = Dv\,[C_i]$ an oscillatory set of A about a.

A continuum may have many oscillatory sets about one of its points. For instance, in the example given in § 4 each of the broken lines BaC, CaD, and DaB is an oscillatory set of A about the point a. If a continuum is unlimited, it may have no oscillatory set about one or more of its points. However, we have the following theorem.

THEOREM. *Let A be a continuum, $a \varepsilon A$, and $\sigma(a)$ be finite. Then there exists at least one oscillatory set of A about a.*

Proof. Let $\delta_1 > \delta_2 > \cdots$ and $\delta_i \to 0$. Since $\sigma(a)$ is finite, so is $\tau(a)$. Hence there is a sub-continuum C_1 of A which is limited and irreducible about $V_{\delta_1}(a)$. Likewise there is a sub-continuum C_2 of C_1 which is irreducible about $V_{\delta_2}(a)$; etc. Thus we have a monotone decreasing sequence

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots,$$

and $C = Dv\,[C_i]$ is an oscillatory set of A about a by definition.

* C. Kuratowski, *L'élimination des nombres transfinis*, Fundamenta Mathematicae, vol. 3, pp. 88–89.

† F. Hausdorff, *Grundzüge der Mengenlehre*, p. 302.

8. THEOREM. *Let A be a continuum and $a \varepsilon A$. If $\tau(a)$ is finite and C denotes any oscillatory set of A about a, then $\tau(a) =$* Min Diam $[C]$. *If $\tau(a) = \infty$, and any oscillatory set C of A about a exists,* Diam $C = \infty$.

Proof. Let Min Diam $[C] = k$. Then there is at least one oscillatory set C for which

$$k \leqq \text{Diam } C \leqq k + \frac{\varepsilon}{2}, \tag{1}$$

for an arbitrarily chosen positive ε. Now suppose that $C = Dv[C_i]$, where each C_i is a sub-continuum of A irreducible about $V_{\delta_i}(a)$, $\delta_i \to 0$. It follows at once that there is an i_0 such that for all $i > i_0$ every point of the set C_i has a distance from C not greater than $\varepsilon/4$. Hence

$$\text{Diam } C_i \leqq \text{Diam } C + \frac{\varepsilon}{2} \leqq k + \varepsilon. \tag{2}$$

Thus we have at least one sub-continuum C_i of A which contains a $V_\delta(a)$ and whose diameter is not greater than $k + \varepsilon$. Hence $\tau(a) \leqq k + \varepsilon$, whatever ε may be. Thus

$$\tau(a) \leqq k. \tag{3}$$

On the other hand, if $\tau(a)$ is finite, there is a limited sub-continuum B of A containing a $V_{\delta_1}(a)$ and of diameter not greater than $\tau(a) + \varepsilon$, where $\varepsilon > 0$ is arbitrary. Now B contains a sub-continuum C_1 irreducible about $V_{\delta_1}(a)$ and as in § 7 there is a monotone decreasing sequence $\{C_i\}$, whose divisor C is an oscillatory set about a. Since Diam $B \leqq \tau(a) + \varepsilon$, Diam $C \leqq \tau(a) + \varepsilon$. Thus

$$\text{Min Diam } [C] \leqq \tau(a). \tag{4}$$

Relations (3) and (4) give the theorem for the case that $\tau(a)$ is finite. The case that $\tau(a) = \infty$ is disposed of by (3).

9. **The two kinds of oscillatory sets.** We have at once as a corollary of the previous theorem that $\sigma(a) \leqq$ Min Diam $[C]$. It is also evident that these sets furnish an independent means of defining the function $\tau(a)$.

From the two previous sections it is clear that the oscillatory sets C of a continuum A may be of two kinds:

I. C is the divisor of a *descending* sequence of sub-continua $\{C_i\}$, where each C_i is irreducible about a $V_{\delta_i}(a)$, $\delta_i \to 0$.

II. C is irreducible about every $V_\delta(a)$ for every δ less than or equal to some δ_0.

In particular C is of the second kind if, in the monotone decreasing sequence $\{C_i\}$ determining C, all the sets C_i from some i_0 on are identical.

That such sets exist follows at once from the fact that if A is indecomposable there exist in any $V_\delta(a)$ points x such that A is irreducible between a and x. If, however, C is an oscillatory set of the second kind, it is not necessarily indecomposable.

If there is only a finite number of oscillatory sets of A about a, the diameter of one of them is $\tau(a)$. It is also easy to show that, if $\sigma(a) = 0$, one of the oscillatory sets of A about a is the point a itself.

10. THEOREM. *Let A be a continuum and $a \varepsilon A$. Let C be an oscillatory set about a with a finite diameter $k \neq 0$. Then for any $\varepsilon > 0$, however small, there exists a continuum of condensation of A containing a and forming a part of C whose diameter is greater than or equal to $k - \varepsilon$.*

Proof. Let $\delta_1 > \delta_2 > \cdots$ and $\delta_i \to 0$, and let $\{C_i\}$ be a monotone decreasing sequence of sub-continua of A with each C_i irreducible about $V_{\delta_i}(a)$. Let $C = Dv[C_i]$. Since C is limited, there is no loss in generality in assuming that every C_i is also limited.*

Now let x be any point of C different from a, and let h denote the distance from a to x. Let $\varepsilon < h/4$, and let Γ_σ be the interior of a sphere of center x and of radius $\sigma < \varepsilon/2$. Then, for i greater than some i_0, $\bar{\Gamma}_\sigma \cdot \bar{V}_{\delta_i}(a) = 0$. Let

$$D_i = C_i - C_i \cdot \Gamma_\sigma. \tag{1}$$

Each D_i is closed and contains $V_{\delta_i}(a)$.

Let $D = C - C \cdot \Gamma_\sigma$ and C_σ be that component of D containing a; it also contains points of the frontier of Γ_σ, which we denote by Front Γ_σ. This follows from Janiszewski's theorem.† There is at least one point z_i of D_i in $V_{\delta_i}(a)$ not lying on C_σ, for otherwise C_i would not be irreducible about $V_{\delta_i}(a)$.

Let S_{z_i} be that component of D_i which contains one of these points z_i. As in the case of C_σ we see that S_{z_i} has a point on Front Γ_σ. Now if $S_{z_i} \cdot C_\sigma \neq 0$, then $C_\sigma \subseteq S_{z_i}$ since $C \subseteq C_i$. If this were true for every z_i

* For, since C is closed and limited, there is a finite closed sphere S such that every point of C is an inner point of S. A set C_i which is unlimited must contain points without S. Then, since it is a continuum and contains both inner and outer points of S, it must have points in common with $F = \text{Front}\, S$. Let $F_i = F \cdot C_i$. The sets F_i are closed and limited; also the sequence $\{F_i\}$ is monotone decreasing. Then either $F_i = 0$ for all i greater than some i_0 or $F_i \neq 0$ for all values of i. In the latter case $Dv[F_i] \neq 0$. But, since each $F_i \subset C_i$, $Dv[F_i] \subset C$, which makes $F \cdot C \neq 0$, contrary to our assumption regarding S.

† Janiszewski, *Sur les continus irréducibles entre deux points*, Journal de l'Ecole Polytechnique, ser. 2, vol. 16 (1912), p. 100. "Let C be a limited continuum and a be one of its points. Let A be closed and a be an inner point of A. Then there is a subcontinuum of C containing a and a frontier point of A (unless C is contained in A) and contained in A."

in $V_{\delta_i}(a)$, the union of the sets S_{z_i}, which we denote by U, would be connected and $\bar{U}$ would be a real sub-continuum of C_i containing $V_{\delta_i}(a)$. This, however, is contrary to the hypothesis that C_i is irreducible about $V_{\delta_i}(a)$. Hence in each $V_{\delta_i}(a)$ there is a point z_i for which there is a sub-continuum S_{z_i} of C_i containing z_i and a point of Front Γ_σ, but containing no point of C_σ.

As $i \to \infty$, $\delta_i \to 0$ and $z_i \to a$. Let K be the aggregate of accumulation of $\{S_{z_i}\}$. K is a continuum or reduces to the point a by virtue of a theorem of Janiszewski*. As each S_{z_i} has a point on Front Γ_σ, K also has a point x' on Front Γ_σ. Hence

$$(2) \qquad \text{Diam } K \geqq h - \sigma \geqq h - \frac{\varepsilon}{2}.$$

Since $S_{z_i} \subseteq C_i$, K is a part of every C_i. Hence $K \subseteq C$. Obviously K has no points in Γ_σ. Thus

$$(3) \qquad K \subseteq C - C \cdot \Gamma_\sigma \subseteq C_\sigma.$$

But $C_\sigma \cdot S_{z_i} = 0$; this with (3) shows that K is a continuum of condensation. We have shown then that if x is any point of C different from a there is a continuum of condensation of A which forms a part of C and which contains a and at least one point x' whose distance from x is less than $\varepsilon/2$.

Now since C is a closed set it contains two points, x and y, whose distance apart is k. If one of these is a, the previous paragraph together with relations (2) and (3) give the theorem, for then $h = k$. If not, C contains a continuum of condensation K_1 of A joining a and x', a point whose distance from x is less than $\varepsilon/2$, and a K_2 joining a and y', a point whose distance from y is less than $\varepsilon/2$. Let $K = K_1 + K_2$. Since $K_1 \cdot K_2 \neq 0$, K is a continuum of condensation. Moreover,

$$\text{Diam } K \geqq \text{Dist } (x', y') \geqq \text{Dist } (x, y) - \varepsilon \geqq k - \varepsilon.$$

11. One of the fundamental theorems in S. Mazurkiewicz' paper† is to the effect that if a point of a continuum is of the second genre, it lies on a continuum of condensation. By means of § 10 we are able to show a relation between the size of this continuum of condensation and the oscillation of the given continuum at the point.

THEOREM. *Let A be a continuum, $a \varepsilon A$, and $\sigma(a) = k$ be finite and different from zero. Then for any $\varepsilon > 0$ there exists a continuum of condensation of A containing a and of diameter greater than or equal to $k - \varepsilon$.*

* Loc. cit., p. 97.

† Loc. cit., p. 176.

Proof. This is a corollary of § 10. For $\tau(a)$ has a value between k and $2k$, and hence there is an oscillatory set of A about a whose diameter is greater than or equal to k.

12. THEOREM. *Let A be a continuum, $a \varepsilon A$, and $\sigma(a) = \infty$. Then for every $G > 0$ there exists a continuum of condensation of A containing a and of diameter greater than or equal to G.*

Proof. Since $\sigma(a) = \infty$, A is not limited. Let Γ_r be a closed sphere of center a and radius G. Let $D = A \cdot \Gamma_r$.

Let D_a be that component of D containing a. In any $V_\delta(a)$ there are points z of D not lying on D_a. For otherwise $\sigma(a) \leqq \text{Diam } D_a \leqq 2G$. Let D_z be that component of D containing one of these points. It is easily seen that D_a and each D_z have points on Front Γ_r. Obviously $D_a \cdot D_z = 0$ for every z.

Since there is a z in every $V_\delta(a)$, there is a sequence $\{z_i\}$ converging to a. Let K be the aggregate of accumulation of $\{D_{z_i}\}$. It contains a and a point on Front Γ_r. Hence

$$\text{Diam } K \geqq G. \tag{1}$$

Since $K \cdot D_a$ contains a,

$$K \subseteq D_a. \tag{2}$$

But $D_a \cdot D_{z_i} = 0$; hence K is a continuum of condensation. As its diameter is greater than or equal to G by (1), the theorem is proved.

13. THEOREM. *Let A be a continuum, $a \varepsilon A$, and $\sigma(a) = k$ be finite and not zero. Then for any $\varepsilon > 0$ there exists a continuum of condensation of A of diameter not less than $k/2 - \varepsilon$ containing a and contained in an oscillatory set C of A about a, and having no point where the oscillation is zero.*

Proof. Let $\sigma = k/2 - \varepsilon/2$ and $\varrho = k/2 - \varepsilon$. Let Γ_σ and Γ_ϱ be closed spheres of center a and radii σ and ϱ respectively. Let A_a be that component of $A \cdot \Gamma_\sigma$ which contains a. For any $\delta > 0$ there are points z of $V_\delta(a)$ which do not lie on A_a. For otherwise $\sigma(a) \leqq \text{Diam } A_a \leqq k - \varepsilon$. Let A_z be that component of $A \cdot \Gamma_\sigma$ which contains z. It is obvious that A_a and each A_z contains one or more points on Front Γ_σ and that

$$A_a \cdot A_z = 0. \tag{1}$$

Since $\sigma(a)$ is finite, there is a monotone decreasing sequence of limited sub-continua $\{C_i\}$ of A, each C_i irreducible about a $V_{\delta_i}(a)$, $\delta_i \to 0$, and $C = Dv\,[C_i]$ is an oscillatory set of A about a of diameter not less than k. Let $C_{1,a}$ be that component of $C_1 \cdot \Gamma_\sigma$ containing a, and $C_{1,z}$ be that component containing z, where z is one of the points of the previous paragraph. Since Diam $C_1 \geqq k$, each of these components has points on Front Γ_σ. Also, since $C_1 \subseteq A$, we have

$$(2) \qquad C_{1,a} \subseteq A_a \quad \text{and} \quad C_{1,z} \subseteq A_z.$$

Now let $\{z_i\}$ be a sequence of the points z converging to a. Let K' be the aggregate of accumulation of the sets $\{C_{1,z_i}\}$. Then K' is a continuum containing a and a point on Front Γ_σ. Since $C_1 \cdot \Gamma_\sigma$ is closed, $K' \subseteq C_1 \cdot \Gamma_\sigma$. Since $K' \cdot C_{1,a} \supseteq a$, $K' \subseteq C_{1,a}$. But $C_{1,a} \cdot C_{1,z_i} = 0$. Hence K' is a continuum of condensation.

Now let K_1 be the saturated sub-continuum of K' contained in Γ_ϱ and containing a. It has a point on Front Γ_ϱ, but no points of $\Gamma_\sigma - \Gamma_\varrho$. Hence Diam $K_1 \geqq \varrho = k/2 - \varepsilon$. If $\sigma(x) = 0$ for some point of K_1, there is an $\eta > 0$ such that $V_\eta(x)$ lies in a sub-continuum $C(x)$ of A of diameter not greater than $\varepsilon/4$. Since $\sigma - \varrho = \varepsilon/2$, $C(x) \subseteq A \cdot \Gamma_\sigma$ aud hence

$$(3) \qquad V_\eta(x) \subset C(x) \subseteq A_a.$$

But each point of K_1 is a point of accumulation of the sets $\{C_{1,z_i}\}$ and hence every $V_\eta(x)$ contains a point of some C_{1,z_i}. Thus we have $A_a \cdot C_{1,z_i} \neq 0$, and $C_{1,z_i} \subseteq A_a$. This, however, contradicts relations (1) and (2). Hence $\sigma(x) \neq 0$ for every point in K_1.

Since $\delta_2 < \delta_1$ and $C_2 \subseteq C_1$, the same argument applied to C_2 will yield a continuum of condensation K_2, which is a part of K_1 and of $C_{2,a}$, contains a and a point of Front Γ_ϱ, and has no point for which $\sigma(x) = 0$.

Let this be continued and set K equal to the divisor of the monotone decreasing sequence $\{K_i\}$. Since every $K_i \subseteq C_i$, $K \subseteq C$. Since $\sigma(x) \neq 0$ in every K_i, $\sigma(x) \neq 0$ in K. Since every continuum K_i contains a and a point on Front Γ_ϱ, and is limited, the divisor K is a continuum of diameter not less than $k/2 - \varepsilon$. Since every point of K_1 is a limit point of points not in K_1 and $K \subseteq K_1$, this is also true of K, and therefore K is a continuum of condensation.

It should be noted that the proof can be considerably shortened, if we do not attempt to prove that the continuum of condensation is a part of C.

14. THEOREM. *Let A be a continuum, $a \varepsilon A$, and $\sigma(a) = \infty$. Then for any $G > 0$ there is a continuum of condensation of A containing a and of diameter not less than G, at each point of which $\sigma(x) > 0$.*

Proof. The theorem may be established by reasoning similar to that in § 13, by taking $\sigma = G$ and substituting for C_1 the set A itself.

15. Theorem. *Let A be a continuum, $a \epsilon A$, and $\tau(a) = k$, finite and not zero. Then a lies on a continuum of condensation K of A such that as $x \to a$ on K, $\overline{\lim}\, \tau(x) \leqq k$ and $\underline{\lim}\, \tau(x) \geqq k/2$.*

Proof. The first statement holds by virtue of § 5. To prove the second let K be the continuum of condensation obtained in § 13. Now if $\underline{\lim}\, \tau(x) < k/2$ as $x \to a$ on K, there is in every $V_\delta(a)$ a point x on K, such that for any ε sufficiently small and positive

$$(1) \qquad \tau(x) = \lambda < \frac{k}{2} - \varepsilon.$$

Now there is an $\eta > 0$ so small that $V_\eta(x)$ lies in a sub-continuum $C(x)$ of A of diameter not greater than $\lambda + \varepsilon/4$. If $z \epsilon C(x)$,

$$(2) \qquad \begin{aligned} \text{Dist }(a, z) &\leqq \text{Dist }(a, x) + \text{Dist }(x, z) \\ &\leqq \delta + \lambda + \frac{\varepsilon}{4} \\ &\leqq \lambda + \frac{\varepsilon}{2} \\ &\leqq \frac{k}{2} - \frac{\varepsilon}{2}, \end{aligned}$$

if $\delta \leqq \varepsilon/4$. This shows that $C(x)$ lies in the closed sphere Γ_σ of § 13, since $\sigma = k/2 - \varepsilon/2$. Hence $C(x) \subseteq A \cdot \Gamma_\sigma$, and therefore

$$(3) \qquad C(x) \subseteq A_a.$$

But, since $x \epsilon K$, every $V_\eta(x)$ contains a point of some C_{1,z_i}, which would give $A_a \cdot C_{1,z_i} \neq 0$, and as in § 13 we arrive immediately at the contradiction that $C_{1,z_i} \subseteq A_a$.

Hence the assumption that $\underline{\lim}\, \tau(a)$ can be less than $k/2$ is false.

The above theorem can at once be expressed in terms of $\sigma(x)$ by means of § 4. That $\underline{\lim}\, \tau(x)$ can equal $\frac{1}{2}\tau(a)$ is seen by considering the continuum defined as follows. For $x = 0$, y has all the values between ± 1; for $x \neq 0$, $y = \pm \sin^2(1/x)$ according as x is positive or negative. If we set $a = (0, 0)$, the segment of the y-axis between ± 1 is a continuum of condensation containing a and $\tau(a) = 2$, but as $z \to a$ on this continuum of condensation, $\overline{\lim}\, \tau(z) = 1$.

16. The example just given brings out clearly the analogy between the oscillation of a continuum and the oscillation of certain functions of a real variable. If in the definition of the continuum we set $y = 0$ when $x = 0$,

then $y = f(x)$ is a one-valued function, continuous except at $x = 0$. At points of continuity the oscillation of the function is 0; at the origin it is 2. At points of the continuum where $x \neq 0$, $\tau(x) = 0$; at the point $(0,0)$ $\tau(x) = 2$. The oscillatory set at the point $(0,0)$ corresponds to the aggregate of limiting values of $f(x)$ as $x \rightarrow 0$.

If we consider the oscillatory set as a tool in studying continua (as the oscillation of a function is used in studying discontinuous functions), the question at once arises as to whether it possesses any merits differentiating it from continua of condensation. Two are evident from the contents of this paper. First, the diameter of an oscillatory set is at least as great as the oscillation $\sigma(x)$ or $\tau(x)$; this is not in general true for continua of condensation, as can be shown by an example. Second, for such a continuum as the unit square the oscillation at every point is zero and the oscillatory set is the point itself, while any linear continuum containing the point is a continuum of condensation. The obvious defect is that, like the continuum of condensation, the oscillatory set at a point is not unique. This is to be expected on account of the generality of the continua considered here. With the proper restrictions this difficulty should disappear. For example, when a continuum is irreducible between two points and limited, the oscillatory set is unique, as will be shown in a paper at present under preparation.

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