ON A CLASS OF POLYNOMIALS IN THE THEORY OF BESSEL'S FUNCTIONS*

BY

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1. Those values of z for which Bessel's function

$$J_n(z) = \left(\frac{z}{2}\right)^n \left\{ \frac{1}{\Gamma(n+1)\Gamma(1)} - \frac{\left(\frac{z}{2}\right)^2}{\Gamma(n+2)\Gamma(2)} + \dots \right\}$$

vanishes are known to be infinite in number for a general value of n . Their importance in mathematical physics has led to their calculation for positive integral values of n . They may also be regarded as branches of an infinitely many branched function of a complex variable. From this point of view they have been studied but little, and even for the case of real values of n few results are known. For example, it has been proved that if $n > -1$ the roots are all real, and that each root is an increasing function of n if $n > 0$.

From the recursion relations between the functions $J_n(z)$ for consecutive values of n a set of polynomials is derived which has been considered incidentally in the theory. Hurwitz† made a systematic study of these polynomials regarded as functions of z . These polynomials are also polynomials in n but have not been examined as such. In what follows some properties showing their character as functions of n are deduced, and it is found that these properties are susceptible of some applications. In particular, it is shown that the roots of $J_n(z) = 0$ are increasing functions of n when n lies between -1 and 0 , a result that cannot be deduced from Poisson's integral formula employed by Schlöfli in deriving the similar conclusion for $n > 0$.

2. In this paper the notation and results of Hurwitz are used and the following extract gives what is essential for its comprehension.

If

$$f_n(z) = \frac{1}{\Gamma(n+1)\Gamma(1)} + \frac{z}{\Gamma(n+2)\Gamma(2)} + \dots$$

then

$$J_n(z) = \left(\frac{z}{2}\right)^n f_n\left(-\frac{z^2}{4}\right).$$

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† A. Hurwitz, *Mathematische Annalen*, vol. 33 (1889), p. 246.

Between the functions f_n subsists the relation $f_n = g_n f_{n+p} + z h_n f_{n+p+1}$ where g_p and h_p are polynomials in z and n . It is found that $h_{p+1} = g_p$ and $g_{p+1} = (n+p+1)g_p + z g_{p-1}$. It is also found that

$$\frac{g_p}{\Gamma(n+p+1)} = f_n \left[1 + \frac{z}{-n-p} + \dots \right] + \frac{(-1)^{p+1}\pi}{\sin(n+1)\pi} \cdot f_{-n} \\ \frac{z^{p+1}}{\Gamma(n+p+1)\Gamma(n+p+2)} \left(1 + \frac{z}{n+p+2} + \dots \right),$$

$n \neq \text{integer}$.

In any region of the plane of z , $f_n(z)$ is the uniform limit of $g_p(z)/\Gamma(n+p+1)$. In accordance with a more general theorem proved by Hurwitz the roots of $f_n(z) = 0$ are the values given by the derived set of the set of roots of $g_p(z) = 0$.

If Δ_p denotes the Jacobian

$$g_p \frac{dg_{p+1}}{dz} - g_{p+1} \frac{dg_p}{dz}$$

there is the recurrence relation

$$\Delta_{p+2} = (n+p+2)g_{p+1}^2 + z^2 \Delta_p$$

and Δ_p is found to be equal to

$$\left(\frac{\pi}{\sin(n+1)\pi} \right)^2 \left(\frac{1}{\Gamma(-n-p-1)} \right)^2 \left[\frac{1}{n+p+1} + \dots \right].$$

If $f_n(z) = 0$, $\Delta_p = (-z)^{p-1} + \dots$ where the terms not written tend to zero as p tends to infinity. From these equations, making use of the Heine-Borel theorem, it follows, if z is restricted to a given segment of the real axis, that m may be chosen so great that when $p > m$, $\Delta_{2p-1} > 0$. If $n > -1$ the roots of the equation $g_p = 0$ are all real and $\Delta_p > 0$ on any segment of the negative part of the real axis.

3. The functions g_i are polynomials in n as well as z , and the consideration of them as such leads to a number of new results. Let

$$D_p = g_p \frac{dg_{p+1}}{dn} - g_{p+1} \frac{dg_p}{dn};$$

then

$$D_p = g_p^2 - z D_{p-1}.$$

Differentiating $g_{p+1} = (n+p+1)g_p + z g_{p-1}$ with respect to n gives

$$\frac{dg_{p+1}}{dn} = g_p + (n+p+1) \frac{dg_p}{dn} + z \frac{dg_{p-1}}{dn}$$

and combining with

$$\frac{dg_p}{dn} = \frac{dg_p}{dn}$$

it follows that

$$g_p \frac{dg_{p+1}}{dn} - g_{p+1} \frac{dg_p}{dn} = g_p^2 - z \left(g_{p-1} \frac{dg_p}{dn} - g_p \frac{dg_{p-1}}{dn} \right)$$

or $D_p = g_p^2 - zD_{p-1}$, which is the required relation.

Since $D_0 = 1$, $D_1 = (n+1)^2 - z$ it follows that if a negative value is assigned to z , $D_p = g_p^2 + |z|D_{p-1} > 0$ so that for every value of n , $D_p > 0$ and g_{p+1}/g_p is an increasing function of n .

From this property of g_{p+1}/g_p it follows, if z_1 is a negative root of $g_{p+1} = 0$, and p is sufficiently large, that $g_p(z_1) > 0$ or < 0 according as g_{p+1} changes from negative to positive or the reverse when z increases through z_1 . For by the results of Hurwitz if $z_1 < 0$ it may be shown that $\Delta_p > 0$ for the value z_1 of z and, p being sufficiently large, g_{p+1}/g_p is an increasing function of z . Writing $g_{p+1} = g_{p+1}(z, n)$ and taking n' a value slightly greater than n , it follows that $g_{p+1}(z_1, n') > 0$ in the first case and < 0 in the second. This is evident because in both cases

$$\frac{g_{p+1}(z_1, n')}{g_p(z_1, n')} > \frac{g_{p+1}(z_1, n)}{g_p(z_1, n)} = 0,$$

since $z_1 < 0$ and $g_{p+1}(z_1)/g_p(z_1)$ is an increasing function of n . If z'_1 denotes the root of $g_{p+1}(z, n') = 0$ which differs slightly from z_1 it follows in both cases that $z'_1 < z_1$.

According to a general theorem of Hurwitz the roots of $f_n = 0$ are the limits of the roots of the polynomials $g_{p+1}(z, n) = 0$. It follows from the preceding discussion, if r_k denotes the k th root of $f_n = 0$ and r'_k the k th root of $f_{n'} = 0$, that $r'_k \leq r_k$. The equality may be excluded since the functions f_n and $f_{n'}$ cannot have a common root for a range of values of n .

Taking account of the relation between f_n and J_n , the inequality $r'_k < r_k$ gives the following theorem: *The absolute values of the real roots of $J_n = 0$ are increasing functions of n , whatever real value n may have. In particular if $n > -1$ the roots of $J_n = 0$ are all real and increase in absolute value with n .* It seems that this property of the roots cannot be deduced from the more familiar methods of treating such a question; these methods are, in fact, restricted to the range $n > 0$.

4. Another property of the functions g_p is found as follows. Differentiate the equation

$$f_n = g_p(z, n)f_{n+p} + zg_{p-1}(z, n)f_{n+p+1}$$

with respect to z and replace f_{n+p} by $(n+p+1)f_{n+p+1}+zf_{n+p+2}$. The result is

$$f_{n+1} = g_p(z, n)f_{n+p+1} + zg_{p-1}(z, n)f_{n+p+2} + g_{p-1}(z, n)f_{n+p+1} \\ + z \frac{dg_{p-1}(z, n)}{dz} f_{n+p+1} + \frac{dg_p(z, n)}{dz} [(n+p+1)f_{n+p+1} + zf_{n+p+2}] .$$

Since $f_{n+1} = g_p(z, n+1)f_{n+p+1} + zg_{p-1}(z, n+1)f_{n+p+2}$ it follows that

$$g_p(z, n+1) = g_p(z, n) + (n+p+1) \frac{dg_p(z, n)}{dz} + \frac{d}{dz} [zg_{p-1}(z, n)]$$

and

$$g_{p-1}(z, n+1) = g_{p-1}(z, n) + \frac{dg_p(z, n)}{dz} .$$

The first of these equations reduces to the second which may be retained in the form

$$g_p(z, n+1) - g_p(z, n) = \frac{dg_{p+1}(z, n)}{dz} .$$

An application of this equation may be made to the calculation of the roots of $f_n = 0$ when $n > -1$. Let z_1, z_2, z_3 be the k th roots respectively of $g_{p+1}(z, n), g_p(z, n), g_p(z, n+1)$; then $z_1 > z_2 > z_3$ and the functions all change in the same sense from negative to positive or the reverse when z passes through the root corresponding to the function. If z' is the k th root of

$$\frac{dg_{p+1}(z, n)}{dz} = 0$$

it follows from the equation

$$g_p(z, n+1) - g_p(z, n) = \frac{dg_{p+1}(z, n)}{dz}$$

that $g_p(z', n+1) = g_p(z', n)$ and it can be seen that

$$\frac{dg_{p+1}(z_2, n)}{dz} > 0 \text{ or } < 0$$

according as $g_{p+1}(z, n)$ changes from negative to positive or the reverse when z passes through z_1 . This allows a series of approximations to be made to any root of $f_n = 0$, say the first, as follows: Let z_2 be the root of $g_2 = 0$; form the quantities

$$z_3 = z_2 - \frac{g_3(z_2)}{g'_3(z_2)} , \quad z_4 = z_3 - \frac{g_4(z_3)}{g'_4(z_3)} , \text{ etc.}$$

For example, if $n=0$, $z_4 = -29/20$ which gives for the least positive root of $J_0=0$ the value 2.40. These approximations to the first root are rational functions of n , the value of z_4 being

$$-\frac{1}{2}(n+1)(n+3) + \frac{(n+1)^3}{4(2n+5)}.$$

The consideration of the convergence of this process of approximation is waived.

5. A class of arithmetical equations may be deduced from the equation

$$g_p(z, n+1) - g_p(z, n) = \frac{dg_{p+1}(z, n)}{dz}.$$

Omitting z from the notation for g_p and distinguishing p even from p odd,

$$\begin{aligned} g_{2p}(n) = & z^p + {}_{p+1}C_{p-1}(n+p+1)(n+p)z^{p-1} + \dots \\ & + {}_{p+r}C_{p-r}(n+p+r) \dots (n+p-r+1)z^{p-r} + \dots \\ & + (n+2p) \dots (n+1). \end{aligned}$$

Let $g_{2p}(n)$ be expanded in powers of n so that

$$g_{2p}(n) = \sum_{l=0}^{2p} B_l n^l;$$

then

$$\begin{aligned} B_{2p-2k} = & \sum_{1 \dots 2p} i_1 \dots i_{2k} + (2p-1) \sum_{2 \dots 2p-1} i_1 \dots i_{2k-2} z + \dots \\ & + {}_{p+r}C_{p-r} \sum_{p-r+1 \dots p+r} i_1 \dots i_{2k-2(p-r)} z^{p-r} + \dots + {}_{2p-k}C_k z^k, \\ B_{2p-(2k-1)} = & \sum_{1 \dots 2p} i_1 \dots i_{2k-1} + (2p-1) \sum_{2 \dots 2p-1} i_1 \dots i_{2k-3} z + \dots \\ & + {}_{p+r}C_{p-r} \sum_{p-r+1 \dots p+r} i_1 \dots i_{2k-2(p-r)-1} z^{p-r} + \dots + {}_{2p-k+1}C_{k-1} \sum_{k \dots 2p-k+1} i_1 z^{k-1} \end{aligned}$$

where

$$\sum_{1 \dots 2p} i_1 \dots i_{2k}$$

denotes the sum of the products of the numbers $1 \dots 2p$ taken $2k$ at a time.

Similarly

$$\begin{aligned} g_{2p+1}(n) = & {}_{p+1}C_p(n+p+1)z^p + \dots + {}_{p+r+1}C_{p-r}(n+p+r+1) \\ & \dots (n+p-r+1)z^{p-r} + \dots + (n+2p+1) \dots (n+1). \end{aligned}$$

Expanding as before it is found that

$$\begin{aligned}
 B_{2p-2k} &= \sum_{1 \dots 2p+1} i_1 \dots i_{2k+1} + {}_{2p}C_1 \sum_{2 \dots 2p} i_1 \dots i_{2k-1} z + \dots \\
 &\quad + {}_{p+r+1}C_{p-r} \sum_{p-r+1 \dots p+r+1} i_1 \dots i_{2k-2(p-r)+1} z^{p-r} + \dots \\
 &\quad + {}_{2p+1-k}C_k \sum_{k+1 \dots 2p+1-k} i_1 z^k, \\
 B_{2p-(2k-1)} &= \sum_{1 \dots 2p-1} i_1 \dots i_{2k} + {}_{2p}C_1 \sum_{2 \dots 2p} i_1 \dots i_{2k-2} z + \dots \\
 &\quad + {}_{p+r+1}C_{p-r} \sum_{p-r+1 \dots p+r+1} i_1 \dots i_{2k-2(p-r)} z^{p-r} + \dots + {}_{2p+1-k}C_k z^k.
 \end{aligned}$$

It may be supposed that $g_{2p}(n)$ is expanded in terms of Bernoulli's polynomials

$$g_{2p}(n) = a_0 + a_1 \phi_1 + \dots + a_{2p} \phi_{2p},$$

where $\phi_1 = n$, $\phi_2 = \frac{1}{2} n(n+1)$, etc., and

$$\begin{aligned}
 a_0 &= z^p + \left[\frac{(p+1)p}{1 \cdot 2} \right]^2 z^{p-1} + \dots + \left[\frac{(p+2) \dots (p-r+1)}{1 \cdot 2 \dots 2r} \right]^2 z^{p-r} \\
 &\quad + \dots + [2p(2p-1) \dots 1].
 \end{aligned}$$

The polynomials ϕ_i satisfy the relations $\phi_i(n+1) - \phi_i(n) = n^{i-1}$ so that $g_{2p}(n+1) - g_{2p}(n) = a_1 + a_2 n + \dots + a_{2p} n^{2p-1}$ and since

$$g_{2p}(n+1) - g_{2p}(n) = \frac{dg_{2p+1}(n)}{dz}$$

it follows that $a_i = dB_i/dz$; hence the coefficients of a_i may be found from the above values of B_i , those being selected which belong to the function g_{2p+1} .

Denoting by a_i^q the coefficient of z^q in a_i , and equating coefficients of the q th power of z in the equation

$$g_{2p}(n) = a_0 + a_1 \phi_1 + \dots + a_{2p} \phi_{2p},$$

it follows that

$${}_{2p-q}C_q (n+2p-q) \dots (n+q+1) = a_0^q + a_1^q \phi_1 + \dots + a_{2p}^q \phi_{2p}$$

where

$$\begin{aligned}
 a_0^q &= \frac{[(2p-q) \dots (q+1)]^2}{1 \cdot 2 \dots (2p-2q)}, \\
 &\dots \dots \dots
 \end{aligned}$$

$$a_{2l}^q = (q+1)_{2p-q-1} C_{q+1} \sum_{q+2 \cdots 2p-q-1} i_1 \cdots i_{2p-2q-2l-2},$$

$$a_{2l-1}^q = (q+1)_{2p-q-1} C_{q+1} \sum_{q+2 \cdots 2p-q-1} i_1 \cdots i_{2p-2q-2l-1}.$$

The coefficients B_{2l} , B_{2l-1} involve powers of z up to $p-l$ so that a_{2l}^q , $a_{2l-1}^q = 0$ if $l > p-q-1$. When the preceding equation is reduced by excluding a factor of both sides it becomes

$$(2p-q)(n+2p-q) \cdots (n+q+1) = (2p-q)(2p-q)(2p-q-1) \cdots (q+1) \\ + (2p-2q)(2p-2q-1)[b_1\phi_1 + \cdots + b_{2p}\phi_{2p}],$$

where b_i denotes the summation $\sum$ in a_i^q . By assigning special values to n , or by comparing coefficients of powers of n , or by taking the equation as a congruence, special results may be derived. For example, if $n=1$

$$(2p-q)(2p-q+1) \cdots (q+2) = (2p-q)(2p-q)(2p-q-1) \cdots (q+1) \\ + (2p-2q)(2p-2q-1)[b_1 + \cdots + b_{2p}]$$

since if $n=1$, $\phi_i=1$. If $2p-q$ is prime to $(2p-2q)$ $(2p-2q-1)$ and n is an integer, then

$$b_1\phi_1 + \cdots + b_{2p}\phi_{2p} \equiv 0 \pmod{(2p-2q)}, \text{ etc.}$$

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