SOLUTION OF CERTAIN FUNCTIONAL EQUATIONS RELATIVE TO A GENERAL LINEAR SET*

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I. INTRODUCTION

This paper exhibits the solution of the functional equation

$$(A) \quad \sum_i^{0, n+1} (-1)_{n+1}^i C_i f(x_0 + jx_1) = 0$$

for every x_0 and x_1 belonging to a linear set of a certain type. As is well known, relative to the complex, real or rational number system any polynomial of degree n satisfies (A). Equation (A) simply states that the $(n+1)$ st difference of the function f vanishes for every set of values of the argument with a constant difference.

Both the equation studied and the method pursued are generalizations of the equation and method of Legendre,† Hamel‡ and Schimmack§ in their studies of the equation

$$f(x+y) = f(x) + f(y) .$$

It is obvious that any solution of (1) is also a solution of (A) for the case in which $n=1$.

Hamel shows that if we assume that the system $\mathfrak{R}$ of real numbers can be normally ordered (well ordered) then a set $\mathfrak{R}_0$ of real numbers exists, such that all elements of $\mathfrak{R}_0$ are linearly independent as to finite combinations with rational coefficients and such that every real number is a linear combination of a finite number of elements of $\mathfrak{R}_0$ with rational coefficients.

Consider $\mathfrak{R}_0$ as a range and V the set of all vectors with rational coördinates of which, for any one vector, only a finite number are different from zero. It is obvious that the elements of $\mathfrak{R}$ may be set into one to one correspondence with the elements of V in such a way that the correspondence is preserved under addition and multiplication by rational numbers.

Hamel shows that any solution of (1) is a linear combination of the coördinates of the vectors of V . This paper shows that as a direct generaliza-

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† A. M. Legendre (not verified by direct reference) *Eléments de Géométrie*, 4th edition, Paris, 1802, p. 279.

‡ Hamel, *Mathematische Annalen*, vol. 60 (1905), p. 459.

§ R. Schimmack, *Axiomatische Untersuchungen über die Vektoraddition*, Nova Acta, Halle, 1908.

III. LINEAR SYSTEMS AND TYPES OF FUNCTIONS

In light of the results of the last section we will confine our attention to a system

$$(R, Q, V, F)$$

in which the elements are defined as follows.

R is either the rational number system with modulus ω or a field of integers modulo a prime p .

Q is a range; when Q is the range $1, 2, 3, \dots, m$ we say that Q^m .

$V \equiv$ [all v on Q to R finitely non-zero], i. e., all functions (vectors) on Q to R such that there are only a finite number of elements of Q for which the corresponding functional values are $\neq 0$. Obviously if Q is finite $V =$ [all v on Q to R].

$F \equiv$ [all f on V to V], i. e., all functions on V to V .

The remainder of this paper studies the solutions of (A) when the x 's are replaced by the v 's. By ${}_nC_i v$ we shall mean $v + v + v + \dots$ (${}_nC_i$ terms) and by $(-1)^n$ the element 1 of R if n is even and -1 of R if n is odd.

A function f is of type A^k if f satisfies (A) when $n = k$ but does not satisfy (A) if $n < k$.

We proceed to define a class of functions in F which are analogous to polynomials.

Consider $H \equiv$ [all functions h , finitely non-zero, on Q to the class of integers $1, 2, 3, \dots$]. We define the weight of any such h ,

$$w(h) \equiv \sum_q h(q) .$$

We define, letting $0^0 = 1$,

$$v^h = \prod_q (v(q))^{h(q)} \quad (v, h) . *$$

This notation is of decided convenience due to the many theorems which it suggests, for instance

$$(i) \quad v.) . v^{0_H} = 1 .$$

Here by 0_H we mean the function in H which is identically zero.

$$(ii) \quad h_1 h_2 v.) . v^{h_1} v^{h_2} = v^{h_1 + h_2} .$$

$$(iii) \quad r v h .) . (rv)^h = r^{w(h)} v^h .$$

Define

$${}_{h_1} C_h \equiv \prod_q {}_{h_1(q)} C_{h(q)} (h_1, h) ;$$

* This notation is essentially the same as that used by E. H. Moore in his paper entitled *On power series in general analysis*, *Mathematische Annalen*, vol. 86, p. 32 ff.

then

$$h_1 v_1 v_2 \dots (v_1 + v_2)^{h_1} = \sum_h h_1 C_h v^h v^{h_1 - h}.$$

The symbol $h_1 C_h$ may be given a combinatory interpretation. Consider a class of $h_1(q)$ objects for each element of Q ; then the number of ways $h(q)$ of these elements can be selected from each of the corresponding classes is the number $h_1 C_h$, of course, using the ordinary conventions that

$$n \dots n C_0 = 1 \text{ and } n_1 > n_2 \dots n_2 C_{n_1} = 0.$$

Clearly

$$w(h) > w(h_1) \dots h_1 C_h = 0$$

and

$$h \dots h C_h = 1.$$

We say that a function is of type P^n if it is of the form

$$\sum_h v_h v^h (v)$$

and such that

- (i) $\exists h : \exists w(h) = n \cdot v_h \neq 0_V,$
- (ii) $w(h) > n \dots v_h = 0_V.$

Definition of $m(h)$:

$$m(h) = r : \equiv r = \text{maximum value of } (h(q) (q)).$$

We say that f is of type P_n if it is a function of the form

$$\sum_h v_h v^h (v)$$

and such that

- (i) $\exists h : \exists m(h) = n \cdot v_h \neq 0_V,$
- (ii) $m(h) > n \dots v_h = 0_V.$

Definition of a linear m -fold:

V_0 is a linear m -fold : $\equiv \exists v_{00}, (v_1, \dots, v_m)$ linearly independent

$$\therefore \exists \dots V_0 \supset v : \sim \exists r_1 \dots r_m$$

$$\therefore \exists v = v_{00} + \sum_i^{1..m} v_i r_i.$$

In such a case we say that $V_0 = \{v_{00}; v_1, \dots, v_m\}$.

THEOREM 1. *If r belongs to R and v_1, v_2 , and v_3 belong to a linear m -fold V_0 , then $v_1 + r(v_2 - v_3)$ belongs to V_0 .*

The proof is obvious.

THEOREM 2 (the converse of Theorem 1). *If for every r of R and v_1, v_2 of a set V_0 , $v_1 + r(v_1 - v_2)$ belongs to V_0 and V_0 is contained in a linear m -fold, then V_0 is a linear m_1 -fold where $m_1 \leq m$.*

Proof. Any element of V_0 is effective as v_{00} and the $v_1, \dots, v_m$ of the definition may be readily obtained by the use of the hypothesis. Call the greatest common subset of two sets of vectors their intersection.

THEOREM 3. *The intersection, if it exists, of a linear m_1 -fold and a linear m_2 -fold is a linear m_3 -fold where m_3 is equal to or less than the lesser of m_1 and m_2 ; or it is a vector v .*

In light of Theorems 1 and 2 the proof is obvious.

Definition: A linear m_1 -fold, V_1 , and a linear m_2 -fold, V_2 , $m_1 \geq m_2$, are said to be parallel provided the difference of every two vectors of V_2 is the difference of two vectors of V_1 .

In this case either $V_1 \supset V_2$ or V_1 and V_2 have no intersection.

THEOREM 4.* *If Q is the finite range $(1, 2, 3, \dots, m+1)$, if V_0 is a linear m -fold contained in the linear $(m+1)$ -fold $\{0_V; \delta_1, \dots, \delta_{m+1}\}$ (where δ_i is the function f for which $f(i) = 1$ and $f(j) = 0 (i \neq j)$) then there exists uniquely a k and $r_0, \dots, r_{k-1}$ such that $V_0 = \{v_0; v_1, \dots, v_m\}$ where*

$$\begin{aligned} v_0 &= r_0 \delta_k, \\ v_i &= \delta_i + r_i \delta_k & (j < k), \\ v_j &= \delta_{j+1} & (j \geq k). \end{aligned}$$

THEOREM 5. *Relative to R with a modulus p , in every linear m -fold there exist $(p^{m+1} - p)/(p - 1)$ distinct linear $(m - 1)$ -folds.*

Proof. For each determination of k in Theorem 4 there are p^k possible sets of values for the numbers r_0 to r_{k-1} . Hence the number of linear $(m - 1)$ -folds contained in the linear m -fold $\{0_V; \delta_1, \dots, \delta_m\}$ is $\sum_{k=1}^m p^k = (p^{m+1} - p)/(p - 1)$. Clearly in light of the definition of a linear m -fold and Theorems 1 and 2, this carries with it the theorem for the general case.

One should note here that relative to the matrix (r_{ij}) where i and j run from 1 to n , there exists for every set of vectors $v_j (j = 1, \dots, n)$ of V a set of vectors $v'_i (i = 1, \dots, n)$ such that

$$(1) \quad \sum_i^{1..n} v'_i r_{ij} = v_j \quad (j)$$

* *A general theory of linear sets*, loc. cit., see p. 185 ff.

if and only if the determinant $||r_{ij}||$ does not vanish. This follows at once from the analogous classical theorem, since condition (1) may be written

$$\sum_i^{1,n} v'_i(q) r_{ij} = v_j(q) \quad (j, q),$$

which are purely numerical conditions and for every q the existence of the set $[v'_i(q)]$ ($i=1, \dots, n$) for every set $[v_j(q)]$ ($j=1, \dots, n$) is dependent on the non-vanishing of $||r_{ij}||$.

IV. THE SOLUTION OF (A). PART A

THEOREM 1. *If f is of type P^n then there exists a $k \leq n$ such that f is of type A^k .*

Proof. Clearly the theorem follows from the proof of it for the special case in which f is of the form $(v_2 v^h(v))$, where $w(h)=n$. Clearly for any v , the first difference

$$\begin{aligned} v_2(v+v_1)^h - v_2 v^h \\ = v_2 \sum_{h_1}^{h_1+h} C_{h_1} v^{h_1} v^{h-h_1} \quad (v) \end{aligned}$$

is of the form $\sum_h v_h v^h(v)$, where $v_h \neq 0_{\nu}$, $w(h) < n$. Hence the theorem is true for $n=n_1$ if it is true for $n=n_1-1$. However we know that it holds for the case $n=0$, since in that case f is a constant and the first difference vanishes. Hence by induction the proof is complete.

THEOREM 2. *If f_1 and f_2 are each of the type A^k , $k \leq n$, and for $n+1$ vectors of a linear 1-fold $V_0 = \{v_{00}; v_1\}$ the functional values of f_1 are equal to the functional values of f_2 , then, as on V_0 , f_1 is equal to f_2 .*

Proof. Clearly since the $(n+1)$ st difference vanishes Newton's interpolation formula $f(v_0 + lrv_1) = v_0 + l\Delta_1 + \frac{1}{2}l(l-1)\Delta_2 + \dots$ holds for any set of values of the form $v_0 + lrv_1$ for every integer l and any fixed value of r in R . Hence as on such a set f_1 is of the type P^{k_1} and f_2 is of the type P^{k_2} , where both k_1 and k_2 are equal to or less than n . Hence if $p < n+1$ the theorem is vacuously fulfilled. If $p = n+1$ the theorem is obvious. If $p > n+1$ or R has a modulus ω the well known theorem that a polynomial of degree equal to or less than n is determined by its values for $n+1$ values of the argument carries with it our theorem since both are equivalent to the non-vanishing of the same determinant.

PART B. Q FINITE AND R WITH A MODULUS p

In this section we will consider Q the finite range $1, 2, \dots, m$ and R with a modulus p .

In this section we order the elements of R linearly rather than cyclically. Thus $0 < 1 < 2 \cdots < p-1$.

THEOREM 1. *If two functions f_1 and f_2 are each of type P_{k_1} and P_{k_2} respectively ($k_1 \leq k_2 \leq n < p$), and $V_1 = [\text{all vectors of the form } \sum_{i=1}^m r_i \delta_i (r_i \leq n(i))]$ and, as on V_1 , $f_1 = f_2$, then $f_1 = f_2$ and corresponding coefficients are equal.*

Proof. This theorem is, of course, equivalent to the theorem that if f is of type P_k ($k \leq n < p$) and if f as on V_1 is identically 0 then f is identically 0_V and all the coefficients are 0_V .

Consider the coefficients of the polynomial in r_1 which are themselves polynomials in r_2 to r_m with coefficients in V which are of degree equal to or less than k in r_2 to r_m and vanish for all values of r_2 to r_m less than or equal to n , since the polynomial vanishes for $n+1$ values of r_1 whenever r_2 to r_m are all equal to or less than n ; hence we have a reduction process and every coefficient must be 0_V .

THEOREM 2. *If $V_1 = [\text{all vectors of the form } \sum_{i=1}^m r_i \delta_i (r_i \leq n(i))]$ where $n < p$, and g is a function on V_1 to V , then there exists one and only one function f of type P_k ($k \leq n$) such that f (as on V_1) = g .*

Proof. There are $p^{m(n+1)^m}$ distinct functions of type P_k ($k \leq n$) and since by Theorem 1 these functions as on V_1 are distinct they must as on V_1 be the $p^{m(n+1)^m}$ distinct functions on V_1 to V .

Hence every function f on V to V is of type P_k where $k < p$. Every function is, therefore, of type P^l where $l \leq (p-1)m$.

THEOREM 3. *If f is of type A^k ($k < p-1$) then f is of type P^k and conversely.*

Proof. f is of type P_l where $l < p$, i. e. f is of the form $\sum_h v_h v^h$ where $m(h) > p-1$. $v_h = 0_V$. As on any linear 1-fold with typical element $v_0 + r v_1$, Newton's interpolation formula shows that f is a polynomial in r with coefficients in V of degree less than or equal to k . By substitution of $v = v_0 + r v_1$ it is seen that

$$\begin{aligned} f &= \sum_h v_h (v_0 + r v_1)^h \\ &= \sum_h v_h \sum_{h_1} {}_h C_{h_1} v_0^{h-h_1} r^{(h_1)} v_1^{h_1} \\ &= \sum_h v_h \sum_{k_1}^{w(h_1) \equiv k_1 (p-1)} \sum_{h_1} {}_h C_{h_1} v_0^{h-h_1} r^{w(h_1)} v_1^{h_1}. \end{aligned}$$

Thus, since for any integer α , $r^\alpha = r^{\alpha+p-1}$,

$$p > k_1 > k \therefore v_0 \cdot v_1 : \sum_h \sum_{h_1}^{w(h_1) \equiv (p-1)} v_{hh} {}_h C_{h_1} v_0^{h-h_1} v_1^{h_1} = 0_V.$$

Hence, since this holds for all v_1 by Theorem 1,

$$p > k_1 > k \therefore v_0 \cdot w(h_1) \equiv k_1(p-1) \therefore \sum_h v_{hh} C_{h_1} v_0^{h-h_1} = 0_V.$$

But this holds for every v_0 , so

$$h \cdot p > k_1 > k \therefore w(h_1) \equiv k_1(p-1) \therefore v_{hh} C_{h_1} = 0_V.$$

Moreover,

$$p > k_1 \cdot h \therefore m(h) < p \cdot w(h) \geq k_1 \therefore \exists h_1 \therefore w(h_1) \equiv k_1(p-1) \cdot h C_{h_1} \neq 0(p).$$

Hence,

$$p-1 > k \cdot w(h) > k \therefore v_h = 0_V.$$

Therefore, if f is of type A^k it is of type P^l where $l \leq k$, but, if $l < k$, f would be of type A^{l_1} ($l_1 \leq l$) (see Part A, Theorem 1) which is in contradiction to our hypothesis and, hence, f is of type P^k . The converse follows at once in light of Theorem 1 of Part A.

PART C. Q FINITE AND R RATIONAL

In this part we will consider Q a finite range and R isomorphic with the field of all rational numbers.

In order to pass from the case where R has a modulus p to the case in which R is isomorphic with the field of rational numbers one should note the following theorem.

THEOREM 1. *Relative to an m by m non-singular matrix $u = (r_{ij})$ whose elements are rational integers, and a set of integers c_i ($i=1, \dots, m$) there exists a proportionality factor λ and an integer n such that the solution of the equations*

$$\sum_i r_{ij} x_j = \lambda c_i \quad (i)$$

is the reduced solution of the congruences

$$\sum_i r_{ij} x_j \equiv \lambda c_i(p) \quad (i)$$

for every $p > n$.

By reduced solution is meant the solution for which each x is in absolute value equal to or less than $p/2$.

Proof. The system of linear equations

$$(1) \quad \sum_i r_{ij} x_j = c_i ||\mu|| \quad (i)$$

may be reduced by multiplication by integers (cofactors of the elements of $||\mu||$) and addition to the system

$$||\mu||x_j = \Delta_j ||\mu|| \quad (j=1, 2, \dots, m)$$

where Δ_j is the determinant of the matrix secured by replacing the j th column of μ by the c 's. If n is greater than twice the absolute value of any of the multipliers or the coefficients of the unknowns in the equations of this reduction process, then for $p > n$ the reduction in the sense of a congruence modulo p would be formally identical. Since n is chosen greater in absolute value than $2||\mu||$, $2\Delta_j$ and $2\Delta_j||\mu||$, for every j , the reduced solution of the congruence

$$||\mu||x_j \equiv \Delta_j ||\mu|| (p) \text{ where } p > n$$

is $x_j = \Delta_j$.

Hence $||\mu||$ is effective as λ .

THEOREM 2. *If on $V_1 = [\text{all vectors of the form } \sum_{i=1}^m r_i \delta_i (0 \leq r_i \leq n \text{ and integers } i)]$ two functions f_1 and f_2 of type $A^k (k \leq n)$ are equal, the two functions are equal.*

Proof. This can be readily arrived at by repeated application of Newton's interpolation formula. Obviously if f_1 is determined on V_1 it is determined for all vectors such that $v(i)$ is an integer or 0, less than or equal to n if $i \neq 1$ and unrestricted as to the value of $v(1)$. By another application the restriction on $v(2)$ may be removed, etc.

Moreover, from consideration of Newton's interpolation formula, if, as on V_1 , f_1 has integral values for its coördinates, it has integral values for its coördinates for all P whose coördinates are integers.

THEOREM 3. *If g is of type $A^k (k \leq n)$ then there exists one and only one function f of type $P_l (l \leq n)$ such that f (as on V_1) = g (as on V_1).*

Proof. This theorem holds provided the $(n+1)^m$ by $(n+1)^m$ matrix of the system of equations for the coefficients of the terms of f

$$(1) \quad \sum_h v_h v^h = g(v) \quad (v \text{ in } V_1),$$

summation extending over all h such that $m(h) \leq n$, is non-singular.

But this matrix reduced modulo $p \geq n$ is non-singular since Theorem 2, Part B, holds.

Since g is of type $A^k (k \leq n)$, there exists a λ such that $g_1 \equiv \lambda g$ has integral coördinates for all vectors v whose coördinates are integers. If for all vectors v with integral coördinates we reduce the coördinates of $g_1(v)$ modulo p ($p > n$) we have a new function g'_1 (on all v with integral coördinates to all v with integral coördinates).

Moreover, g'_1 is of type A^{k_1} ($k_1 \leq k$) and has the property that for every v_1 and v_2 with integral coördinates $g'_1(v_1 + pv_2) = g'_1(v_1)$. Hence g'_1 can be used to define a function of type A^{k_1} in the case where R has a modulus p . Moreover, if p is chosen sufficiently large, $k_1 = k$, since there will exist some values of v_0 and v_1 relative to which equation (A) holds for $n = k$ but not for $n = k - 1$ and if p is sufficiently large the same will be true for g'_1 .

Hence by Theorem 1 and Theorems 2, 3, Part B, if p is sufficiently large the solution of equation (1) with g replaced by g'_1 yields a function of type P^k and is the same as the solution of (1) with g replaced by g_1 . But g_1 differs from g only by a constant factor. Hence f is of the type P^k .

But by Theorem 1, Part A, f is of type A^{k_2} ($k_2 \leq k$) and therefore by Theorem 2 of this part $f = g$. Hence we may conclude

THEOREM 4. *If V is the set of all vectors on a finite range Q to R then any function of type A^n is of type P^n if R is isomorphic with the field of all rational numbers or has a modulus $p > n + 1$. If $p = n + 1$ then f is of type P^k where $k \geq n + 1$ and P_l where $l < p$. There are no functions of type A^k where $k \geq p$.*

Note: Another proof of the above theorem for the case in which R is rational exhibits a set of vectors such that for every corresponding set of values there exists one and only one function of type A^k ($k \leq n$) having for these vectors the assigned functional values. This proof is inserted here for the added insight it gives into the general problem of setting up solutions of equation (A) although as a proof of the above theorem it is much longer than the first one given.

Let ${}_nE_m$ be the number of ways that n or any integer less than n may be expressed as the sum of m integers or zeros, regarding each change of order as a new way. ${}_nE_m$ is also the number of distinct vectors h on the range 1, 2, 3, $\dots$, m to the range 0, 1, 2, 3, $\dots$ such that $w(h) \leq n$ and hence is the maximal number of non-zero terms in a polynomial of degree n in S_m , that is, the maximum number of non-zero terms in a function of type P^n when Q^m .

It is readily shown that

$${}_nE_m = \sum_i^{0,n} {}_iE_{m-1}.$$

Consider a set E of ${}_nE_m$ vectors arranged in $n + 1$ linear $(m - 1)$ -folds $S_0, S_1, \dots, S_n$, such that every pair S_i, S_j ($i \neq j$) intersect in a linear $(m - 2)$ -fold and these linear $(m - 2)$ -folds are pair by pair non-parallel, and such that no three intersect in the same $(m - 2)$ -fold. Let S_0 contain a set E_0 of ${}_nE_{m-1}$ vectors of E . Let S_1 contain a set E_1 of ${}_{n-1}E_{m-1}$ vectors of E not

in S_0 . Let S_2 contain a set E_2 of ${}_{n-2}E_{m-1}$ vectors of E not in S_0 or S_1 , etc. Moreover, let the ${}_{n-i}E_{m-1}$ vectors of E_i be arranged in $n-i+1$ linear $(m-2)$ -folds $S_{i0}, \dots, S_{i, n-i}$ contained in S_i and not parallel to the intersection of any S_j ($j < i$) with S_i and such that pair by pair they intersect in non-parallel linear $(m-3)$ -folds and such that S_{i0} contains a set E_{i0} of ${}_{n-i}E_{m-2}$ vectors of E_i , S_{i1} contains a set E_{i1} of ${}_{n-i-1}E_{m-2}$ vectors of E_i not in S_{i0} , etc. Continue this mode of subdivision until all vectors of E are distributed in linear 1-folds.

The theorem that relative to any function g on E to V there exists one and only one function f of type P^k ($k \leq n$) such that, as on E , $f = g$, is equivalent to the non-vanishing of the determinant D of the matrix $\mu = (r_{vh})$ where $r_{vh} = v^h$ and v runs through the ${}_nE_m$ vectors of E while h runs through the ${}_nE_m$ vectors on the range 1, 2, 3, $\dots$, m to the integers and 0 such that $w(h) \leq n$.

Clearly by a reversible transformation S_0 may be taken as the linear $(m-1)$ -fold $\{0_V; \delta_1, \dots, \delta_{m-1}\}$. Hence, the minor corresponding to E_0 and the set H_0 of H such that, for any vector h_0 of H_0 , $h_0(m) \neq 0$, is composed of zero elements only. Hence D is the product of the determinant D_1 of the minor μ_1 of μ which has elements r_{vh} such that $v \subset E_0$ and $h(m) = 0$ and the determinant D_2 of the minor μ_2 which has elements r_{vh} such that v is not contained in E_0 and $h(m) \neq 0$.

The non-vanishing of D_1 is the condition that the theorem holds for $Q_1 = 1, 2, \dots, m-1$.

$D_2 = (\prod_v v(m))D_3$, where the product extends over all v in E not in E_0 and hence not in S_0 and hence is different from zero, whereas the non-vanishing of D_3 is equivalent to our theorem for the case of order $n-1$.

Hence by a finite number of steps we can reduce our theorem to the two special cases

- (1) the case in which Q is singular;
- (2) the case in which $n = 0$.

In the first case the theorem is equivalent to the fact that a polynomial of degree less than n is determined by its value for $n+1$ values of the argument. In the second case f is a constant and the theorem is obvious.

If this work is to lead to an independent proof of Theorem 3 we should note that if two functions f_1 and f_2 are of type A^k ($k \leq n$), and, as on $S_0, \dots, S_n$, $f_1 = f_2$, then $f_1 = f_2$. For consider any vector v not in any of the sets $S_0, \dots, S_n$; then there exists a linear 1-fold V_0 containing v and $n+1$ distinct vectors $v_0, \dots, v_n$ such that $S_i \supset v_i$. Hence, by Theorem 2, as on V_0 , $f_1 = f_2$ and hence $f_1(v) = f_2(v)$ for every v , i. e., $f_1 = f_2$. However, the determination of a function of type A^k ($k \leq n$) on S_0 depends, in a similar manner, on the determination

of the function on $S_{00}, \dots, S_{0n}$, the determination of the function on S_1 depends on its determination on $S_{10}, \dots, S_{1n-1}$ and on the intersection of S_0 and S_1 ; its determination on S_2 depends on its determination on $S_{20}, \dots, S_{2n-2}$ and the intersection of S_2 with S_1 and S_0 ; hence, we reduce the case for each linear m -fold considered to the case for $n+1$ linear $(m-1)$ -folds and finally to the vectors of E . Hence any function of type A^k ($k \leq n$) must be identical with the function of type P^k having the same functional values on E .

This proof does not hold in general for the case where R has a modulus p a prime, as can be readily seen from the Theorem 5 of § III.

PART D. Q GENERAL

In this part we consider Q as a general range.

We shall show that any solution of (A) is of the form

$$\sum_h v_h v^h.$$

For any h_1 consider the set $Q_{h_1} = [\text{all } q. \ni. h_1(q) \neq 0]$; Q_{h_1} is finite. Consider Q_1 and Q_2 , finite subsets of Q such that $Q_1 \supset Q_2 \supset Q_{h_1}$. By the theorem of the last section, f as on Q_1 is of the form

$$\sum_h v_{1h} v^h,$$

and as on Q_2 is of the form

$$\sum_h v_{2h} v^h.$$

From the unique determination of the v 's as shown in the above sections it is readily seen that v_{1h_1} (as on Q_2) = v_{2h_1} , since v_{1h_1} as on Q_2 must be completely determined by the value of f for v 's for which $v(q) \neq 0$. $Q_2 \supset q$. Let the set Q_2 swell.* It is readily seen that a v_{h_1} is determined uniquely as a limit. Consider, using the above determined v 's, the function

$$\sum_h v_h v^h;$$

we readily see that it is identical with f , since for any v all terms except those for which $Q_h \subset [\text{all } q. \ni. v(q) \neq 0]$ vanish and, for any such h , v_h as on Q_v is the same on any finite range containing Q_v as the coefficient we

* The consideration of this kind of limit has been given by E. H. Moore in his courses in General Analysis and a theory of limits including this as a special case has been discussed by E. H. Moore and H. L. Smith in their joint paper entitled *A general theory of limits*, American Journal of Mathematics, vol. 44 (1922).

where r is rational, then any continuous solution will be a polynomial as on $\{v_0; v_1\}$.

Moreover, in considering the case in which Q^n it is readily seen that any continuous solution is determined if the solution is determined for all v such that the functional values of v are rational. Hence, one may readily see from the results of the preceding sections

THEOREM 1. *Any continuous function of type A^n is a function of type P^n (where the terms are now understood to be relative to the $Q, \mathfrak{R}, V$ system).*

To feel the force of the last theorem the following three corollaries are worth noting.

COROLLARY 1. *If $\mathfrak{A}$ is the real number system any continuous function of type A^n is a polynomial of degree n .*

This, of course, is obviously provable without reference to the theorem.

COROLLARY 2. *If $\mathfrak{A}$ is the complex number system any continuous function of type A^n is a polynomial of degree n in the coördinates x and y of $z = x + iy$, the coefficients of which are complex numbers.*

COROLLARY 3. *If $\mathfrak{A}$ is the system of real quaternions any continuous function of type A^n is a polynomial of degree n in the coördinates x, y, z , and w of $q = x + iy + jz + kw$, in which the coefficients are quaternions.*

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