# ON THE EXPANSION OF ANALYTIC FUNCTIONS OF THE COMPLEX VARIABLE IN GENERALIZED TAYLOR'S SERIES*

BY

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**1. Introduction.** In a previous paper‡ in these Transactions the representation of real functions by generalized Taylor's series was discussed. The methods there employed are not applicable to the corresponding representation of functions of the complex variable. In the present paper new methods are developed, and the representation of an arbitrary function of the complex variable is obtained.

Let us denote the function to be represented by $f(z)$ and the functions in terms of which the representation is to be made by

$$u_0(z),\ u_1(z),\ \cdots . \tag{1}$$

As in the previous paper we begin by a sort of "normalization"§ of this sequence, replacing it by a sequence

$$g_0(z),\ g_1(z),\ \cdots , \tag{2}$$

where the function $g_n(z)$ is a linear combination of $u_0(z)$, $u_1(z)$, $\cdots$, $u_n(z)$, and has a zero of order $n$ exactly at the origin. The possibility of such a normalization imposes, of course, a restriction on the sequence (1) which we shall specify in §2. Indeed, a number of other conditions must be imposed, as one would expect, in order to insure the representation of an arbitrary analytic function $f(z)$ by means of linear combinations of the functions (1). These conditions are most conveniently expressed in terms of the set (2), and amount briefly to demanding that they be analytic and that $|1-n!g_n(z)/z^n|$ should decrease at least as rapidly as $1/n$ when $n$ becomes infinite. It is then found that the region of convergence of a series

$$\sum_{n=0}^{\infty} c_n g_n(z) \tag{3}$$

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‡ D. V. Widder, *A generalization of Taylor's series*, these Transactions, vol. 30 (1928), p. 126.

§ This process corresponds to orthogonalizing and normalizing the sequence of prescribed functions by the Schmidt method, when the method of approximation employed is that of least squares.

is circular, and that an analytic function $f(z)$ can be represented in a series of this type inside a circle with center at the origin and reaching out to the nearest singularity of $f(z)$, just as in the case of Taylor's series. These facts we shall prove in §§2 and 3.

In §4 we compare the result just stated with one of G. D. Birkhoff* on generalized Taylor's series. We show that the latter result demands implicitly that the functions $v_n(z)$ of Birkhoff* should have exactly $n$ zeros inside the circle $|z|<1$. In that respect the function is like the function $g_n(z)$ of the present paper, which has $n$ zeros coincident at the origin and no others in the unit circle if $n$ is sufficiently large. But it is found that the conditions imposed on the mode of decrease of $|h_n(z)|$ in the present paper are of such a nature that a class of expansions arise which can not be treated by Birkhoff's theorem. When the expansion can be treated by both methods a comparison of the two methods of determining the coefficients (for a common function $f(z)$) yields an interesting identity.

Finally, we cite several other papers concerned with problems related to our own, but which do not treat the case considered by us:

S. Kakeya, *On the expansion of analytic function*, Proceedings of the Physico-Mathematical Society of Japan, (3), vol. 2 (1920), p. 96.

Y. Okada, *On a certain expansion of analytic function*, Tôhoku Mathematical Journal, vol. 22 (1923), p. 325.

J. L. Walsh, *On the expansion of analytic functions in series of polynomials*, these Transactions, vol. 26 (1924), p. 155.

S. Izumi, *On the expansion of analytic function*, Tôhoku Mathematical Journal, vol. 28 (1927), p. 97.

R. F. Graesser, *A certain general type of Neumann expansion and expansions in hypergeometric functions*, American Journal of Mathematics, vol. 49 (1927), p. 577.

J. L. Walsh, *On the expansion of analytic functions in series of polynomials and in series of other analytic functions*, these Transactions, vol. 30 (1928), p. 307.

We wish to emphasize the fact that the series of the present paper converge throughout the circular region of analyticity of the function $f(z)$ and represent $f(z)$ there, a property not shared by the series of Izumi and Graesser, for example. This property is retained from Taylor's series in the generalization, and is also shared by the series of Birkhoff and Walsh.

2. **The region of convergence.** Let the functions (1) and $f(z)$ be analytic

* G. D. Birkhoff, *Sur une généralisation de la série de Taylor*, Comptes Rendus, vol. 164, p. 942. For a statement of the theorem see §4 of the present paper.

in the circle $|z| \leqq R$. Denote by $s_n(z)$ a linear combination of the first $n+1$ functions (1):

$$s_n(z) = c_0u_0(z) + c_1u_1(z) + c_2u_2(z) + \cdots + c_nu_n(z),$$

and determine the constants of combination $c_0, c_1, \cdots, c_n$ in such a way that $f(z)-s_n(z)$ shall have a zero of order $n+1$ at $z=0$. This is possible if the Wronskian

$$W_n(z) = W[u_0(z), u_1(z), \cdots, u_n(z)] = \begin{vmatrix} u_0(z) & u_1(z) & \cdots u_n(z) \\ u_0'(z) & u_1'(z) & \cdots u_n'(z) \\ \cdot & \cdot & \cdots \cdot \\ \cdot & \cdot & \cdots \cdot \\ u_0^{(n)}(z) & u_1^{(n)}(z) & \cdots u_n^{(n)}(z) \end{vmatrix}$$

is different from zero at $z=0$. We assume that $W_n(z) \neq 0$ for $|z| \leqq R$ and for all* $n$. The computations of the previous paper are evidently applicable here, so that $s_n(z)$ takes the form

$$s_n(z) = L_0f(0)g_0(z) + L_1f(0)g_1(z) + \cdots + L_nf(0)g_n(z),$$

where

$$g_n(z) = \frac{1}{W_n(0)} \begin{vmatrix} u_0(0) & u_1(0) & \cdots u_n(0) \\ u_0'(0) & u_1'(0) & \cdots u_n'(0) \\ \cdot & \cdot & \cdots \cdot \\ \cdot & \cdot & \cdots \cdot \\ u_0^{(n-1)}(0) & u_1^{(n-1)}(0) & \cdots u_n^{(n-1)}(0) \\ u_0(z) & u_1(z) & \cdots u_n(z) \end{vmatrix},$$

$$L_nf(z) = \frac{W[u_0(z), u_1(z), \cdots, u_{n-1}(z), f(z)]}{W_{n-1}(z)} \quad (n = 0,1,2, \cdots).$$

Since the functions $g_n(z)$ are linear combinations of the functions (1), it is clear that $L_nf(z)$ may equally well be expressed in terms of $g_n(z)$:

$$L_nf(z) = \frac{W[g_0(z), g_1(z), \cdots, g_{n-1}(z), f(z)]}{W[g_0(z), g_1(z), \cdots, g_{n-1}(z)]}.$$

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* J. L. Walsh has called the author's attention to the fact that if we admit the rearrangement of the order of the functions in the sequence (1), then it is only necessary to assume here that $W_n(0) \neq 0$ for *n sufficiently large*. The proof of this fact becomes evident by use of a lemma of Walsh to be found in the American Journal of Mathematics, vol. 42 (1920), p. 93.

It is seen that the functions $g_n(z)$ have the properties enunciated in the introduction:

$$g_n^{(k)}(0) = 0 \qquad (k = 0,1,2,\cdots,n-1), \qquad g_n^{(n)}(0) = 1.$$

Since $g_n(z)$ is analytic for $|z| \leqq R$, it must have the form

$$g_n(z) = z^n[1 + h_n(z)]/n!,$$

where $h_n(z)$ is analytic in the same circle and vanishes at the center. We further restrict the sequence (2) by the

CONDITIONS A: (a) *The functions* $g_n(z)$ *are analytic for* $|z| \leqq R$; (b) $g_n(z)=z^n[1+h_n(z)]/n!$, *where* $h_n(z)$ *is analytic for* $|z|\leqq R$ *and vanishes at* $z=0$; (c) *a constant* $M$ *independent of* $n$ *exists such that* $|h_n(z)|\leqq M/(n+1)$ *for* $|z|\leqq R$.

We can now state

THEOREM I. *Let the functions* $g_n(z)$ *satisfy Conditions* A *in* $|z|\leqq R$. *Then if the series*

$$\sum_{n=0}^{\infty} c_n g_n(z) \tag{3}$$

*converges for one value* $z=\xi$, $|\xi|\leqq R$, *it converges uniformly for* $|z|\leqq r$ *for any* $r<|\xi|$ *to an analytic function.*

To prove this, note first that the convergence of (3) for $z=\xi$ implies the existence of a constant $N$ independent of $n$ such that

$$|c_n g_n(\xi)| \leqq N,$$

so that a dominant series for (3) can be obtained at once:

$$\sum_{n=0}^{\infty} c_n g_n(z) \ll N \sum_{n=0}^{\infty} \frac{|g_n(z)|}{|g_n(\xi)|}.$$

By Conditions A,

$$|g_n(z)| \leqq \frac{|z|^n}{n!}\left(1 + \frac{M}{n+1}\right), \qquad |z| \leqq R,$$

and

$$|g_n(\xi)| = \frac{|\xi|^n}{n!}\left|1 + h_n(\xi)\right| \geqq \frac{|\xi|^n}{n!}\left(1 - \frac{M}{n+1}\right).$$

Hence

$$\sum_{n=m}^{\infty} c_n g_n(z) \ll N \sum_{n=m}^{\infty} \frac{|z|^n}{|\xi|^n} \frac{\left(1+\dfrac{M}{n+1}\right)}{\left(1-\dfrac{M}{n+1}\right)} \ll N \sum_{n=m}^{\infty} \frac{r^n}{|\xi|^n} \frac{\left(1+\dfrac{M}{n+1}\right)}{\left(1-\dfrac{M}{n+1}\right)}.$$

Here $m$ is chosen so large that $m+1>M$. If $r<|\xi|$, (3) clearly converges uniformly for $|z|\leqq r$, and the theorem is established.

One may show, just as for power series, that if (3) diverges for a value $z=\xi$ with $|\xi|<R$, it diverges for $|\xi|\leqq|z|\leqq R$. It follows then that (3) converges nowhere except at $z=0$, or converges for $|z|<r<R$ and diverges for $r<|z|\leqq R$, or converges for $|z|\leqq R$. It is easy to construct examples to show that all three possibilities actually arise. It should be pointed out that in the third case we have not proved that the true region of convergence is circular, for (3) may converge outside of the circle $|z|\leqq R$ in which Conditions A hold. In case the region of convergence is a circle we define the circle as the *circle of convergence* and its radius as the *radius of convergence.*

THEOREM II. *The radius of convergence $\rho$ of* (3), *if it is less than $R$, is given by the equation*

$$\frac{1}{\rho}=\overline{\lim_{n=\infty}}(|c_n|/n!)^{1/n}. \tag{4}$$

For, if (4) holds then to every positive $\epsilon$ there corresponds an integer $m$ such that

$$|c_n|<(1+\epsilon)^n n!/\rho^n, \qquad n\geqq m,$$

and by virtue of Conditions A we have

$$\sum_{n=m}^{\infty}c_n g_n(z)\ll\sum_{n=m}^{\infty}\frac{(1+\epsilon)^n}{\rho^n}|z|^n\left(1+\frac{M}{n+1}\right).$$

Hence (3) converges for $|z|<\rho$. It diverges for $|z|>\rho$, for by (4) it follows that

$$|c_n|>\frac{(1-\epsilon)^n}{\rho^n}n!, \qquad \epsilon>0,$$

for an infinite number of integers $n$. For these integers

$$|c_n g_n(z)|>\frac{|z|^n}{\rho^n}\left(1-\frac{M}{n+1}\right)(1-\epsilon)^n,$$

and if $|z|>\rho$ and $(1-\epsilon)/|z|>\rho$ the general term of (3) can not approach zero as $n$ becomes infinite. Hence $\rho$ must be the radius of convergence. A similar proof shows that if $\overline{\lim}\,(|c_n|/n!)^{1/n}=\infty$ then $\rho=0$, and we have the first case mentioned above.

3. The representation of an arbitrary analytic function. Let us suppose that $f(z)$ is analytic in the circle $|z|<\rho\leqq R$. We shall show that the series

$$(5) \qquad \sum_{n=0}^{\infty} L_n f(0) g_n(z)$$

converges and represents $f(z)$ in that circle provided that the functions $g_n(z)$ satisfy Conditions A in $|z| \leqq R$. To prove this we first prove that (5) converges for $|z| < \rho$. This is done by use of the following formula:

$$(6) \quad f^{(n)}(0) = L_0 f(0) g_0^{(n)}(0) + L_1 f(0) g_1^{(\ )}(0) + \cdots + L_{n-1} f(0) g_{n-1}^{(n)}(0) + L_n f(0),$$

taken from the previous paper.* A moment's consideration will show that the proof there given for the real variable applies equally well here. By its use we can determine the mode of increase of $|L_n f(0)|$. We first transform the formula so as to involve the functions $h_n(z)$ instead of $g_n(z)$. By the definition of $h_n(z)$ it is clear that

$$g_n^{(k)}(z) = \sum_{p=0}^{k} \binom{k}{p} \left(\frac{z^n}{n!}\right)^{(p)} (1 + h_n(z))^{(k-p)},$$

where

$$\binom{k}{p} = \frac{k!}{p!(k-p)!}.$$

For formula (6) we shall want $k > n$. Then

$$g_n^{(k)}(0) = \binom{k}{n} (1 + h_n(z))^{(k-n)} \Big|_{z=0} = \binom{k}{n} h_n^{(k-n)}(0).$$

Hence (6) reduces to

$$(7) \qquad L_n f(0) = f^{(n)}(0) - \sum_{p=0}^{n-1} \binom{n}{p} h_p^{(n-p)}(0) L_p f(0).$$

Let $r$ be an arbitrary positive constant less than $\rho$. The analyticity of $f(z)$ in $|z| < \rho$ implies the existence of a constant $L$ independent of $n$ such that

$$(8) \qquad |f^{(n)}(0)| \leqq \frac{Ln!}{r^n}, \qquad |z| \leqq r.$$

Since $h_n(z)$ is analytic in $|z| \leqq r$ and satisfies the inequality

$$|h_n(z)| \leqq M/(n+1), \qquad |z| \leqq R,$$

it follows that

$$(9) \qquad |h_n^{(k)}(0)| \leqq \frac{M}{n+1} \frac{k!}{r^k}.$$

Now

* Loc. cit., p. 131.

$$L_0f(0) = f(0), \qquad |L_0f(0)| \leqq L,$$
$$L_1f(0) = f'(0) - L_0f(0)h_0'(0),$$
$$|L_1f(0)| \leqq L/r + LM/r = L(1+M)/r.$$

Proceeding in this way one is led to induce the inequality

$$|L_nf(0)| \leqq L(M+1)(M+2)\cdots(M+n)/r^n.$$

To establish this result we may assume it true for $n=0, 1, 2, \cdots, m-1$, and show as a consequence that it is also true for $n=m$. By (7), (8), (9), and (10)

$$|L_mf(0)| \leqq |f^{(m)}(0)| + \sum_{p=0}^{m-1}\binom{m}{p}|L_pf(0)|\cdot|h_p^{(m-p)}(0)|$$
$$\leqq \frac{Lm!}{r^m} + \sum_{p=0}^{m-1}\binom{m}{p}\frac{L}{r^p}(M+1)(M+2)\cdots(M+p)\frac{M}{p+1}\,\frac{(m-p)!}{r^{m-p}}.$$

It is easily verified that the expression on the right of this inequality is precisely

$$(10) \qquad L(M+1)(M+2)\cdots(M+m)/r^m,$$

so that the induction is complete.

We can now establish the convergence of (5) for $|z|<\rho$. For, we have

$$\sum_{n=0}^{\infty}L_nf(0)g_n(z) \ll L\sum_{n=0}^{\infty}\frac{(M+1)(M+2)\cdots(M+n)}{r^n}\,\frac{|z|^n}{n!}\left(1+\frac{M}{n+1}\right).$$

Simple tests show that the dominant series converges for $|z|<r$. Since $r$ was arbitrary it follows that (5) converges absolutely for $|z|<\rho$. It is equally evident that (5) converges uniformly for $|z|\leqq r$ if $r<\rho$. Consequently the sum of the series, which we denote by $\phi(z)$, must be an analytic function in the circle $|z|<\rho$. Moreover, the series may be differentiated term by term as often as desired by a familiar theorem of Weierstrass.

It will now be established that $\phi(z)=f(z)$ for $|z|<\rho$. Setting $z=0$ it is seen immediately that $\phi(0)=f(0)$. Differentiating (5) term by term $k$ times, it is clear that

$$\phi^{(k)}(0) = \sum_{n=0}^{\infty}L_nf(0)g_n^{(k)}(0) = \sum_{n=0}^{k}L_nf(0)g_n^{(k)}(0).$$

But by formula (6)

$$f^{(k)}(0) = \sum_{n=0}^{k}L_nf(0)g_n^{(k)}(0).$$

Hence

$$f^{(k)}(0) = \phi^{(k)}(0) \qquad (k = 0,1,2, \cdots).$$

Since $f(z)$ and $\phi(z)$ are both analytic in the neighborhood of the origin, thev must coincide throughout their region of analyticity, so that

$$f(z) = \sum_{n=0}^{\infty} L_n f(0) g_n(z), \qquad |z| < \rho.$$

We have thus proved not only that the expansion is possible but also that it is unique (the functions $g_n(z)$ supposed given). We sum up the results in

THEOREM III. *Let the functions $g_n(z)$ satisfy Conditions* A *in the circle $|z| \leqq R$. Then any function $f(z)$ analytic in $|z| < \rho \leqq R$ can be expanded in one and only one way by a series of the form*

$$f(z) = \sum_{n=0}^{\infty} c_n g_n(z)$$

*convergent in $|z| < \rho$. Moreover, the coefficients $c_n$ are determined by the formula*

$$c_n = \left. \frac{W[g_0(z), g_1(z), \cdots, g_{n-1}(z), f(z)]}{W[g_0(z), g_1(z), \cdots, g_{n-1}(z)]} \right|_{z=0},$$

*or by the recursion formula*

$$c_n = f^{(n)}(0) - c_0 g_0^{(n)}(0) - c_1 g_1^{(n)}(0) - \cdots - c_{n-1} g_{n-1}^{(n)}(0).$$

It is scarcely necessary to point out that (3) reduces to a Taylor's series if $h_n(z) \equiv 0$, when Conditions A are surely satisfied. The existence of other functions $g_n(z)$ satisfying these conditions can not be held in question, so that (3) is a bona fide generalization of Taylor's series.

**4. Relation of the series to Birkhoff's series.** In order to make the proposed comparison we begin by stating

BIRKHOFF'S THEOREM.* *Let $v_0(z), v_1(z), \cdots$ be a sequence of functions each of which is analytic in a circle $|z| \leqq 1$. If the series*

(11) $$|v_0(z) - 1| + |v_1(z) - z| + |v_2(z) - z^2| + \cdots$$

*converges uniformly for $|z| = 1$ to a value less than unity, every function $f(z)$ analytic in $|z| \leqq 1$ can be represented by an absolutely and uniformly convergent series of the form*

(12) $$c_0 v_0(z) + c_1 v_1(z) + \cdots.$$

* We have altered the notation in order to avoid confusion. Also we state the result for the unit circle rather than for a circle of arbitrary radius in order to facilitate the comparison.

*Here $c_n$ is given by*

$$c_n = \frac{1}{2\pi i}\int_C \frac{g(t)}{t^{n+1}}dt$$

*where the integration is taken over the circle $C$, $|t|=1$, in the positive sense, and where $g(z)$ is the solution of the integral equation*

$$f(z) = g(z) + \frac{1}{2\pi i}\int_C \left(\sum_{n=0}^{\infty} \frac{v_n(z) - z^n}{t^{n+1}}\right) g(t)dt.$$

We wish to point out the resemblance of series (12) to series (3) by showing that under the conditions imposed on (11) $v_n(z)$ must have exactly $n$ zeros in the circle $|z|<1$. For, since the sum of the series (11) is less than unity on the unit circle, the general term must surely be less than unity there,

$$(13) \qquad |v_n(z) - z^n| < 1, \quad |z| = 1.$$

Let $z$ trace the unit circle once in the counter-clockwise sense. The change in the argument of $v_n(z)$ thus produced multiplied by $1/(2\pi)$ gives the number of zeros of $v_n(z)$ inside the circle. Clearly

$$\operatorname{arc} v_n(z) = \operatorname{arc} z^n[1 + \psi(z)/z^n] = n \operatorname{arc} z + \operatorname{arc}[1 + \psi(z)/z^n],$$

where

$$\psi(z) = v_n(z) - z^n.$$

By (13)

$$|\psi(z)/z^n| < 1, \qquad |z| = 1,$$

and hence the change in the argument of $[1+\psi(z)/z^n]$ must be zero as $z$ traces the unit circle. The change in arc $z^n$ is $2n\pi$, so that $v_n(z)$ must have just $n$ zeros inside the unit circle. These zeros need not be coincident at the origin,* however, as is the case with $g_n(z)$. We shall show that for the case in which all the zeros of $v_n(z)$ *are* concentrated at the origin, Conditions A include cases not included in Birkhoff's Conditions. This may be done by the simple example $h_n(z)=z/(n+1)$. Then

$$|n!g_n(z) - z^n| = 1/(n+1), \qquad |z| = 1.$$

Hence series (11) diverges if $v_n(z)=n!g_n(z)$. But Conditions A are satisfied. The expansion of $f(z)$ in a series (3) is assured by Theorem III, whereas Birkhoff's theorem is not applicable.

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* The example $v_n(z)=z^n-1/2^n$ shows this.

If we restrict the functions $h_n(z)$ further by the inequality

$$| h_n(z) | < 1/2^{n+1}, \quad | z | = 1, \quad h_n(0) = 0,$$

both theorems are applicable. Since the expansion of a function $f(z)$ in a series (3) was shown to be unique, one may obtain an interesting identity by comparing the two determinations of the coefficients. That is, if

$$g_n(z) = \frac{z^n}{n!}[1 + h_n(z)], \quad | h_n(z) | < 1/2^{n+1}, \quad | z | = 1, \quad h_n(0) = 0,$$

then

$$\left. \frac{W[g_0(z), g_1(z), \cdots, g_{n-1}(z), f(z)]}{W[g_0(z), g_1(z), \cdots, g_{n-1}(z)]} \right|_{z=0} = \frac{n!}{2\pi i} \int_C \frac{g(t)}{t^{n+1}} dt,$$

where $g(z)$ satisfies the integral equation

$$f(z) = g(z) + \frac{1}{2\pi i} \int_C \left[ \sum_{n=0}^{\infty} \frac{n! g_n(z) - z^n}{t^{n+1}} \right] g(t) dt.$$

$C$ is the circle $z = 1$, and the integration is in the positive sense.

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