RIEMANN INTEGRATION AND TAYLOR'S THEOREM IN GENERAL ANALYSIS*

BY

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The importance and usefulness of Taylor's theorem need not be dwelt upon here. We are interested in it for functions whose arguments and functional values belong to abstract spaces of the Fréchet type. Consequently no Rolle's theorem can be even stated, and the proofs will be somewhat different from the usual proofs in the theory of numerically-valued functions. It is also to be expected that a slight strengthening of hypotheses will be required. However, it is not necessary to assume the existence of a uniformly continuous nth differential, as was done in the first announcement of these results.‡ It is sufficient that the function should have an nth *variation* (in the sense of Gateaux), with certain limitations on its discontinuities.

The functions we discuss will be one-valued functions whose arguments x and functional values y belong to *linear metric spaces* $\mathfrak{X}$ and $\mathfrak{Y}$ respectively, briefly, functions F on $\mathfrak{X}$ to $\mathfrak{Y}$. Linear metric spaces correspond to the spaces called by Fréchet, "espaces (D) vectoriels." For the notations, postulates and fundamental propositions, we refer the reader to Part 1 of a paper by T. H. Hildebrandt and the author,§ to avoid practically entire repetition of that section. No other parts of that paper are needed here.

We shall consistently use the letters r, s, a, b, c to refer to real numbers, and the German $\mathfrak{R}$ to refer to the real axis. The notation $\mathfrak{R}_0$ will then refer to a set of open intervals of that axis. By the notation (ab) we shall mean the bounded closed interval of $\mathfrak{R}$ with end points a and b. Whenever the space $\mathfrak{Y}$ is required to be *complete*, this will be specifically mentioned.

The form of remainder obtained in our Taylor's formula is a generalization of that given by Jordan‖ and analogous at least to those obtained in

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‡ Comptes Rendus, vol. 180 (1925), p. 1719.

§ *Implicit functions and their differentials in general analysis*, in the present number of these Transactions.

‖ *Cours d'Analyse*, 3d edition, vol. I, p. 251.

other special cases by Hart,* and involves a Riemann integral of a function G on $\mathfrak{R}$ to a linear metric space $\mathfrak{Y}$. Hence it is necessary to develop a theory of Riemann integration for such functions. This is on the whole parallel to, or rather a generalization of, the theory for numerically valued functions.† A theory of line integrals taken along curves in an abstract space $\mathfrak{X}$ could be developed in a similar way, as well as a theory of Stieltjes integrals. We shall content ourselves here with discussing Riemann integrals.

1. **Derivatives and variations.** For functions F on $\mathfrak{X}$ to $\mathfrak{Y}$ there are several useful definitions of differentiability. We are concerned to base the present theory on the least restrictive one, in order to gain for it the widest range of applicability. We have selected a definition of variations generalizing the one given by Gateaux.‡ Gateaux's definition is less restrictive than the definition of variation used by Lévy in his *Analyse Fonctionnelle*§ and much less restrictive than the definition of differential used by Fréchet.||

Derivatives. We say that a function F on a region $\mathfrak{R}_0$ of the real axis, to $\mathfrak{Y}$, has a derivative, or more specifically, a first derivative,

$$F'(r_0) = \frac{dF}{dr}\bigg|^{r=r_0}$$

at a point r_0 of $\mathfrak{R}_0$ in case

$$\lim_{r=r_0} \left\| \frac{F(r) - F(r_0)}{r - r_0} - F'(r_0) \right\| = 0.$$

In case F is defined on a closed interval (ab), we define the derivatives at the end points by one-sided limits, as is commonly done in the classical theory of real functions. We define nth derivatives inductively, as in the classical theory.

* These Transactions, vol. 18 (1917), p. 138, vol. 23 (1922), p. 39; Annals of Mathematics, (2), vol. 24 (1922), pp. 28; 32.

† N. Wiener has discussed functions F on $\mathfrak{C}_0$ to $\mathfrak{Y}$, where $\mathfrak{C}_0$ is a domain of the realm $\mathfrak{C}$ of complex numbers and $\mathfrak{Y}$ is a linear metric space having $\mathfrak{C}$ as its associated number system, and has briefly treated differentiation, integration, and Taylor's series for such functions. See his paper in Fundamenta Mathematicae, vol. 4 (1923), p. 136.

‡ Bulletin de la Société Mathématique de France, vol. 47 (1919), p. 83.

§ See pp. 50–52.

|| Cf. these Transactions, vol. 15 (1914), p. 139; vol. 16 (1915), p. 216; Annales Scientifiques de l' École Normale Supérieure, vol. 42 (1925), p. 293. Also § III of the paper on implicit functions by Hildebrandt and Graves, already cited.

Variations. We say that a function F on a region $\mathfrak{X}_0$ of $\mathfrak{X}$ to $\mathfrak{Y}$ has an nth variation at a point x_0 of $\mathfrak{X}_0$ in case, for every element δx of $\mathfrak{X}$, the function of r, $F(x_0+r\delta x)$, has an nth derivative at $r=0$. We denote this nth derivative at $r=0$ by $\delta^n F(x_0, \delta x)$. Regarded as a function on $\mathfrak{X}$ to $\mathfrak{Y}$, $\delta^n F$ is the nth variation of F at x_0. We say that F has an nth variation on $\mathfrak{X}_0$ in case it has an nth variation at each point of $\mathfrak{X}_0$.

LEMMA 1.1. *If a function F on $\mathfrak{R}_0$ to $\mathfrak{Y}$ has a first derivative at a point r_0 of $\mathfrak{R}_0$, then F is continuous at r_0.*

It is not true in general, however, that a function F on a region $\mathfrak{X}_0$ to $\mathfrak{Y}$ which has a first *variation* is therefore continuous, as numerous examples from the calculus of variations show.

Some properties of the nth variation (which is readily shown to be unique) are stated in

LEMMA 1.2. *Suppose the functions F on $\mathfrak{X}_0$ to $\mathfrak{Y}$, G on $\mathfrak{X}_0$ to $\mathfrak{Y}$, and H on $\mathfrak{X}_0$ to $\mathfrak{R}$, all have nth variations $\delta^n F(x_0, \delta x), \cdots$, at a point x_0 of $\mathfrak{X}_0$. Then*

(a) F has variations of all lower orders at x_0, and for every positive integer $k<n$ and every point δx in $\mathfrak{X}$ we have

$$\delta^n F(x_0, \delta x) = \frac{d^k}{dr^k}(\delta^{n-k}F(x_0+r\delta x, \delta x))\Big|^{r=0};$$

(b) the sum function $F+G$ and the product function FH have nth variations at x_0, and $\delta^n(FH)$ is given by a generalization of Leibnitz's formula;

(c) the variations $\delta^n F$, etc., are homogeneous of the nth degree in δx, i. e.,

$$\delta^n F(x_0, s\delta x) = s^n \delta^n F(x_0, \delta x)$$

for every point δx of $\mathfrak{X}$ and every real number s.

Property (a) follows at once from the definition of derivative and variation, and property (b) is proved in the usual way, considering derivatives first. To prove property (c), we proceed by induction. We have first, if $s\neq 0$,

$$\|\delta F(x_0, s\delta x) - s\delta F(x_0, \delta x)\| \leqq \left\|\delta F(x_0, s\delta x) - \frac{F(x_0+(r/s)s\delta x) - F(x_0)}{(r/s)}\right\|$$
$$+ |s| \left\|\frac{F(x_0+r\delta x) - F(x_0)}{r} - \delta F(x_0, \delta x)\right\|.$$

Since the terms on the right approach zero with r, the left hand side equals zero. The case $s=0$ follows from the fact that $\delta F(x_0, x_*)=y_*$ for every function F. The manipulation to complete the induction proceeds in the same way as above.

2. **Riemann integration.** The theory of Riemann integration of bounded functions here presented is on the whole parallel to the classical theory. A noteworthy hiatus is the failure to show that for the existence of the integral it is *necessary* that the Lebesgue measure of the set of discontinuities of the integrand shall be zero. That this condition is not necessary is shown by the following example, where the integrand function is discontinuous at every point of the interval $0 \leqq r \leqq 1$. Let the space $\mathfrak{Y}$ consist of all functions y on the interval $0 \leqq t \leqq 1$ to $\mathfrak{R}$ which are bounded on that interval, and let $\|y\|$ = the upper bound of $|y(t)|$. Consider the system of points y_r defined by

$$y_r(t) = \begin{cases} 0 \text{ for } 0 \leqq t \leqq r, \\ 1 \text{ for } r \leqq t \leqq 1, \end{cases} \quad 0 \leqq r \leqq 1.$$

Let $F(r)=y_r$. Then F is integrable and yet everywhere discontinuous.

Definition of integrals. Consider a bounded function F on (ab) to $\mathfrak{Y}$. (By bounded we mean that $\|F(r)\|$ is bounded on (ab)). Let π be a partition of (ab) into sub-intervals Δ_i of lengths Δ_i. Denote the norm of the partition by $N\pi$. The lengths Δ_i are understood to have the same sign as $(b-a)$. Let r_i be an arbitrary point of the interval Δ_i. Then if the limit

$$\lim_{N\pi=0} \sum_{\pi} F(r_i)\Delta_i$$

exists, we say that F is integrable on (ab), and denote the limit by the usual symbol

$$\int_a^b F(r)dr.$$

The limit is taken in the sense that, for every positive ϵ there exists a positive δ such that, for every partition π with $N\pi \leqq \delta$ and every choice of the points r_i in the intervals Δ_i of π, we have

$$\left\| \sum_{\pi} F(r_i)\Delta_i - \int_a^b F(r)dr \right\| \leqq \epsilon.$$

In a complete linear metric space we have the usual necessary and sufficient condition for the existence of the integral, stated in

Lemma 2.1 *If the space* $\mathfrak{Y}$ *is complete, then a necessary and sufficient condition that a bounded function* F *on* (ab) *to* $\mathfrak{Y}$ *be integrable on* (ab), *is that*

$$\lim_{\substack{N\pi_1=0\\ N\pi_2=0}} \left\| \sum_{\pi_1} F(r_{i1})\Delta_{i1} - \sum_{\pi_2} F(r_{i2})\Delta_{i2} \right\| = 0.$$

We find it convenient in proving Theorem 1, on the sufficiency of certain conditions for the existence of the integral, to derive the lemmas numbered 2.2 and 2.3 relating to oscillation and the sets of discontinuities of functions.

Oscillation. For an interval (ab) and a function F on (ab) to $\mathfrak{Y}$, we define the oscillation $O_F(a, b)$ to be the upper bound of $\|F(r_1)-F(r_2)\|$ for r_1 and r_2 in (ab). For an interior point c of (ab) we define the point oscillation $O_F(c)$ by the equation

$$O_F(c) = \lim_{r=0} O_F(c-r, \ c+r).$$

If c is an end point, say the left hand end point, of the interval, then we put c in place of $(c-r)$ in the limitand. The oscillation functions $O_F(a, b)$ and $O_F(r)$ are evidently always well defined for bounded functions F. A necessary and sufficient condition for the continuity of a function F at a point c is that $O_F(c)=0$.

Content and measure. Let $\mathfrak{E}$ be a set of points of the real axis $\mathfrak{R}$. Then we shall mean by *content* $\mathfrak{E}$, the Jordan measure of $\mathfrak{E}$,* and by *measure* $\mathfrak{E}$, the Lebesgue measure of $\mathfrak{E}$,* if these measures exist.

For a function F on (ab) to $\mathfrak{Y}$, we shall denote by $\mathfrak{E}_{\epsilon F}$ the set of points r of (ab) at which $O_F(r) \geqq \epsilon$, and by $\mathfrak{D}_F$ the set of points of (ab) at which F is discontinuous.

Lemma 2.2. *For every bounded function* F *on* (ab) *to* $\mathfrak{Y}$, *the statements "content* $\mathfrak{E}_{\epsilon F}=0$ *for every* $\epsilon>0$," *and "measure* $\mathfrak{D}_F=0$" *are equivalent.*

Consider a sequence of positive numbers $\{\epsilon_k\}$ with $\lim \epsilon_k=0$. Then $\mathfrak{D}_F=\sum_1^\infty \mathfrak{E}_{\epsilon_k F}$. By definition of content, each $\mathfrak{E}_{\epsilon_k F}$ is enclosable interiorly in a finite number of non-overlapping intervals the sum of whose lengths is less than $\epsilon/2^k$, where ϵ is arbitrary. Hence $\mathfrak{D}_F$ is enclosable in a denumerable infinity of intervals the sum of whose lengths is arbitrarily small, so that measure $\mathfrak{D}_F=0$. For the converse, we prove first that every set $\mathfrak{E}_{\epsilon F}$ is closed. Let c be a limit point of $\mathfrak{E}_{\epsilon F}$. Then every interval $(c-r, c+r)$ encloses a point of $\mathfrak{E}_{\epsilon F}$, so that $O_F(c-r, c+r) \geqq \epsilon$ for every positive r, and

*See Jordan, *Cours d'Analyse*, 3d edition, vol. I, p. 28; Lebesgue, *Leçons sur l'Intégration*, p. 102 and p. 28.

hence $O_F(c) \geqq \epsilon$. Now by definition of "measure $\mathfrak{D}_F = 0$," for every positive number ω there exists a denumerable set of intervals enclosing $\mathfrak{D}_F$ interiorly with the sum of their lengths less than ω. Since each set $\mathfrak{E}_{\epsilon F}$ is a part of the set $\mathfrak{D}_F$ and since each $\mathfrak{E}_{\epsilon F}$ is closed, we can apply the Heine-Borel-Lebesgue theorem in generalized form* to show that each $\mathfrak{E}_{\epsilon F}$ is enclosed interiorly by a finite number of intervals the sum of whose lengths is less than ω. Since ω is arbitrary, each $\mathfrak{E}_{\epsilon F}$ has content zero.

LEMMA 2.3. *Let F be a function on (ab) to $\mathfrak{Y}$ such that $O_F(r) \leqq \epsilon$ on (ab). Then for every constant $\omega > 0$ there exists a constant $\delta > 0$ such that, for every pair π_1, π_2 of partitions of (ab) with norms less than δ we have*

$$\left\| \sum_{\pi_1} F(r_{i1})\Delta_{i1} - \sum_{\pi_2} F(r_{i2})\Delta_{i2} \right\| \leqq (\epsilon + \omega) \; |b - a| \, . \tag{2.1}$$

The hypothesis $O_F(r) \leqq \epsilon$ on (ab) implies that F is bounded on (ab). By definition of the point oscillation $O_F(r)$, we know that if we fix ω arbitrarily, then for every point r of (ab) there is a constant $2\alpha_r$ such that

$$O_F(r - 2\alpha_r, \; r + 2\alpha_r) \leqq \epsilon + \omega. \tag{2.2}$$

Since (ab) is a closed interval, by the Heine-Borel-Lebesgue theorem there is a finite subset $(r_i - \alpha_{r_i}, \; r_i + \alpha_{r_i})$ of the set of intervals $(r - \alpha_r, \; r + \alpha_r)$, which also covers (ab). Let 4δ be the minimum length of the intervals of this finite subset, i. e., $2\delta = \text{minimum } \alpha_{r_i}$. Then if π_1 and π_2 have norms less than δ, and Δ_1 and Δ_2 are intervals of π_1 and π_2 respectively having a point in common, there is an interval $(r_i - 2\alpha_{r_i}, \; r_i + 2\alpha_{r_i})$ to which Δ_1 and Δ_2 are both interior. Let π_3 be the partition of (ab) obtained by using all the division points of π_1 and π_2. Then by the inequality (2.2) we have

$$\begin{aligned} \left\| \sum_{\pi_1} F(r_{i1})\Delta_{i1} - \sum_{\pi} F(r_{j2})\Delta_{j2} \right\| &= \left\| \sum_{\pi_3} \left[F(r_{k1}) - F(r_{k2}) \right] \Delta_{k3} \right\| \\ &\leqq (\epsilon + \omega) \, | \, b - a \, | \, . \end{aligned}$$

THEOREM 1. *Existence theorem. Suppose that the space $\mathfrak{Y}$ is complete, and that the function F on (ab) to $\mathfrak{Y}$ is bounded on (ab), and has measure $\mathfrak{D}_F = 0$. Then F is integrable on (ab).*

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*For a simple proof of this theorem, see Lebesgue, loc. cit., p. 105. Only a slight modification of Lebesgue's reasoning is necessary to prove the theorem when the interval (ab) is replaced by an arbitrary bounded closed set E. See Hildebrandt, Bulletin of the American Mathematical Society, vol. 32 (1926), pp. 423 ff.

By Lemma 2.2, we may replace the hypothesis "measure $\mathfrak{D}_F=0$" by "content $\mathfrak{C}_{\epsilon F}=0$ for every positive ϵ." Select positive numbers ϵ, ω, and α arbitrarily. Since content $\mathfrak{C}_{\epsilon F}=0$, there is a finite set $\mathfrak{S}$ composed of n intervals, the sum of whose lengths is less than α, and which enclose the set $\mathfrak{C}_{\epsilon F}$ in their interiors. Let $\mathfrak{T}$ be the set of intervals complementary to $\mathfrak{S}$ on (ab). Then $O_F(r)<\epsilon$ on $\mathfrak{T}$. By Lemma 2.3, there is a positive δ_ω corresponding to ω such that the inequality (2.1) holds for all partitions π_1 and π_2 of the set of intervals $\mathfrak{T}$, having norm less than δ_ω. Select $\delta_1<\delta_\omega$ and $<\alpha/n$, and let π_3 and π_4 be partitions of (ab) of norm less than δ_1. Let π_5 and π_6 be the partitions formed from π_3 and π_4 respectively by inserting the end points of the intervals of the set $\mathfrak{S}$ as division points. The number of intervals of π_3 which are not identical with intervals of π_5 is not greater than $2n$, where n is the number of intervals in $\mathfrak{S}$. We specify that on an interval Δ_{j5} identical with a Δ_{i3}, $r_{j5}=r_{i3}$, so that

$$\left\| \sum_{\pi_3} F(r_{i3})\Delta_{i3} - \sum_{\pi_5} F(r_{j5})\Delta_{j5} \right\| \leqq 4Mn\delta_1 \leqq 4M\alpha,$$

where M is the upper bound of $\|F(r)\|$ on (ab). We do similarly for π_4 and π_6. Next we have

$$\left\| \sum_{\pi_5}^{\mathfrak{S}} F(r_{j5})\Delta_{j5} \right\| + \left\| \sum_{\pi_6}^{\mathfrak{S}} F(r_{j6})\Delta_{j6} \right\| \leqq 2M\alpha,$$

where the sums are taken over the intervals of π_5 and π_6 contained in the set $\mathfrak{S}$. Since $\delta_1<\delta_\omega$, we have finally

$$\left\| \sum_{\pi_5}^{\mathfrak{T}} F(r_{j5})\Delta_{j5} - \sum_{\pi_6}^{\mathfrak{T}} F(r_{j6})\Delta_{j6} \right\| \leqq (\epsilon+\omega)\,|\,b-a\,|\,.$$

By combination of these inequalities we obtain

$$\left\| \sum_{\pi_3} F(r_{i3})\Delta_{i3} - \sum_{\pi_4} F(r_{i4})\Delta_{i4} \right\| \leqq 10M\alpha + (\epsilon+\omega)\,|\,b-a\,|\,.$$

Since α, ϵ and ω are arbitrary, we may apply Lemma 2.1 to obtain the desired conclusion.

The next theorem contains the ordinary formulas for definite integrals, which extend immediately to our general case.

THEOREM 2. *Suppose that the functions F, G, and H, on (ab) to $\mathfrak{Y}$, $\mathfrak{Y}$, and $\mathfrak{R}$ respectively, are integrable on (ab). Let c, d, and e be points of (ab). Then the following equalities and inequality are valid*:

$$\int_c^d F\,dr + \int_d^e F\,dr = \int_c^e F\,dr \text{ if the space } \mathfrak{Y} \text{ is complete}; \tag{2.3}$$

$$\int_a^b (F + G)dr = \int_a^b F\,dr + \int_a^b G\,dr; \tag{2.4}$$

$$\int_a^b FH\,dr = H\int_a^b F\,dr \text{ if } H \text{ is constant on } (ab); \tag{2.5}$$

$$\int_a^b FH\,dr = F\int_a^b H\,dr \text{ if } F \text{ is constant on } (ab); \tag{2.6}$$

$$\left\|\int_a^b F\,dr\right\| \leqq \left|\int_a^b H\,dr\right| \text{ if } \|F(r)\| \leqq H(r) \text{ on } (ab). \tag{2.7}$$

3. **Relations between derivatives and integrals.** We say that a function F on (ab) to $\mathfrak{Y}$ has a *primitive* H in case the function H is defined on (ab) to $\mathfrak{Y}$ and has F for its derivative on (ab). If a function F on (ab) to $\mathfrak{Y}$ is integrable on every sub-interval of (ab), then the function

$$G(r) = \int_a^r F(r)dr$$

is called an *indefinite integral* of F. In Theorem 4 we show essentially that when a function F has both a primitive and an indefinite integral, these differ at most by a constant element of the space $\mathfrak{Y}$. A particular case where both a primitive and an indefinite integral exist is when the space $\mathfrak{Y}$ is complete, and the function F is continuous. This follows from Theorems 1 and 3. Lemma 3.1 is concerned with integration by parts.

THEOREM 3. *If the space $\mathfrak{Y}$ is complete, and if the function F on (ab) to $\mathfrak{Y}$ is integrable on (ab), then the function G on (ab) to $\mathfrak{Y}$ defined by*

$$G(r) = \int_a^r F\,dr$$

has the properties

(1) $\|G(r_1)-G(r_2)\| \leqq M|r_1-r_2|$ *for every r_1, r_2 in (ab), where M is the upper bound of $\|F(r)\|$ on (ab)*;

(2) *G is continuous on* (ab);

(3) *if F is continuous at a point* r_0 *of* (ab), *then G has a derivative at* r_0, *equal to* $F(r_0)$.

This theorem is proved by application of the formulas of Theorem 2.

THEOREM 4. *If the function F on* (ab) *to* $\mathfrak{Y}$ *has a derivative* F' *on* $(ab$ *which is integrable on* (ab), *then*

$$F(b) - F(a) = \int_a^b F'\,dr.^*$$

For definiteness we assume $a<b$. Let ϵ be an arbitrary positive number. Then since F' is integrable, there exists a $\delta>0$ such that, for every partition π with $N\pi\leqq\delta$ and for every choice of the r_i in the closed intervals Δ_i of π, we have

$$\left\|\sum_\pi F'(r_i)\Delta_i - \int_a^b F'dr\right\| \leqq \epsilon. \tag{3.1}$$

Since F' is the derivative of F on (ab), there is for each point r of (ab) a positive number $\alpha_r\leqq\delta$ such that, for every point r' of (ab) satisfying $|r'-r|\leqq\alpha_r$, we have

$$\|F(r') - F(r) - F'(r)(r' - r)\| \leqq \epsilon\,|\,r' - r\,|. \tag{3.2}$$

The open intervals $I_r\equiv(r-\alpha_r<r'<r+\alpha_r)$ constitute a set covering the closed interval (ab); hence we may apply the Heine-Borel-Lebesgue theorem to show that a finite set $I_1\cdots I_m$ of these, with centers at $\rho_1<\rho_2<\cdots<\rho_m$ respectively, also cover (ab). We may evidently assume that no one of the intervals I_k is wholly contained in another one of them, and that $\rho_1=a$, $\rho_m=b$. For each interval I_k we have by (3.2) the condition that when r is in I_k or at an end point of I_k,

$$\|F(r) - F(\rho_k) - F'(\rho_k)(r - \rho_k)\| \leqq \epsilon\,|\,r - \rho_k\,|\,. \tag{3.3}$$

Now in the interval $\rho_k\leqq r\leqq\rho_{k+1}$ blot out all end points of intervals I_j having $j\neq k$. The remaining end points together with the points ρ_k, determine a partition π of (ab) of norm $\leqq\delta$. Each interval $\Delta_i=(r_i, r_{i+1})$ of π has at least one of the points ρ_k for an end point. If there are two, we choose the left hand one, and in either case denote the ρ_k thus determined for Δ_i by ρ_i. Then by (3.3) we have for every i,

$$\|F(r_{i+1}) - F(r_i) - F'(\rho_i)\Delta_i\| \leqq \epsilon\Delta_i. \tag{3.4}$$

*This theorem in its present generality is due to T. H. Hildebrandt, as is also the second corollary. I have slightly altered his proof.

Thus by (3.1) and (3.4) we obtain

$$\left\| F(b) - F(a) - \int_a^b F'\,dr \right\| \leqq \sum_i \left\| F(r_{i+1}) - F(r_i) - F'(\rho_i)\Delta_i \right\|$$

$$+ \left\| \sum_\pi F'(\rho_i)\Delta_i - \int_a^b F'\,dr \right\| \leqq \epsilon(b - a + 1).$$

Since ϵ is arbitrary, the equality is proved.

That two primitives of a function always differ by a constant is a result of

COROLLARY 1. *If the function F on (ab) to $\mathfrak{Y}$ has a derivative $F'(r) = y_*$ on (ab), then F is constant on (ab).**

For the function F' is evidently integrable on every sub-interval of (ab), and hence

$$F(r) - F(a) = \int_a^r F'\,dr = y_*$$

for every r in (ab).

COROLLARY 2. *If the space $\mathfrak{Y}$ is complete, and if the function F on (ab) to $\mathfrak{Y}$ has a derivative F' on (ab) which is continuous on (ab), then*

$$\lim_{r_1=r_2} \left\| \frac{F(r_1) - F(r_2)}{r_1 - r_2} - F'(r_2) \right\| = 0$$

uniformly on (ab).

Since F' is continuous on (ab), it is bounded on (ab), and since (ab) is closed, F' is continuous uniformly on (ab). This can be shown by classical methods. Then by applying Theorems 1, 4, and 2, we obtain

$$\left\| \frac{F(r_1) - F(r_2)}{r_1 - r_2} - F'(r_2) \right\| = \left\| \frac{\int_{r_2}^{r_1} [F'(r) - F'(r_2)]dr}{r_1 - r_2} \right\| \leqq \epsilon$$

whenever $|r_1 - r_2|$ is sufficiently small.

LEMMA 3.1. *Integration by parts. Suppose the space $\mathfrak{Y}$ is complete, and that the functions F on (ab) to $\mathfrak{Y}$ and G on (ab) to $\mathfrak{R}$ have derivatives F' and G' on (ab) which are bounded on (ab) and whose sets of discontinuities on (ab) each have measure zero. Then*

$$F(b)G(b) - F(a)G(a) = \int_a^b F(r)G'(r)dr + \int_a^b F'(r)G(r)dr.$$

* Cf. Fréchet, Annales de l'École Normale Supérieure, vol. 42 (1925), pp. 313, 316.

For the proof we apply Lemmas 1.2 and 1.1 and Theorems 1, 4, and 2.

4. **Taylor's theorem.** Taylor's theorem is valid only for convex regions as defined below. However, it should be remarked that in a linear metric space every "neighborhood" constitutes a convex region.

Convex regions. A region $\mathfrak{X}_0$ of a linear metric space $\mathfrak{X}$ is *convex* in case, for every pair of points x_1, x_2 of $\mathfrak{X}_0$ and every number r in the interval (0, 1), the point $x_1+(x_2-x_1)r$ is in $\mathfrak{X}_0$.

THEOREM 5. TAYLOR'S THEOREM. *Suppose that the space $\mathfrak{Y}$ is complete, and that the region $\mathfrak{X}_0$ of the space $\mathfrak{X}$ is convex, and suppose that the function F on $\mathfrak{X}_0$ to $\mathfrak{Y}$ has an nth variation on $\mathfrak{X}_0$. Suppose also that for every x_1, x_2 in $\mathfrak{X}_0$ the function of r, $\delta^n F(x_1+(x_2-x_1)r, x_2-x_1)$, is bounded on the interval* (0, 1) *and its set of discontinuities is of measure zero. Then for every x_1, x_2 in $\mathfrak{X}_0$ we have*

$$F(x_2) = F(x_1) + \sum_{i=1}^{n-1} \delta^i F(x_1, x_2 - x_1)/i! \quad + R_n(x_1, x_2)$$

where

$$R_n(x_1, x_2) = \int_0^1 \delta^n F(x_1 + (x_2 - x_1)r, x_2 - x_1)\frac{(1 - r)^{n-1}}{(n-1)!}\, dr.$$

We first note the fact that for $n>1$, the existence of the nth variation of F on $\mathfrak{X}_0$ implies the continuity and hence the boundedness of the function $\delta^{n-1}F(x_1+(x_2-x_1)r, x_2-x_1)$ on the interval (0, 1), by Lemma 1.1. Consider now the case $n=1$. The function $F(x_1+(x_2-x_1)r)$ has a first derivative $\delta F(x_1+(x_2-x_1)r, x_2-x_1)$ which is integrable on (0, 1), by definition of variation and Theorem 1. Then Theorem 4 yields the required result. Now assume the formula true for $n=m$, and that F has an $(m+1)$st variation with the specified properties. Then the function $G(r)=\delta^m F(x_1+(x_2-x_1)r, x_2-x_1)$ has for its derivative $\delta^{m+1}F(x_1+(x_2-x_1)r, x_2-x_1)$. The function $H(r)=-(1-r)^m/m!$ has a continuous derivative $H'(r)$. Therefore we may integrate the remainder R_m by parts (Lemma 3.1) and obtain

$$R_m(x_1, x_2) = \int_0^1 G(r)H'(r)dr = \frac{\delta^m F(x_1, x_2 - x_1)}{m!} + R_{m+1}(x_1, x_2).$$

This completes the induction.

Note that by Lemma 1.2, the Taylor's expansion has the following properties:

$$\text{(a)} \qquad \delta^i F(x, (\delta x)r) = \delta^i F(x, \delta x)r^i \qquad (i = 1, \cdots, n-1),$$

and

$$\text{(b)} \qquad \lim_{s=0} R_n(x, x + (\delta x)s)/s^{n-1} = y_*,$$

holding for every x in $\mathfrak{X}_0$, δx in $\mathfrak{X}$, and r in $\mathfrak{R}$. The uniqueness of an expansion having these properties is shown in

THEOREM 6. *Let the function F be on $\mathfrak{X}_0$ to $\mathfrak{Y}$, and let x_0 be a point of $\mathfrak{X}_0$. Then for each positive integer m there is not more than one expansion*

$$F(x) = F(x_0) + \sum_{i=1}^{m} L_i(x - x_0) + R_{m+1}(x) \tag{4.1}$$

having the properties

$$L_i((\delta x)r) = L_i(\delta x)r^i \qquad (i = 1, \cdots, m), \tag{4.2}$$

$$\lim_{s=0} \frac{R_{m+1}(x_0 + (\delta x)s)}{s^m} = y_*, \tag{4.3}$$

holding for every real number r and every δx in $\mathfrak{X}$.

For each δx of $\mathfrak{X}$, the point $x_0+(\delta x)s$ is in $\mathfrak{X}_0$ when s is sufficiently small. Now assume two expansions (4.1) with terms L_i and M_i and remainders R_{m+1} and S_{m+1} respectively. Assume also that for a certain integer k satisfying $1 \leqq k \leqq m$ we have

$$L_i = M_i \qquad (1 \leqq i < k). \tag{4.4}$$

Then by substituting $x_0+(\delta x)s$ for x in (4.1) and using property (4.2) we obtain the inequality

$$\| L_k(\delta x) - M_k(\delta x) \| \leqq \sum_{i=k+1}^{m} \Big[\| L_i(\delta x) \| + \| M_i(\delta x) \| \Big] | s |^{i-k} + \frac{\| R_{m+1}(x_0 + (\delta x)s)\|}{| s |^k} + \frac{\| S_{m+1}(x_0 + (\delta x)s)\|}{| s |^k}.$$

By property (4.3), the right hand side approaches zero with s. Therefore $L_k = M_k$ is a consequence of the assumption (4.4). By m applications of this result we arrive at the desired conclusion.

5. **Applications.** We shall discuss here only a few simple applications of Taylor's theorem. As a first consequence we state an existence theorem for the expansion whose uniqueness was shown in Theorem 6. We have already noted that the existence of such an expansion follows immediately from Taylor's theorem, but by a simple manipulation it is possible to show that existence under less restrictive hypotheses.

THEOREM 7. *Let the space $\mathfrak{Y}$ be complete, and let x_0 be a point of the space $\mathfrak{X}$. Let F be a function defined on a neighborhood $(x_0)_a$ to $\mathfrak{Y}$, which has an $(m-1)$st variation on $(x_0)_a$ satisfying the hypotheses of Taylor's theorem, and which has also an mth variation at x_0. Then the function R_{m+1} on $(x_0)_a$ to $\mathfrak{Y}$ defined by the equation*

$$F(x) = F(x_0) + \sum_{i=1}^{m} \delta^i F(x_0, x - x_0)/i! + R_{m+1}(x)$$

satisfies the condition

$$\lim_{r=0} \frac{R_{m+1}(x_0 + \delta x r)}{r^m} = y_*$$

*for every δx of $\mathfrak{X}$.**

For the case $m=1$ we omit the hypothesis concerning $\delta^{m-1}F$. The conclusion in this case follows immediately from the definition of variation. For $m>1$, we apply Taylor's theorem with $n=m-1$ to obtain the formula, valid for every fixed δx in $\mathfrak{X}$, for every real value of t sufficiently small,

$$R_{m+1}(x_0 + (\delta x)t) = \int_0^1 S(r, t) \frac{r(1-r)^{m-2}}{(m-2)!} t^m dr,$$

where

$$S(r, t) = \frac{\delta^{m-1}F(x_0 + (\delta x)tr, \delta x) - \delta^{m-1}F(x_0, \delta x)}{tr} - \delta^m F(x_0, \delta x) \qquad (tr \neq 0),$$

$$S(r, t) = y_* \qquad (tr = 0).$$

By definition of mth variation, we have

$$\lim_{t=0} S(r, t) = y_*$$

uniformly on $0 \leqq r \leqq 1$. Then application of Theorem 2 yields the required result.

Symmetry of differentials.† Symmetry has no meaning for variations as we have defined them. But for differentials of order higher than the first, as defined by Fréchet‡ and by Hildebrandt and Graves in the paper previously cited, Part III, we do have a theorem on symmetry. This is a

* In discussing these theorems for the case $\mathfrak{X}=\mathfrak{Y}=\mathfrak{R}$, W. H. Young states, in the Proceedings of the London Mathematical Society, vol. 7 (1909), p. 158, that the existence of an expansion of F at x_0 in the form given in Theorem 6 is *equivalent* to the existence of an mth derivative of F at x_0. That this is erroneous is shown by a simple example such as $F(x)=x^3\sin(1/x)$, $m=2$.

† Cf. Fréchet, these Transactions, vol. 16 (1915), p. 233.

‡ Annales de l'École Normale Supérieure, vol. 42 (1925), p. 321.

generalization of the theorem on inversion of the order of partial differentiation for ordinary functions of several variables. For present purposes it is sufficient to define differentials as follows. We say that a function F on $\mathfrak{X}_0$ to $\mathfrak{Y}$ has a *first differential* at a point x_0 of $\mathfrak{X}_0$ in case there exists a function $dF(x_0, dx)$ on $\mathfrak{X}$ to $\mathfrak{Y}$ with the following properties:

(1) $dF(x_0, (d_1x)a_1+(d_2x)a_2)=dF(x_0, d_1x)a_1+dF(x_0, d_2x)a_2$ for every pair d_1x, d_2x in $\mathfrak{X}$ and every pair of numbers a_1, a_2;

(2) for every $e>0$ there exists a $d>0$ such that, for every Δx in $\mathfrak{X}$ such that $\|\Delta x\|\leqq d$ we have

$$\|F(x_0 + \Delta x) - F(x_0) - dF(x_0, \Delta x)\| \leqq \|\Delta x\|e.$$

We say that F has an *nth differential* at x_0 in case F has an $(n-1)$st differential $d^{n-1}F(x, d_1x, \cdots, d_{n-1}x)$ which is continuous in x in a neighborhood of x_0, and the function $d^{n-1}F$ has a first differential at x_0 for each $d_1x, \cdots, d_{n-1}x$ in $\mathfrak{X} \cdots \mathfrak{X}$. We note that a function which has an nth differential at a point certainly has an nth varation at that point, and that the nth differential if continuous satisfies the requirements of Taylor's theorem on the nth variation. The above definition requires less than the one given by Fréchet, but is sufficient to validate the following theorem on symmetry.

THEOREM 8. *If the space $\mathfrak{Y}$ is complete, and the function F on $\mathfrak{X}_0$ to $\mathfrak{Y}$ has an nth differential at x_0 $(n\geqq 2)$, then $d_nF(x_0, d_1x, \cdots, d_nx)$ is symmetric in each pair of differentials d_ix, d_jx, of the independent variable x.*

Consider first the case $n=2$. Since F has a continuous first differential on a neighborhood of x_0, we can apply Taylor's theorem on such a neighborhood to obtain

$$\begin{aligned} H(a) - d^2F(x_0, d_1x, d_2x) &= \frac{F(x_0 + (d_1x)a + (d_2x)a) - F(x_0 + (d_2x)a)}{a^2} \\ &\quad - \frac{F(x_0 + (d_1x)a) - F(x_0)}{a^2} - d^2F(x_0, d_1x, d_2x) \\ &= \int_0^1 \left[\frac{dF(x_0 + (d_1x)ar + (d_2x)a, d_1x) - dF(x_0, d_1x)}{a}\right. \\ &\quad \left. - d^2F(x_0, d_1x, (d_1x)r + d_2x)\right] dr \\ &\quad - \int_0^1 \left[\frac{dF(x_0 + (d_1x)ar, d_1x) - dF(x_0, d_1x)}{a}\right. \\ &\quad \left. - d^2F(x_0, d_1x, (d_1x)r)\right] dr. \end{aligned}$$

Since dF has a first differential at x_0, we can apply the second condition in the definition of first differential, and Theorem 2 to show that $\| H(a) - d^2F(x_0, d_1x, d_2x)\|$ approaches zero with a. Since $H(a)$ is symmetric in the arguments d_1x, d_2x, so is its limit d^2F.

To complete the proof by induction, we suppose that the proposition is true for $n=m$, and that F has an $(m+1)$st differential at x_0. Then F has an $(m-1)$st differential $d^{m-1}F(x, d_1x, \cdots, d_{m-1}x)$, and an mth differential $d^mF(x, d_1x, \cdots, d_mx)$, which are continuous on a neighborhood of x_0. The differential d^mF is symmetric in the differentials $d_1x, \cdots, d_mx$, and hence $d^{m+1}F(x_0, d_1x, \cdots, d_{m+1}x)$ is also symmetric in the first m differentials. Since $d^{m+1}F$ is the second differential of $d^{m-1}F$, $d^{m+1}F$ is symmetric in its last two arguments. Hence it is symmetric in all its differential arguments, and the induction is complete.

Difference functions. Bliss, Barnett, and Lamson have made some use of the notion of difference function.* Taylor's theorem shows that a function F having continuous differentials up to order n has also continuous difference functions up to order n. The converse is true only for $n=1$. Other relations between different definitions of differentiability in abstract spaces are easily derived.

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* Cf. Bliss, these Transactions, vol. 21 (1920), p. 79; Barnett, American Journal of Mathematics, vol. 44 (1922), p. 172; Lamson, the same Journal, vol. 42 (1920), p. 243.

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