ON LAPLACE'S INTEGRAL EQUATIONS*

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The equation with which this paper is concerned is of the form

$$(\star) \quad \int_{(C)} e^{-zx} F(x) dx = f(z),$$

which is known in the literature as Laplace's integral equation. The contour (C) and the function $f(z)$ are supposed given and $F(x)$ is to be found.

In the case when the contour (C) consists of the positive part of the axis of reals, the solution of the equation $(\star)$ was given by H. Poincaré† and H. Hamburger.‡ Each of these authors considers $F(x)$ as a function of the real variable x .

When the contour (C) consists of the entire axis of reals, a simple substitution reduces $(\star)$ to the form studied by Riemann§ and H. Mellin.|| In the present paper we discuss the equation $(\star)$ in the case of Poincaré, extending the solution $F(x)$ to complex values of x . A certain relation of reciprocity between the functions $f(z)$ and $F(x)$ is thereby revealed.

1. Poincaré obtained the solution of the equation

$$(1) \quad f(z) = \int_0^\infty e^{-zx} F(x) dx$$

in form of a definite integral

$$(2) \quad F(x) = \frac{1}{2\pi i} \int_{(D)} e^{xz} f(\zeta) d\zeta.$$

This solution can be easily verified assuming certain hypotheses concerning the function $f(z)$ and the contour (D) .¶ Setting

$$x = u + iv$$

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† *Sur la théorie des quantas*, Journal de Physique, ser. 5, vol. 2 (1912).

‡ *Ueber eine Riemann'sche Formel . . .*, Mathematische Zeitschrift, vol. 6 (1920), pp. 6-9.

§ *Ueber die Anzahl der Primzahlen . . .*, Werke, 1876, p. 140.

|| *Abriss einer einheitlichen Theorie der Gamma- und der hypergeometrischen Funktionen*, Mathematische Annalen, vol. 68 (1910), pp. 318-324.

¶ The integration over (D) is taken from $\zeta = \lambda - i\infty$ to $\zeta = \lambda + i\infty$. If the integral $\int_{(D)}$ fails to exist, but Cauchy's principal value exists, let it be denoted by the same symbol $\int_{(D)}$, and no further modifications are necessary.

we make the following assumptions:

(A₁) The function $f(z)$ is analytic on the half-plane

$$(3) \quad u > \lambda_0,$$

where λ_0 is a non-negative constant determined by $f(z)$. Further, if λ denotes any number $> \lambda_0$, then $f(z)$ approaches zero, uniformly for $u \geq \lambda, |z| \rightarrow \infty$.

(A₂) If (D) denotes the straight line $u = \lambda$, the integrals

$$\frac{1}{2\pi i} \int_{(D)} e^{z\xi} f(\xi) d\xi, \quad \frac{1}{2\pi i} \int_0^\infty e^{-z\xi} d\xi \int_{(D)} e^{\xi\zeta} f(\zeta) d\zeta$$

exist and, in the latter, the order of integration can be interchanged.

THEOREM 1. *Under the conditions (A₁) and (A₂) the function $F(x)$ given by (2) is a solution of (1) and does not depend on λ .*

We have

$$\int_0^\infty e^{-z\xi} F(\xi) d\xi = \frac{1}{2\pi i} \int_0^\infty e^{-z\xi} d\xi \int_{(D)} e^{\xi\zeta} f(\zeta) d\zeta = \frac{1}{2\pi i} \int_{(D)} f(\zeta) d\zeta \int_0^\infty e^{\xi(\zeta-z)} d\xi.$$

If z is any point in the region $u > \lambda_0$, we can always find $\lambda > \lambda_0$ such that $u > \lambda > \lambda_0$, and then

$$\int_0^\infty e^{\xi(\zeta-z)} d\xi = -\frac{1}{\zeta - z}$$

and

$$(4) \quad \int_0^\infty e^{-z\xi} F(\xi) d\xi = -\frac{1}{2\pi i} \int_{(D)} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

The point z is situated to the right of (D) and by virtue of (A₁) the integral

$$\int \frac{f(\zeta)}{\zeta - z} d\zeta$$

taken round the right half of the circle $|\zeta - \lambda| = R$, approaches 0 as $R \rightarrow \infty$. A simple application of the Cauchy integral theorem shows then that the right hand member of (4) equals $f(z)$. A similar argument shows that the value of the integral (2) remains unchanged under any parallel displacement of (D) in the region (3), in other words that the expression (2) for $F(x)$ does not depend on λ . A special case of the condition (A₂) is the following:

(A₃) The integral

$$(5) \quad \int_{(D)} f(\zeta) d\zeta$$

is absolutely convergent.

In this case the integral

$$\frac{1}{2\pi i} \int_{(D)} e^{zx} f(\xi) d\xi \quad (x \geq 0)$$

exists, and the same is true of the double integral

$$\int_0^\infty \int_{(D)} |e^{\xi(\tau-x)} f(\xi)| |d\xi| |d\tau| = \int_0^\infty \int_{-\infty}^\infty e^{-\xi(u-\lambda)} |f(\lambda + i\tau)| d\xi d\tau,$$

which assures the existence of the repeated integrals of the condition (A₂), and also the legitimacy of interchanging of the order of integration. It should be noted that the condition (A₂) is more general than the condition (A₃). For instance, the equation

$$\int_0^\infty e^{-z\xi} F(\xi) d\xi = z^p \quad (\text{Re } p < 0)$$

admits of a solution

$$F(x) = \frac{1}{\Gamma(\lambda - p)} x^{-p-1} = \frac{1}{2\pi i} \int_{(D)} e^{zx} \xi^p d\xi,$$

whereas it is readily found that the function

$$f(Z) = z^p \quad (\text{Re } p > -1)$$

satisfies both conditions (A₁) and (A₂) but not (A₃). In virtue of this example it is clear that the condition (A₃) may be replaced by the more general

(A₄) The function $f(z)$ is of the form

$$f(z) = \sum_{s=1}^m c_s z^{p_s} + \varphi(z),$$

where c_s are arbitrary constants, the exponents p_s satisfy the conditions $\text{Re } p_s < 0$ ($s=1, 2, \dots, n$) and the function $\varphi(z)$ satisfies the conditions (A₁) and (A₃).

2. Thus far the solution $F(x)$ of the equation (1) which is given by the formula (2) is determined only for real positive values of x . For complex values of x the integral (2) may even become divergent. Let us now consider $F(x)$ as the given function and define $f(z)$ by the formula (1). We set

$$x = \xi + i\eta$$

and suppose that

(B₁) The function $F(x)$ is analytic in the strip

$$(6) \quad \xi \geq 0; \quad |\eta| < \alpha_0,$$

where α_0 is a positive constant determined by $F(x)$. Further, there exists a non-negative constant λ_0 , depending only on α_0 , such that for every pair of numbers λ, α ($\lambda > \lambda_0$; $0 < \alpha < \alpha_0$) the product

$$(7) \quad e^{-\lambda z} F(x)$$

approaches zero uniformly as $|x| \rightarrow \infty$ in the strip $\xi \geq 0, |\eta| \leq \alpha$.

Consider now the function $f(z)$ which is defined by

$$(8) \quad f(z) = \int_0^\infty e^{-z\xi} F(\xi) d\xi.$$

If a positive constant M is suitably chosen, we have, for any $\lambda_1 > \lambda_0$,

$$|e^{-z\xi} F(\xi)| = |e^{(\lambda_1 - z)\xi}| \cdot |e^{-\lambda_1 \xi} F(\xi)| \leq M e^{(\lambda_1 - u)\xi}.$$

Therefore the integral (8) is absolutely convergent in the region $u > \lambda_0$ and uniformly convergent in the closed region $u \geq \lambda > \lambda_0$. Hence the function $f(z)$ is analytic in the region $u > \lambda_0$.

Suppose now that the number $\lambda > \lambda_0$ is fixed. Since the integral (8) is uniformly convergent for

$$(9) \quad u \geq \lambda,$$

it follows that to an arbitrary positive ϵ there corresponds a positive constant X independent of z in (9) such that

$$\left| \int_x^\infty e^{-z\xi} F(\xi) d\xi \right| < \frac{\epsilon}{2}.$$

Further, this constant X being fixed, the integral

$$\int_0^X e^{-z\xi} F(\xi) d\xi = \frac{e^{-z\xi}}{-z} F(\xi) \Big|_0^X + \frac{1}{z} \int_0^X e^{-z\xi} F'(\xi) d\xi$$

likewise is $< \epsilon/2$ in absolute value for z in (9) and $|z|$ sufficiently large. Thus the condition (A₁) is satisfied by $f(z)$. Denote now by α_1 an arbitrary positive number $< \alpha_0$. If (B₁) is satisfied, the formula (8) may be rewritten in either of the following forms:

$$(10_1) \quad f(z) = \int_0^{\alpha_1} e^{-z i \eta} F(i \eta) i d\eta + \int_0^\infty e^{-z(\xi + i \alpha_1)} F(\xi + i \alpha_1) d\xi \equiv \varphi_1(z) + f_1(z);$$

$$(10_2) \quad f(z) = - \int_0^{\alpha_1} e^{z i \eta} F(-i \eta) i d\eta + \int_0^\infty e^{-z(\xi - i \alpha_1)} F(\xi - i \alpha_1) d\xi \equiv \varphi_2(z) + f_2(z).$$

We shall proceed by associating the first form with the case $v \leq 0$ and the second form with the case $v \geq 0$. Accordingly, the occurrence of the functions

$\varphi_k(z)$, $f_k(z)$ in a formula shall imply the former case when $k=1$, and the latter case when $k=2$.

Let us suppose for the moment that $F(0)=0$. Integration by parts gives

$$\varphi_1(z) = \int_0^{\alpha_1} e^{-z i \eta} F(i \eta) i d\eta = -\frac{e^{-z i \alpha_1}}{z} F(i \alpha_1) + \frac{i}{z} \int_0^{\alpha_1} e^{-z i \eta} F'(i \eta) d\eta = O\left(\frac{1}{z}\right),$$

because

$$|e^{-z i \eta}| = e^{v \eta} \leq 1 \text{ for } v \leq 0 \leq \eta.$$

Integrating by parts once more we find easily for large $|z|$ and $|v|$:

$$(11_1) \quad \varphi_1(z) = O\left(\frac{e^{-\alpha_1 |v|}}{z}\right) + O\left(\frac{1}{z^2}\right).$$

The same is true of the function $\varphi_2(z)$:

$$(11_2) \quad \varphi_2(z) = O\left(\frac{e^{-\alpha_1 |v|}}{z}\right) + O\left(\frac{1}{z^2}\right).$$

It is obvious finally that $\varphi_1(z)$ and $\varphi_2(z)$ are entire transcendental functions of z .

The formulas (10_{1,2}) and (11_{1,2}) show that the functions $f_1(z)$ and $f_2(z)$ satisfy the condition (A₁). Moreover, denoting by λ_1 any number between λ_0 and λ we can write

$$f_1(z) = \int_0^\infty e^{-s(\xi + i\alpha_1)} F(\xi + i\alpha_1) d\xi = e^{-s i \alpha_1} \int_0^\infty e^{(\lambda_1 - s)\xi} e^{-\lambda_1 \xi} F(\xi + i\alpha_1) d\xi,$$

which gives

$$(12) \quad |f_1(z)| \leq e^{-\alpha_1 |v|} \int_0^\infty e^{(\lambda_1 - \lambda)\xi} |e^{-\lambda_1 \xi} F(\xi + i\alpha_1)| d\xi.$$

By virtue of (B₁)

$$|e^{-\lambda_1 \xi} F(\xi + i\alpha_1)| = |e^{-\lambda_1 (\xi + i\alpha_1)} F(\xi + i\alpha_1)| \rightarrow 0 \text{ as } \xi \rightarrow \infty.$$

Then (12) shows that

$$(13_1) \quad f_1(z) = O(e^{-\alpha_1 |v|})$$

for large values of $|v|$. In precisely the same way we find that

$$(13_2) \quad f_2(z) = O(e^{-\alpha_1 |v|}).$$

Suppose now that

$$F(0) = c \neq 0.$$

We have

$$f(z) = \frac{c}{z} + \int_0^\infty e^{-z\xi} \{F(\xi) - F(0)\} d\xi.$$

The integral being of the form discussed above, we see that

$$f(z) = \frac{c}{z} + \begin{cases} \bar{\varphi}_1(z) + \bar{f}_1(z) & \text{for } v \leq 0, \\ \bar{\varphi}_2(z) + \bar{f}_2(z) & \text{for } v \geq 0, \end{cases}$$

where the functions $\bar{\varphi}_1, \bar{\varphi}_2, \bar{f}_1, \bar{f}_2$ possess the same properties as the functions $\varphi_1, \varphi_2, f_1, f_2$.

Hence the function $f(z)$ satisfies the condition (A₄) and a fortiori the condition (A₂).

3. In this section we return to the point of view that the function $F(x)$ is unknown and $f(z)$ is given. We suppose now, however, that the function $f(z)$ satisfies the condition

(C) The function $f(z)$ may be represented in the form

$$f(z) = \frac{c}{z} + \begin{cases} \psi_1(z) + \theta_1(z) & \text{for } v \leq 0, \\ \psi_2(z) + \theta_2(z) & \text{for } v \geq 0, \end{cases}$$

where $\psi_1(z), \psi_2(z)$ are entire transcendental functions which in the corresponding half-planes are of the order

$$(14) \quad \psi_k(z) = O\left(\frac{e^{-\alpha_1|v|}}{z}\right) + O\left(\frac{1}{z^2}\right) \quad (k = 1, 2)$$

for large $|z|$ and $|v|$, and where $\theta_1(z)$ and $\theta_2(z)$ satisfy the condition (A₁) and are of the order

$$(15) \quad \theta_k(z) = O(e^{-\alpha_1|v|}) \quad (k = 1, 2),$$

α_1 being an arbitrary positive number $< \alpha_a$.

Obviously $f(z)$ satisfies the conditions (A₁) and (A₂). Hence the solution of the equation (1) is given by the formula

$$F(x) = \frac{1}{2\pi i} \int_{(D)} e^{xz} f(\zeta) d\zeta.$$

We shall show now that, because of the special properties of the function $f(z)$, this solution can be extended to complex values of x and is analytic in x .

Suppose for the sake of simplicity that $c=0$ and denote by (D_1) the part of the contour (D) which is situated below the axis of reals and by

(D_2) the part which is above the axis of reals. The formulas (15), in which α_1 is an arbitrary positive number $< \alpha_0$, show that both integrals

$$(16) \quad F_k(x) = \frac{1}{2\pi i} \int_{(D_k)} e^{xz} \theta_k(\zeta) d\zeta \quad (k = 1, 2)$$

converge absolutely for $|\eta| < \alpha_0$ and uniformly in any finite part of the strip $|\eta| \leq \alpha$, α being an arbitrary positive number $< \alpha_0$. Thus $F_k(x)$ are analytic on the strip $|\eta| < \alpha_0$.

We shall prove that both functions $F_k(x)$ satisfy the condition (B_1). Let λ_1 be any number $> \lambda_0$; we can locate the contour (D) so that $\lambda_0 < \lambda < \lambda_1$, and then, denoting by M a suitable positive constant, we have

$$|e^{-\lambda_1 x} F_k(x)| \leq M e^{(\lambda - \lambda_1)x} \int_0^\infty e^{v(\eta - \alpha_1)} dv \rightarrow 0 \text{ as } \xi \rightarrow \infty,$$

the limit being approached uniformly in any region

$$|\eta| \leq \alpha < \alpha_1 < \alpha_0.$$

Thus, α_1 and therefore α being arbitrary numbers $< \alpha_0$, the property (B_1) is proved.

We turn now to the functions

$$(17) \quad \Phi_k(x) = \frac{1}{2\pi i} \int_{(D_k)} e^{xz} \psi_k(\zeta) d\zeta \quad (k = 1, 2).$$

In this form the functions $\Phi_k(x)$ can not be extended to complex values of x . This becomes possible, however, if the contour of integration is suitably transformed. We shall show that

$$(18) \quad \Phi_1(x) = \frac{1}{2\pi i} \int_{-\infty}^{\lambda} e^{xz} \psi_1(\zeta) d\zeta; \quad \Phi_2(x) = \frac{1}{2\pi i} \int_{\lambda}^{-\infty} e^{xz} \psi_2(\zeta) d\zeta.$$

In order to prove this we observe that both functions ψ_1, ψ_2 are entire transcendental functions and that the integral

$$\int e^{xz} \psi_k(\zeta) d\zeta$$

taken round the quarter of the circle $|\zeta - \lambda| = R$ which lies to the left of (D_k), approaches zero as $R \rightarrow \infty$, as follows from (14) using Jordan's lemma.*

* Whittaker and Watson, *Modern Analysis*, 1920, p. 115.

The formulas (18) combined with (14) show that the $\Phi_k(x)$ are analytic for $\xi > 0$, and it is readily found that, for $\lambda_1 > \lambda$,

$$|e^{-\lambda_1 x} \Phi_k(x)| < M e^{(\lambda - \lambda_1)\xi} \rightarrow 0 \text{ as } \xi \rightarrow \infty,$$

the convergence being uniform on any strip $|\eta| < \alpha < \alpha_0$, $\xi > 0$. The case when $c \neq 0$ does not involve any substantial change in the reasoning above. Thus the following theorem is proved:

THEOREM 2. *If the function $f(z)$ satisfies the condition (C), the solution $F(x)$ of the equation (1) satisfies the condition (B) which is identical with the condition (B₁) above except that the strip $\xi \geq 0$, $|\eta| < \alpha_0$, closed at the left hand end, is replaced by the open strip*

$$\xi > 0, \quad |\eta| < \alpha_0.$$

We turn now to the converse theorem:

THEOREM 3. *If the function $F(x)$ satisfies the condition (B₁), the function $f(z)$ defined by*

$$f(z) = \int_0^\infty e^{-z\xi} F(\xi) d\xi$$

*is a solution of the equation (2) for $\xi > 0$ and satisfies the condition (C).**

This theorem establishes the reciprocity between $F(x)$ and $f(z)$, which was mentioned in the beginning of the paper.

It was proved in § 2 that the function

$$f(z) = \int_0^\infty e^{-z\xi} F(\xi) d\xi$$

satisfies the condition (C). It remains only to prove that $f(z)$ is a solution of the equation

$$F(x) = \frac{1}{2\pi i} \int_{(D)} e^{xz} f(z) dz.$$

We suppose again that

$$c = F(0) = 0$$

and consider first the case when x is real and positive. We have then

$$\frac{1}{2\pi i} \int_{(D)} e^{xz} f(z) dz = \frac{1}{2\pi i} \lim_{V \rightarrow \infty} \int_{\lambda - iV}^{\lambda + iV} e^{xz} dz \int_0^\infty e^{-z\xi} F(\xi) d\xi. \dagger$$

* If x is complex, the right hand member of (2) must be transformed as was indicated above.

† See last foot note on p. 417.

Because of the uniform convergence of the interior integral we are justified in interchanging the order of integration, which gives

$$\begin{aligned} \frac{1}{2\pi i} \int_{(D)} e^{xz} f(\xi) d\xi &= \lim_{V \rightarrow \infty} \int_0^\infty F(\xi) e^{\lambda(x-\xi)} \int_{-V}^V e^{iv(x-\xi)} i dv \\ &= \frac{1}{\pi} \lim_{V \rightarrow \infty} \int_0^\infty F(\xi) e^{\lambda(x-\xi)} \frac{\sin V(x-\xi)}{x-\xi} d\xi = F(x). \end{aligned}$$

This last relation can be easily proved by using the Dirichlet formula.* Thus the equation (2) is proved for real positive values of x . The right hand member of this equation is an analytic function of x on the region $\xi > 0$, as was proved above. The same is true by hypothesis of the left hand member. Hence the equation holds true on the whole region $\xi > 0$.

The case $c \neq 0$ does not involve any substantial change in the applied reasoning; neither does the case in which finite sums of terms of the form

$$c_n z^{p_n}; \quad a_n x^{-p_n-1} \quad (\text{Re } p_n < 0)$$

are present in $f(z)$ and $F(x)$ respectively.

Under the restrictions made we can easily prove the formulas

$$\begin{aligned} F(x) &= \frac{1}{2\pi i} \int_{(D)} e^{xz} d\xi \int_0^\infty e^{-t\xi} F(\xi) d\xi; \\ f(z) &= \frac{1}{2\pi i} \int_0^\infty e^{-t\xi} d\xi \int_{(D)} e^{t\xi} f(\xi) d\xi. \end{aligned}$$

There is no difficulty either in proving that the solutions of the equations (1), (2), obtained above, are unique under the same restrictions.

* See Hamburger, loc. cit.