ON SETS OF FUNCTIONS OF A GENERAL VARIABLE*

BY

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A theory of functions of a general variable is due to E. H. Moore.† By a general variable is meant a variable of which the range is a class of elements

$$\mathfrak{Q} \equiv [q]$$

entirely unconditioned. Particular instances of the theory may be obtained by specializing the class $\mathfrak{Q}$. For example the class $\mathfrak{Q}$ may consist of a finite number of elements, a denumerable infinitude of elements, or a continuous infinitude of elements. The elements themselves may be numbers, real, complex, or hypercomplex; or they may be without numerical character.

A real (single-valued) function μ on a general range $\mathfrak{Q}$ is a correspondence between the elements of $\mathfrak{Q}$ and a class of real numbers, such that for every element q of $\mathfrak{Q}$ there is a definite corresponding real number, notationally $\mu(q)$.

A property of such a function which is in no way dependent for its definition on any special character which the range $\mathfrak{Q}$ may have in special instances is said to be a *property of general reference*. For example, a function may be: (a) everywhere zero on $\mathfrak{Q}$, or (b) everywhere positive on $\mathfrak{Q}$, or (c) everywhere negative on $\mathfrak{Q}$, or (d) somewhere positive and nowhere negative on $\mathfrak{Q}$, or (e) somewhere negative and nowhere positive on $\mathfrak{Q}$. We shall use the following symbolic statements to indicate that a function μ has these properties respectively:

(a) $\mu = 0 (\mathfrak{Q}),$	(d) $\mu \geq ' 0 (\mathfrak{Q}),$
(b) $\mu > 0 (\mathfrak{Q}),$	(e) $\mu \leq ' 0 (\mathfrak{Q}).$
(c) $\mu < 0 (\mathfrak{Q}),$	

In the present paper we shall be particularly interested in functions which have the property (d) or the property (e). A function which has either of these properties will be said to be *M-definite*. In other words an *M-definite* function is one which is not identically zero and does not change sign on $\mathfrak{Q}$.

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† The theory is called by Professor Moore "General Analysis," and is developed in his *New Haven Mathematical Colloquium Lectures*, New Haven, 1910.

Properties of general reference may pertain to a *set* of real functions

$$\mu_1, \mu_2, \dots, \mu_m,$$

each on the general range $\mathfrak{Q}$.* For example, if there exists a set of real constants $c_1, c_2, \dots, c_m$ (not all zero) such that

$$(1) \quad \sum_{i=1}^m c_i \mu_i = 0 \quad (\mathfrak{Q}),$$

the set of functions is said to be linearly dependent; otherwise it is linearly independent. Other properties of general reference for a set of functions are obtained by replacing the sign = in (1) by the signs used in (b)-(e). For the special instance in which $\mathfrak{Q}$ consists of a finite number of elements, these properties have been considered by the author.†

The object of the present paper is to study the condition

$$(2) \quad \sum_{i=1}^m c_i \mu_i \geq 0 \quad (\mathfrak{Q}),$$

that is, the condition that a given set of functions on a general range admit an M -definite linear combination.

The central feature of the theory is a certain integral-valued function of the set of functions $\{\mu_i\}$ which we shall call the M -rank of the set. In terms of it may be stated a necessary and sufficient condition that (2) admit a solution $(c_1, c_2, \dots, c_m)$ and the maximum number of c 's that may be zero in such a solution. These results are stated in §4, the earlier sections being preparatory to the definition of M -rank.

1. Reduction and composition of a general range $\mathfrak{Q}$ relative to a function on that range. We consider a class $\mathfrak{Q}$ of elements q , notationally

$$\mathfrak{Q} \equiv [q].$$

Let μ be any real single-valued function on $\mathfrak{Q}$. We shall have occasion to consider three subclasses of $\mathfrak{Q}$, relative to μ , defined as follows:

$$\begin{aligned} \mathfrak{Q}_P^{(\mu)} &\equiv [\text{all } q \text{ such that } \mu(q) > 0] \equiv [p^{(\mu)}], \\ \mathfrak{Q}_N^{(\mu)} &\equiv [\text{all } q \text{ such that } \mu(q) < 0] \equiv [n^{(\mu)}], \\ \mathfrak{Q}_Z^{(\mu)} &\equiv [\text{all } q \text{ such that } \mu(q) = 0] \equiv [z^{(\mu)}]. \end{aligned}$$

* It may be noted, however, that a property of such a *set* of functions can be considered as a property of a *single* function μ' on a composite range $\mathfrak{Q}'$, the elements q' of the range $\mathfrak{Q}'$ being bipartite elements of the form $q' = (q, j)$, the first part q having the range $\mathfrak{Q}$ and the second part j having the finite range consisting of the numbers $1, 2, \dots, m$.

† Annals of Mathematics, (2), vols. 20, 27, and 28.

From these subclasses we form a certain composite range relative to the function μ ,

$$\mathfrak{Q}^{(\mu)} \equiv \mathfrak{Q}_P^{(\mu)} \mathfrak{Q}_N^{(\mu)} + \mathfrak{Q}_Z^{(\mu)},$$

consisting of the logical sum of the two classes

$$\mathfrak{Q}_P^{(\mu)} \mathfrak{Q}_N^{(\mu)} \equiv [p^{(\mu)} n^{(\mu)}] \text{ and } \mathfrak{Q}_Z^{(\mu)} \equiv [z^{(\mu)}].$$

The elements of the new class

$$\mathfrak{Q}^{(\mu)} \equiv [q^{(\mu)}]$$

will therefore be of two kinds: (1) bipartite elements $p^{(\mu)} n^{(\mu)}$ of which the first part $p^{(\mu)}$ ranges over $\mathfrak{Q}_P^{(\mu)}$ and the second part $n^{(\mu)}$ ranges over $\mathfrak{Q}_N^{(\mu)}$ independently; and (2) unipartite elements $z^{(\mu)}$ ranging over $\mathfrak{Q}_Z^{(\mu)}$.

The process here indicated may evidently be repeated. If σ is a real single-valued function on the new range $\mathfrak{Q}^{(\mu)}$, it determines three subclasses of the range, which may be denoted by $\mathfrak{Q}_P^{(\mu\sigma)}$, $\mathfrak{Q}_N^{(\mu\sigma)}$, and $\mathfrak{Q}_Z^{(\mu\sigma)}$; and from these may be formed the composite class

$$\mathfrak{Q}^{(\mu\sigma)} \equiv \mathfrak{Q}_P^{(\mu\sigma)} \mathfrak{Q}_N^{(\mu\sigma)} + \mathfrak{Q}_Z^{(\mu\sigma)}.$$

The process may be repeated indefinitely, provided at each stage a reducing function is available.

It will be noted that if at any stage the reducing function is everywhere positive or everywhere negative, the new composite range will be a null class; while if the reducing function is identically zero, the new composite range will be identical with the old range.

2. Reduced outer multiplication. The reducing function μ determines with any second function ν on the range $\mathfrak{Q}$, a real single-valued function on the composite range $\mathfrak{Q}^{(\mu)}$ which we will call their *reduced outer product*, and denote by $((\mu\nu))$. It is defined as follows:*

$$((\mu\nu)) \equiv \begin{cases} \mu(p)\nu(n) - \nu(p)\mu(n) & \text{for } pn \text{ on } \mathfrak{Q}_P^{(\mu)} \mathfrak{Q}_N^{(\mu)}, \\ \nu(z) & \text{for } z \text{ on } \mathfrak{Q}_Z^{(\mu)}. \end{cases}$$

This multiplication is not commutative. Its most obvious property is that $((\mu\mu)) = 0$ on $\mathfrak{Q}^{(\mu)}$. Other properties are developed in the next section.

* The outer product of two functions $f(x)$ and $g(x)$, where x is a real variable on a closed interval, has been defined as $f(x)g(y) - g(x)f(y)$. See Kowalewski's *Funktionenräume*, Wiener Sitzungsberichte, vol. 120.

3. Reduction of a set of functions. Consider a set of real, single-valued functions

$$\{\mu_i\} \quad \mu_1, \mu_2, \dots, \mu_m,$$

on a general range $\mathfrak{Q}$.

Relative to any one of the functions, say μ_k , we may determine a second set of m functions

$$\{\mu_i^{(k)}\} \quad ((\mu_k\mu_1)), ((\mu_k\mu_2)), \dots, ((\mu_k\mu_m)),$$

each of which is the reduced outer product of the corresponding function in the given set by μ_k . This new set of functions on the composite range $\mathfrak{Q}^{(\mu_k)}$ will be called a reduced set, or more explicitly, the reduction of the set $\{\mu_i\}$ with respect to μ_k . It has the notable property that its k th constituent is identically zero. The usefulness of this type of reduction lies in the following lemma:

If the reducing function μ_k is not M -definite, then the set $\{\mu_i\}$ admits an M -definite linear combination if and only if the same is true of the reduced set $\{\mu_i^{(k)}\}$. More explicitly, every set of constants $c_1, c_2, \dots, c_m$, satisfying the condition

$$(2) \quad \sum_{i=1}^m c_i \mu_i \geq 0 \quad (\mathfrak{Q}),$$

will also satisfy the reduced condition

$$(3) \quad \sum_{i=1}^m c_i ((\mu_k\mu_i)) \geq 0 \quad (\mathfrak{Q}^{(\mu_k)});$$

and conversely, every solution $c_1, c_2, \dots, c_m$ of (3) yields a solution of (2) if the constant c_k , which is arbitrary in a solution of (3), be suitably chosen.

We note first that if $\mu_k = 0(\mathfrak{Q})$, the proposition is true though trivial, since in that case the two sets $\{\mu_i\}$ and $\{\mu_i^{(k)}\}$ are identically the same. We may then assume in the proof that the function μ_k changes sign on $\mathfrak{Q}$.

Suppose that the condition (2) has a solution $\bar{c}_1, \bar{c}_2, \dots, \bar{c}_m$.

Taking account of the three subclasses of the range $\mathfrak{Q}$ relative to the function μ_k , we obtain from this hypothesis the three statements

$$(4) \quad \sum_{i=1}^m \bar{c}_i \mu_i(p) \geq 0 \quad (p \text{ on } \mathfrak{Q}_P^{(\mu_k)}),$$

$$(5) \quad \sum_{i=1}^m \bar{c}_i \mu_i(n) \geq 0 \quad (n \text{ on } \mathfrak{Q}_N^{(\mu_k)}),$$

$$(6) \quad \sum_{j=1}^m \bar{c}_j \mu_j(z) \geq 0 \quad (z \text{ on } \mathfrak{Q}_z^{(\mu_k)}),$$

with the understanding that the sign $\geq$ has the significance of $\geq'$ in at least one of the three statements.

Multiplying (4) by $-\mu_k(n)$ and (5) by $\mu_k(p)$ and adding the results, we have

$$\sum_{j=1}^m \bar{c}_j [\mu_k(p) \mu_j(n) - \mu_j(p) \mu_k(n)] \geq 0 \quad (pn \text{ on } \mathfrak{Q}_P^{(\mu_k)} \mathfrak{Q}_N^{(\mu_k)}).$$

This together with (6) may be written

$$(7) \quad \sum_{j=1}^m \bar{c}_j ((\mu_k \mu_j)) \geq' 0 \quad (\mathfrak{Q}^{(\mu_k)}),$$

which proves the first part of the proposition.

Suppose conversely that the condition (3) has a solution $\bar{c}_1, \bar{c}_2, \dots, \bar{c}_m$, as expressed by (7).

We note first that the constant $\bar{c}_k$ is arbitrary, since its coefficient is identically zero. We may therefore omit the term corresponding to $j=k$ from the summation, indicating the omission by an apostrophe, and write (7) in the two statements

$$(8) \quad \sum_{j=1}^{m'} \bar{c}_j [\mu_k(p) \mu_j(n) - \mu_j(p) \mu_k(n)] \geq 0 \quad (pn \text{ on } \mathfrak{Q}_P^{(\mu_k)} \mathfrak{Q}_N^{(\mu_k)}),$$

$$(9) \quad \sum_{j=1}^{m'} \bar{c}_j \mu_j(z) \geq 0 \quad (z \text{ on } \mathfrak{Q}_z^{(\mu_k)}),$$

one of the signs $\geq$ having the significance of $\geq'$.

Since $-\mu_k(p) \mu_k(n)$ is positive, we may obtain from (8) an equivalent statement

$$(10) \quad \sum_{j=1}^{m'} \bar{c}_j \frac{\mu_j(p)}{\mu_k(p)} \geq \sum_{j=1}^{m'} \bar{c}_j \frac{\mu_j(n)}{\mu_k(n)} \quad (\mathfrak{Q}_P^{(\mu_k)} \mathfrak{Q}_N^{(\mu_k)}).$$

Now the values on the left side of (10) must have a greatest lower bound, and those on the right a least upper bound, which bounds may or may not coincide. In any case we may choose the arbitrary $\bar{c}_k$ so that

$$\sum_{j=1}^{m'} \bar{c}_j \frac{\mu_j(p)}{\mu_k(p)} \geq -\bar{c}_k \geq \sum_{j=1}^{m'} \bar{c}_j \frac{\mu_j(n)}{\mu_k(n)}.$$

And from this double relation we obtain

$$\sum_{j=1}^m{}' c_{ij}\mu_j(p) + c_{hk}\mu_k(p) \geq 0, \quad \sum_{j=1}^m{}' c_{ij}\mu_j(n) + c_{hk}\mu_k(n) \geq 0,$$

which together with (9) may be written

$$\sum_{j=1}^m c_{ij}\mu_j \geq' 0 \quad (\mathfrak{Q}).$$

This completes the proof of the lemma.

The process of reduction may be repeated. As reduction of the set $\{\mu_j\}$ with respect to μ_k yields the set $\{\mu_j^{(k)}\}$, so reduction of this latter set with respect to one of its constituents $\mu_i^{(k)}$ yields a set which we shall denote by $\{\mu_j^{(kl)}\}$.

In general, we define the set

$$(11) \quad \{\mu_j^{(k_1 k_2 \dots k_s)}\}$$

as the reduction of the set

$$\{\mu_j^{(k_1 k_2 \dots k_{s-1})}\}$$

with respect to the function $\mu_{k_s}^{(k_1 k_2 \dots k_{s-1})}$.

The set (11) will be called an *sth* reduction of the set $\{\mu_j\}$. Clearly there are many *sth* reductions, depending on the choice of the sequence $k_1, k_2, \dots, k_s$. If the integers in this sequence are distinct, *s* constituent functions of the *sth* reduction are identically zero.

4. **The *M*-rank of a set of functions.** We recall that a function μ is said to be *M*-definite if either of the conditions

$$\mu \geq' 0 \quad (\mathfrak{Q}) \quad \text{or} \quad \mu \leq' 0 \quad (\mathfrak{Q})$$

is satisfied.

*A set of m functions $\{\mu_j\}$ is said to be of *M*-rank r if at least one of its $(m-r)$ th reductions contains an *M*-definite constituent function while no one of its $(m-r-1)$ th reductions contains such a constituent function. If the given set contains an *M*-definite function the set is of *M*-rank m . If neither it nor any of its reductions contains such a function it is of *M*-rank zero.*

THEOREM. *A necessary and sufficient condition that the set of m functions admit an *M*-definite linear combination is that its *M*-rank be greater than zero.*

If the M -rank is r ($0 < r < m$), then there is a subset of $m-r+1$ of the functions which admits an M -definite linear combination, but there is no subset of $m-r$ functions for which this is true.

First, if the given set $\{\mu_i\}$ admits an M -definite linear combination, its M -rank is greater than zero. For otherwise the $(m-1)$ th reduction $\{\mu_i^{(1,2,\dots,m-1)}\}$ would contain no M -definite function, while the corresponding reduced condition

$$\sum_{i=1}^m c_i \mu_i^{(1,2,\dots,m-1)} \geq 0 \quad (\mathfrak{Q}^{(1,2,\dots,m-1)})$$

must admit a solution $c_1, c_2, \dots, c_m$ by the lemma of §3. These two requirements are incompatible since all functions of the $(m-1)$ th reduction are zero except one.

Conversely, suppose the M -rank of the given set is r (> 0). Then there is an $(m-r)$ th reduction of the set which contains an M -definite function. Suppose, for simplicity of notation, it is $\{\mu_i^{(1,2,\dots,m-r)}\}$, and suppose the $(m-r+1)$ th function of this set is M -definite. Then the condition

$$\sum_{i=1}^m c_i \mu_i^{(1,2,\dots,m-r)} \geq 0 \quad (\mathfrak{Q}^{(1,2,\dots,m-r)})$$

admits a solution $c_1, c_2, \dots, c_m$, in which

$$c_i = 0 \text{ for } i > m-r+1,$$

$$c_i \text{ is arbitrary for } i < m-r+1,$$

$$c_{m-r+1} = +1 \text{ or } -1 \text{ according as } \mu_{m-r+1}^{(1,2,\dots,m-r)} \text{ is positive or negative.}$$

Hence by repeated application of the lemma of §3, we find that the condition

$$\sum_{i=1}^m c_i \mu_i \geq 0 \quad (\mathfrak{Q})$$

admits a solution in which $c_i = 0$ ($i > m-r+1$), suitable values being assigned to c_i ($i < m-r+1$). The given set therefore admits an M -definite linear combination; indeed a subset of $m-r+1$ of them has this property.

It remains to be proved that when the M -rank is r , no subset of $m-r$ of the given functions admits an M -definite linear combination. Suppose for definiteness that the subset

$$(12) \quad \mu_1, \mu_2, \dots, \mu_{m-r}$$

did this property. Then by the first proposition of the theorem (already established) the M -rank of this subset must be greater than zero, call it r' . That means that some $(m-r-r')$ th reduction of the subset (12) would contain an M -definite constituent function. The corresponding $(m-r-r')$ th reduction of the original set of m functions would contain the same M -definite function, and hence the M -rank of the set would be $r+r'$, contrary to our assumption that it was r .

This completes the proof of the theorem.

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