THE SINGULAR POINTS OF ANALYTIC SPACE-CURVES*

BY

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1. The singular points of analytic space-curves have been investigated by various writers.† The object of this paper is to simplify the theory and also to make it self-dual by introducing the dual counterparts of the ordinary curvature and spherical curvature. Let

$$x = x(t), \quad y = y(t), \quad z = z(t) \tag{1}$$

be the equations of a twisted curve, the functions being analytic. Then in the neighborhood of any given point P_0 of the curve, at which $t=t_0$, there exists a certain region of the complex t-plane, within which x, y, z are expressible as series of powers of $t-t_0$ (with positive integral exponents), and such that to different values of t in the region will correspond different points.

The point P_0 will be *singular*, in the broader sense which we shall adopt,‡ if

$$\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix} = 0, \tag{2}$$

when $t=t_0$; here $x'=dx/dt$, etc. The points for which $x'=y'=z'=0$, to which the term singular is sometimes confined, are therefore only a special case.

2. With every point P_0, singular or non-singular, there is associated a triad of positive integers (α, β, γ), such that if we put

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† Fine, American Journal of Mathematics, vol. 8 (1886), pp. 156–177; Björling, Archiv der Mathematik und Physik, (2), vol. 8 (1890), pp. 83–91; Wölffing, Archiv der Mathematik und Physik, (2), vol. 15 (1897), pp. 146–158; Burali-Forti, Atti di Torino, vol. 36 (1901), pp. 935–938; Mehmke, Zeitschrift für Mathematik und Physik, vol. 49 (1903), pp. 62–83; Saurel, Annals of Mathematics, (2), vol. 7 (1905), pp. 3–9; von Lilienthal, *Vorlesungen über Differentialgeometrie*, vol. 1, Leipzig, 1908, pp. 242–262; Meder, Crelle's Journal, vol. 137 (1910), pp. 83–144; Zindler, Wiener Sitzungsberichte, vol. 127I (1918), pp. 871–918; Study, Mathematische Annalen, vol. 87 (1922), pp. 207–228.

‡ The definitions used by Lilienthal (loc. cit., p. 182), Study (p. 212), Burali-Forti (p. 935) and Mehmke (p. 71) are all practically the same as this; but no writer, so far as I am aware, writes the condition in this simple form. Lilienthal gives three possibilities, but the first two are included in the third.

$$(3) \qquad \Delta = \Delta(\alpha,\beta,\gamma) = \begin{vmatrix} x^{(\alpha)} & y^{(\alpha)} & z^{(\alpha)} \\ x^{(\alpha+\beta)} & y^{(\alpha+\beta)} & z^{(\alpha+\beta)} \\ x^{(\alpha+\beta+\gamma)} & y^{(\alpha+\beta+\gamma)} & z^{(\alpha+\beta+\gamma)} \end{vmatrix}$$

then

$$(3') \qquad \Delta \neq 0, \quad \text{when} \quad t = t_0,$$

whereas every determinant $\Delta(\alpha', \beta', \gamma')$ vanishes, if the triad $(\alpha', \beta', \gamma')$ precedes (α, β, γ) lexicographically, which means that either

$$\alpha' < \alpha,$$

or

$$\alpha' = \alpha, \quad \beta' < \beta,$$

or

$$\alpha' = \alpha, \quad \beta' = \beta, \quad \gamma' < \gamma.*$$

Hence P_0 is a non-singular point, if $\alpha=\beta=\gamma=1$, and is singular, if $\alpha+\beta+\gamma>3$. We shall call (α, β, γ) the *type*† of the point P_0 or of the curve (namely the region in the neighborhood of P_0), writing $P_0=(\alpha, \beta, \gamma)$.

By the use of homogeneous coördinates it has been shown‡ that the type of a curve is invariant under the group of collineations, and that every correlation transforms a curve of type (α, β, γ) into one of type (γ, β, α), so that these two types are dual. Hence the type of a curve is a concept belonging to projective, as well as to metric, differential geometry.

We define w and W by the equations

$$(4) \qquad \begin{aligned} w &= \left(\sum (x^{(\alpha)})^2 \right)^{1/2}, \\ W &= \left(\sum (y^{(\alpha)} z^{(\alpha+\beta)} - z^{(\alpha)} y^{(\alpha+\beta)})^2 \right)^{1/2}, \end{aligned}$$

where the summations are cyclic with respect to x, y, z, and where t is put equal to t_0.

3. From now on we shall assume that t is real and that x, y, z are real functions of t. Then the condition (3′) shows that $wW\neq 0$; and by choosing the positive square roots we have $w>0$, $W>0$.

If we substitute a new parameter t for $t-t_0$ and also choose the coördinate axes so that the origin O coincides with P_0, the x-axis with the tangent at P_0, and the xy-plane with the osculator (osculating plane) at P_0, then the equations of the curve (in the given region or interval) become

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* See Lilienthal, pp. 242–251; Mehmke, pp. 68–70; Study, p. 210.

† Lilienthal appears to be the only writer who uses the integers α, β, γ, which he calls ν_1, ν_2, ν_3; Study calls the triad $(\alpha-1, \beta-1, \gamma-1)$ the *characteristic* of the point P_0, writing it (k_1, k_2, k_3).

‡ Björling, p. 85; Study, p. 212.

$$
\begin{aligned}
x &= at^{\alpha} + \cdots, \\
y &= bt^{\alpha+\beta} + \cdots, \\
z &= ct^{\alpha+\beta+\gamma} + \cdots,
\end{aligned} \tag{5}
$$

where $abc \neq 0$ and we can choose $a>0$, $b>0$, $c>0$. Then

$$w = \alpha!a, \quad W = \alpha!(\alpha+\beta)!ab, \quad |\Delta| = \alpha!(\alpha+\beta)!(\alpha+\beta+\gamma)!abc, \tag{6}$$

and therefore

$$a = \frac{w}{\alpha!}, \quad b = \frac{W}{w(\alpha+\beta)!}, \quad c = \frac{|\Delta|}{W(\alpha+\beta+\gamma)!}. \tag{7}$$

The exponents of t in the terms actually present in the expansions (5) will have no common factor; but of course α, β, γ may have a common factor.

The positive branch of the curve, corresponding to small positive values of t, will lie in the octant $[+1, +1, +1]$, in which x, y, z are all positive; and the negative branch will lie in the octant $[(-1)^{\alpha}, (-1)^{\alpha+\beta}, (-1)^{\alpha+\beta+\gamma}]$, which is therefore determined by the evenness or oddness of α, β, and γ. This gives the first rough classification of singular points into eight *categories*.

We also arrive at the same classification in a more significant, because self-dual, manner by considering the curve to be generated by three *elements*, namely a point, a line (the tangent), and a plane (the osculator), and allowing each of these elements to be either *ordinary* or a *reversal*-element (Rückkehrelement).

It is known* that as t decreases (or increases) through the value zero, then at the origin O:

the generating point continues or reverses the direction of its motion along the tangent, according as α is odd or even;

the generating osculator continues or reverses the direction of its rotation about the tangent, according as γ is odd or even;

and the generating tangent continues or reverses the direction of its rotation in the osculator (or about the point), according as β is odd or even.

4. These results are expressed in the following table. In each case, as t decreases through the value zero, the generating point emerges from the first octant $[+1, +1, +1]$ into the octant indicated, whose number, given in the last column, is also the number of the category to which the curve (or point) belongs. The octants 1, 2, 3, 4, for which $z>0$, are numbered like the corresponding quadrants in the xy-plane, and the octants 5, 6, 7, 8 are

* Fine, pp. 166, 173; Saurel, pp. 6–7.

symmetric to the respective octants 1, 2, 3, 4 with respect to the origin, and *not* with respect to the xy-plane, as Lilienthal (p. 202) numbers them.

Type α	β	γ	Octant ($t<0$) $(-1)^\alpha$	$(-1)^{\alpha+\beta}$	$(-1)^{\alpha+\beta+\gamma}$	Number of Octant and of Category
even	even	even	$+1$,	$+1$,	$+1$	1
odd	odd	even	-1,	$+1$,	$+1$	2
odd	even	odd	-1,	-1,	$+1$	3
even	odd	odd	$+1$,	-1,	$+1$	4
odd	even	even	-1,	-1,	-1	5
even	odd	even	$+1$,	-1,	-1	6
even	even	odd	$+1$,	$+1$,	-1	7
odd	odd	odd	-1,	$+1$,	-1	8

In the case of a singular point of the 2d, 3d, or 4th category, one of the three generating elements is a reversal-element and the other two are ordinary; the 2d and 4th categories are dual and the 3d is self-dual. Similarly the 5th and 7th are dual and the 6th is self-dual. The 1st and 8th are also self-dual; the 8th includes the non-singular points (1, 1, 1).*

The positive and negative branches of the curve will lie on the same side of the osculating plane (categories 1, 2, 3, 4) or on opposite sides (categories 5, 6, 7, 8), according as the number of reversal-elements is odd or even.

5. The interval $(-h<t<h)$ within which the curve is defined by (5) can be chosen such that every point except the origin O is non-singular. At any such point $P=(x, y, z)$ consider the six variables

$$\begin{array}{ccccc} s & & \theta & & \eta \\ & \rho & & \sigma & \\ & & \tau & & \end{array}$$

where $1/\rho$ and $1/\tau$ are the ordinary curvature and torsion and where

$$s=\int_0^t\left(\sum x'^2\right)^{1/2}dt, \quad \theta=\int_0^t ds/\rho, \quad \eta=\int_0^t ds/\tau \quad \text{and} \quad \sigma=\rho/\tau.$$

Then $\rho=ds/d\theta$, $\tau=ds/d\eta$ and $\sigma=d\eta/d\theta$; ds, $d\theta$ and $d\eta$ are the element of arc,

* Mehmke (pp. 65–67) and Zindler (pp. 876–879) give distinctive names to these eight categories; Zindler's names seem to be highly appropriate, except that in accordance with his definition a point of the 8th category is called non-singular (gewöhnlich); it would be better to call it a "Gegenhenkel."

the angle of contingence and the angle of torsion, respectively; s is dual to η, ρ to σ, and θ, τ are self-dual. Following Plucker* we shall call σ the *conical curvature* at P.

We define the value of ρ at a singular point O to be the limit of its value at P as $t\to 0$, including the case where $\rho\to\infty$; similarly for τ and σ.

6. We now proceed to find a second classification of the singular points O into thirteen *classes*, depending on whether the values of ρ, τ and σ at O are zero, finite and different from zero, or infinite. Since $\sigma=-\cot\omega$, where ω is the angle between the tangent and the rectifying line, σ will be zero, infinite, or neither, according as the rectifying line coincides with the binormal, with the tangent, or with neither one.

From (5) by means of known formulas we easily derive for the values of ρ, τ and σ at the non-singular point P the following expansions in ascending integral powers of t, in which the exponents of t in the initial terms are sometimes negative:

$$
\begin{aligned}
\rho &= \frac{\alpha^2a^2}{\beta(\alpha+\beta)b}t^{\alpha-\beta}+\cdots, \\
\tau &= -\frac{\alpha\beta(\alpha+\beta)ab}{\gamma(\beta+\gamma)(\alpha+\beta+\gamma)c}t^{\alpha-\gamma}+\cdots, \qquad (8)\\
\sigma &= -\frac{\alpha\gamma(\beta+\gamma)(\alpha+\beta+\gamma)ac}{\beta^2(\alpha+\beta)^2b^2}t^{\gamma-\beta}+\cdots.
\end{aligned}
$$

We also find that

$$
\begin{aligned}
s' &= \alpha a t^{\alpha-1}+\cdots, \\
\theta' &= \frac{\beta(\alpha+\beta)b}{\alpha a}t^{\beta-1}+\cdots, \qquad (9)\\
\eta' &= -\frac{\gamma(\beta+\gamma)(\alpha+\beta+\gamma)c}{\beta(\alpha+\beta)b}t^{\gamma-1}+\cdots,
\end{aligned}
$$

and therefore that

$$
\begin{aligned}
s &= at^{\alpha}+\cdots, \\
\theta &= \frac{(\alpha+\beta)b}{\alpha a}t^{\beta}+\cdots, \qquad (10)\\
\eta &= -\frac{(\beta+\gamma)(\alpha+\beta+\gamma)c}{\beta(\alpha+\beta)b}t^{\gamma}+\cdots.
\end{aligned}
$$

* *Geometrie des Raumes*, 1846, pp. 34–35. See also E. Müller, Wiener Sitzungsberichte, vol. 126 (1917), pp. 311–312. In the Quarterly Journal of Mathematics, vol. 46 (1915), p. 36, I called $\sigma(=1/r')$ the plane-curvature.

The leading coefficient can never be zero in any of these nine expansions. The leading terms in (8_1), (8_2) and (8_3), under the respective hypotheses $\alpha=\beta$, $\alpha=\gamma$ and $\beta=\gamma$, become equal to l, m and n, respectively, where

$$(11)\qquad \begin{gathered} l = \frac{a^2}{2b}, \quad m = -\frac{\beta ab}{(2\alpha+\beta)c}, \\ n = -\frac{2\alpha(\alpha+2\beta)ac}{(\alpha+\beta)^2 b^2}. \end{gathered}$$

If $\alpha=\beta=\gamma$, then m and n become simpler in form and we have

$$l = a^2/(2b), \quad m = -ab/(3c), \quad n = -3ac/(2b^2).$$

We now pass to the limit by letting $t\to 0$ and obtain for the values of ρ, τ and σ at the origin O the following:

$$(12)\qquad \begin{aligned} \rho &= \begin{Bmatrix} 0 \\ l \\ \infty \end{Bmatrix} \text{ according as } \alpha \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \beta, \\ \tau &= \begin{Bmatrix} 0 \\ m \\ \infty \end{Bmatrix} \text{ according as } \alpha \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \gamma, \\ \sigma &= \begin{Bmatrix} 0 \\ n \\ \infty \end{Bmatrix} \text{ according as } \gamma \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \beta, \end{aligned}$$

where l, m and n, given by (11), are finite and $\neq 0$. By means of (7), (3) and (4) we can, if we wish, express l, m and n in terms of the original equation (1) of the curve.

By examining (12_2) and (11_2) we see that if a point O is of type (α, β, α), then the absolute value of τ at O is always less than ab/c, and is *greater than, equal to, or less than* $ab/(3c)$, *according as* β *is greater than, equal to, or less than* α; also if α is fixed and $\beta\to\infty$, then $|\tau|\to ab/c$, and if β is fixed and $\alpha\to\infty$, then $|\tau|\to 0$.

By a similar examination of (12_3) and (11_3) we see that if a point is of type (α, β, β), then $|\sigma|$ is always less than $2ac/b^2$, and is $>$, $=$, or $<3ac/(2b^2)$, according as β is $<$, $=$, or $>\alpha$. Hence ω (an acute angle) is always greater than $\cot^{-1}(2ac/b^2)$, and is $<$, $=$, or $>$ $\cot^{-1}(3ac/(2b^2))$, according as β is $<$, $=$, or $>\alpha$; also if $\alpha\to\infty$, then $\omega\to\cot^{-1}(2ac/b^2)$, and if $\beta\to\infty$, then $\omega\to\pi/2$.

7. Formulas (12) show us that the class to which any real and finite point of type (α, β, γ) belongs depends only on the relative magnitudes of α, β, and γ. From this fact or the relation $\rho=\sigma\tau$ we see that the number of classes, instead of being 3^3, reduces to 13, as is shown in the following table:

Class-Number	Type	ρ	τ	σ
1	$\alpha=\beta=\gamma$	$\dfrac{a^2}{2b}$	$-\dfrac{ab}{3c}$	$-\dfrac{3ac}{2b^2}$
2	$\alpha=\gamma<\beta$	∞	m	∞
3	$\alpha=\gamma>\beta$	0	m	0
4	$\alpha=\beta<\gamma$	l	∞	0
5	$\gamma=\beta<\alpha$	0	0	n
6	$\alpha=\beta>\gamma$	l	0	∞
7	$\gamma=\beta>\alpha$	∞	∞	n
8	$\alpha<\beta<\gamma$	∞	∞	0
9	$\gamma<\beta<\alpha$	0	0	∞
10	$\alpha<\gamma<\beta$	∞	∞	∞
11	$\gamma<\alpha<\beta$	∞	0	∞
12	$\beta<\alpha<\gamma$	0	∞	0
13	$\beta<\gamma<\alpha$	0	0	0

In each of the six classes 2–7 exactly two of the type-integers α, β, γ are equal and therefore exactly one of the variables ρ, τ, σ is finite and $\neq 0$. In each of the last six classes 8–13 the three integers are all distinct and ρ, τ, σ are all $=0$ or ∞. Classes 1, 2 and 3, in which τ is finite and $\neq 0$, are self-dual; and 4 is dual to 5, 6 to 7, 8 to 9, 10 to 11, and 12 to 13. In any pair of dual classes the values of ρ and σ in one are interchanged in the other (except for the difference between l and n).

8. From our cross-classification of singular points into eight categories and thirteen classes, by combination, we obtain a classification into *species*; the species of a point depends, therefore, on the evenness or oddness of α, β, γ and also on their relative magnitudes. Since these two properties are not entirely independent, the number of species is less than $8 \cdot 13=104$.

First, a singular point of class 1 must clearly be of category 1 or 8. Hence *if ρ, τ and σ are all finite and different from zero, the curve must either emerge (for $t<0$) in the 8th octant or return to the 1st octant.* This gives two species.

Next, in each of the six classes 2–7 two of the integers α, β, γ, being equal, must be either both odd or both even, while the third may be odd or even. This gives $6\cdot 4=24$ species.

In particular, putting $\alpha=\beta$ $(\rho=l)$, we see that *the only categories of singular points at which the ordinary curvature $1/\rho$ can be finite and different from zero are the 1st, 8th, 2d and 7th, for which the negative branch of the curve lies in one of the octants* $[(-1)^{\alpha}, +1, (-1)^{\gamma}]$.

Similarly $(\alpha=\gamma,\ \tau=m)$ *the only categories of points at which the torsion $1/\tau$ can be finite and different from zero are the 1st, 8th, 3d and 6th, for which* *the negative branch of the curve lies in one of the octants* $[(-1)^\alpha, (-1)^{\alpha+\beta}, (-1)^\beta]$. Of these four octants no two are adjacent (symmetric with respect to a plane).

Again ($\beta=\gamma$, $\sigma=n$) *the only categories of points at which the conical curvature* σ *can be finite and different from zero are the* 1*st*, 8*th*, 4*th*, *and* 5*th*, *for which the negative branch of the curve lies in one of the octants* $[(-1)^\alpha, (-1)^{\alpha+\beta}, (-1)^\alpha]$.*

In other words, *if the negative branch of the curve lies in the* 2*d or* 7*th octant*, τ *and* σ *must be zero or infinite*; *if in the* 3*d or* 6*th octant*, ρ *and* σ *must be zero or infinite*; *if in the* 4*th or* 5*th octant*, ρ *and* τ *must be zero or infinite.*

Finally, in each of the six classes 8–13 the integers α, β, γ are unrestricted as to oddness or evenness; that is, a point at which ρ, τ, σ are all zero or infinite may belong to any one of the eight categories. This gives $6 \cdot 8 = 48$ species. *The total number of species is therefore* $2+24+48=74$.†

9. Since every point belonging to a self-dual class ($\alpha=\gamma$, $\tau=m$) will also belong to a self-dual category, the self-dual species are precisely those that belong to the classes 1, 2, 3, and their number is $2+4+4=10$.

It is evident that the species of a point is invariant under all the real collineations that carry the point into a *finite* point.‡

If the curve is imaginary, our entire classification falls to the ground, classes as well as categories; for by (4), (7) and (11) we see that w and W may vanish and that a, b, c, l, m, n may not be determined. For instance, the origin is a non-singular point (1, 1, 1) of the curve $x=it$, $y=t+t^2$, $z=t^3$, and yet since $w=0$, $W=2i$ and $\Delta=12i$, we have $\rho=\sigma=0$, $\tau=i/3$, which contradicts (12).

The simplest singular points are (2, 1, 1), (1, 2, 1) and (1, 1, 2); the first is said to be a *stationary* point (of the first degree), at the second the tangent is said to be stationary, and at the third the osculator is said to be stationary. Fine (p. 174) has shown that if the curve is algebraic, then with respect to its effect on the order, class, rank, and genus of the curve, a singular point of type (α, β, γ) is equivalent to $\alpha-1$ stationary points, $\beta-1$ stationary tangents and $\gamma-1$ stationary osculators.

Study (p. 214) has also shown that (α, β, γ) is an α-fold point of the curve, the corresponding tangent is β-fold, and the osculator is γ-fold.

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* The first two parts of this triple theorem were proved by Mehmke (pp. 75, 77).

† Burali-Forti (p. 938) and Mehmke (p. 78), by ignoring σ, obtain a classification that is not self-dual. Instead of 13 classes and 74 species they obtain only 9 classes and 50 species. For them our classes 5, 9 and 13 coincide; also 7, 8 and 10. For them the points (1, 2, 4), (1, 2, 2) and (1, 4, 2) are of the same species, although the corresponding values of σ are 0, n and ∞, respectively.

‡ Meder (loc. cit.) studies the "Ordnungszahlen" of ρ, τ and various other variables at an *infinite* point.

PROJECTIONS ON THE COÖRDINATE PLANES

10. Now consider the orthogonal projections of a curve C on the osculating, rectifying and normal planes at a point $O=(\alpha, \beta, \gamma)$. From (5) we see that the types of these three plane curves at their common point are

$$
\begin{aligned}
O_{xy} &= (\alpha,\beta), \\
O_{xz} &= (\alpha,\beta+\gamma), \qquad (13)\\
O_{yz} &= (\alpha+\beta,\gamma),
\end{aligned}
$$

respectively. Also consider a curve C' dual to C and on it the point O' that corresponds to O in a correlation; then $O'=(\gamma, \beta, \alpha)$, $O_{xz}'=(\gamma, \alpha+\beta)$ and $O_{yz}'=(\beta+\gamma, \alpha)$. But in plane geometry a point (λ, μ) is dual to a point (μ, λ). Hence O_{yz}' is dual to O_{xz} and O_{xz}' to O_{yz}. That is, if two space-curves are dual, the projection of either on the *normal plane* at a point O is dual (planar duality) to the projection of the other on the *rectifying plane* at the corresponding point O'.

If $\alpha=\beta+\gamma$, O_{xz} (and therefore also O_{yz}') is self-dual, and conversely. If $\alpha+\beta=\gamma$, O_{yz} (and therefore also O_{xz}') is self-dual, and conversely.

If $\alpha=\gamma$, so that O is self-dual (spatial duality), then $O_{xz}=(\alpha, \alpha+\beta)$ and $O_{yz}=(\alpha+\beta, \alpha)$, which are therefore dual (planar duality), and conversely. That is, *if the torsion at a point O is finite and different from zero, the projections of the curve on the normal and rectifying planes at O are dual, and conversely.*

11. It is known* that the singular points of plane curves are divided into four categories (quadrants), three classes ($\rho=0$, ∞, or neither) and ten species. Consider the possible combinations of categories (classes, species) of the three projections (13). The combinations of categories are obviously eight in number, and there is a one-to-one correspondence between the eight octants in which the negative branch of the curve C may lie and the eight combinations of quadrants in which the negative branches of the projections may lie.†

12. When we come to the combinations of classes or species, we find a very different situation. Let ρ_{xy}, ρ_{xz} and ρ_{yz} be the respective radii of curvature of the three projections (13) at their common point. Of course $\rho_{xy}=\rho$, and its value is determined by (12_1) and (11_1). Similarly ρ_{xz} and ρ_{yz} are determined as follows:

* See Burali-Forti, p. 938.

† See the figures drawn by Mehmke, pp. 65–67.

$$\text{(14)}\qquad \begin{aligned} \rho_{xz} &= \left\{\begin{matrix} 0 \\ k_1 \\ \infty \end{matrix}\right\}, \text{ according as } \alpha \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} \beta + \gamma, \\ \rho_{yz} &= \left\{\begin{matrix} 0 \\ k_2 \\ \infty \end{matrix}\right\}, \text{ according as } \alpha + \beta \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} \gamma, \end{aligned}$$

where

$$\text{(15)}\qquad k_1 = a^2/(2c) \neq 0, \quad k_2 = b^2/(2c) \neq 0.$$

The number of combinations of classes is only eleven. For if $\alpha \leqq \beta$, then $\alpha < \beta + \gamma$ and γ may be either $<$, $=$, or $> \alpha + \beta$; this gives $2 \cdot 3 = 6$ combinations. But if $\alpha > \beta$, γ may be either $\geqq \alpha + \beta$ or $\leqq \alpha - \beta$, or finally we may have $\alpha + \beta > \gamma > \alpha - \beta$; this gives 5 combinations. Altogether $6 + 5 = 11$ combinations, which are exhibited in the following table.

	Type			ρ_{xy} $(=\rho)$	ρ_{xz}	ρ_{yz}
1	$\alpha<\beta$	$\alpha<\beta+\gamma$	$\alpha+\beta<\gamma$	∞	∞	∞
2	"	"	$\alpha+\beta=\gamma$	∞	∞	k_2
3	"	"	$\alpha+\beta>\gamma$	∞	∞	0
4	$\alpha=\beta$	"	$\alpha+\beta<\gamma$	l	∞	∞
5	"	"	$\alpha+\beta=\gamma$	l	∞	k_2
6	"	"	$\alpha+\beta>\gamma$	l	∞	0
7	$\alpha>\beta$	"	$\alpha+\beta<\gamma$	0	∞	∞
8	"	"	$\alpha+\beta=\gamma$	0	∞	k_2
9	"	"	$\alpha+\beta>\gamma$	0	∞	0
10	"	$\alpha=\beta+\gamma$	"	0	k_1	0
11	"	$\alpha>\beta+\gamma$	"	0	0	0

If $\alpha \geqq \beta + \gamma$, then $\alpha > \beta$ and $\alpha + \beta > \gamma$. Hence by (14) and (12_1) or by an inspection of the table, we see that *the only points O for which the radius of curvature ρ_{xz} of the projection of the curve on the rectifying plane at O is not infinite are those for which the radii of curvature ρ_{xy} and ρ_{yz} of the projections on the osculating and normal planes are both zero.*

Putting $\alpha = \beta + \gamma$ or $\alpha + \beta = \gamma$, we easily find, by §4 or by a known theorem on plane curves, that *if the two branches of the curve at a point O lie on opposite sides of the osculating plane, then the radii of curvature ρ_{xz} and ρ_{yz} of its projections on the rectifying and normal planes must be zero or infinite.*

13. The number of combinations of *species* is easily seen to be only 62, although each projection by itself is entirely unrestricted in species (ten cases). There is obviously no one-to-one correspondence between the 13 classes (74 species) of curves and the 11 combinations of classes (62 combinations of species) of their projections. Not only may two curves of different

species have projections belonging to the same combination of species, but two curves of the same species may have projections belonging to different combinations of species (their projections O_{xy} on the osculator must, of course, be of the same species).

The first possibility is illustrated by the types (4, 2, 3) and (4, 2, 5), for which $\tau=0$ and ∞, respectively, whereas their projections are of types (4, 2), (4, 5), (6, 3) and (4, 2), (4, 7), (6, 5), respectively, for which in both cases $\rho_{xy}=0$, $\rho_{xz}=\infty$, $\rho_{yz}=0$. The second possibility is illustrated by (4, 2, 3), just considered, and (6, 2, 3); these two types are clearly of the same species, whereas the projections of the latter type of curve are of types (6, 2), (6, 5), (8, 3), for which $\rho_{xy}=\rho_{xz}=\rho_{yz}=0$.

In spite of this lack of uniformity in the correspondence, there is, of course, a close connection between the values of ρ, τ, σ and the values of $\rho_{xy}(=\rho)$, ρ_{xz}, ρ_{yz}. For instance:

If $\alpha \geqq \beta+\gamma$, then $\alpha>\beta$ and $\alpha>\gamma$. By (14) and (12) this says that if $\rho_{xz}=k_1$ or 0, then $\rho=\tau=0$.

Hence, *for every point O except those at which $\rho=\tau=0$, the projection of the curve on the rectifying plane at O has an infinite radius of curvature ρ_{xz}.*

If $\gamma \geqq \beta+\alpha$, then $\gamma>\beta$ and $\gamma>\alpha$. By (14) and (12) this says that if $\rho_{yz}=k_2$ or ∞, then $\sigma=0$ and $\tau=\infty$.

Hence, *for every point O except those at which $\sigma=0$ and $\tau=\infty$, the projection of the curve on the normal plane at O has a vanishing radius of curvature ρ_{yz}.*

Sections of the tangent surface by the coördinate planes

14. The osculating, rectifying and normal planes at a point $O=(\alpha, \beta, \gamma)$ will cut the tangent surface of the curve in three plane curves $\bar{O}_{xy}$, $\bar{O}_{xz}$ and $\bar{O}_{yz}$, whose character at their common point O we shall now consider. The parametric equations of the tangent surface of the curve (5), §3, are

$$\begin{aligned} x &= (at^{\alpha}+\cdots)+u(\alpha a t^{\alpha-1}+\cdots), \\ y &= (bt^{\alpha+\beta}+\cdots)+u[(\alpha+\beta)bt^{\alpha+\beta-1}+\cdots], \\ z &= (ct^{\alpha+\beta+\gamma}+\cdots)+u[(\alpha+\beta+\gamma)ct^{\alpha+\beta+\gamma-1}+\cdots], \end{aligned} \tag{16}$$

where t and u are the parameters. The osculator $z=0$ will intersect the surface in points for which

$$u = -\frac{1}{\alpha+\beta+\gamma}t+\cdots.$$

Substituting this expansion of u in (16_1) and (16_2), we find the equations of the section $\bar{O}_{xy}$ to be

$$(17)\qquad x = \frac{\beta+\gamma}{\alpha+\beta+\gamma}at^{\alpha}+\cdots,\quad y = \frac{\gamma}{\alpha+\beta+\gamma}bt^{\alpha+\beta}+\cdots.$$

Similarly the equations of $\bar{O}_{xz}$ are found to be

$$(18)\qquad x = \frac{\beta}{\alpha+\beta}at^{\alpha}+\cdots,\quad z = -\frac{\gamma}{\alpha+\beta}ct^{\alpha+\beta+\gamma}+\cdots,$$

and the equations of $\bar{O}_{yz}$ to be

$$(19)\qquad y = -\frac{\beta}{\alpha}bt^{\alpha+\beta}+\cdots,\quad z = -\frac{\beta+\gamma}{\alpha}ct^{\alpha+\beta+\gamma}+\cdots.$$

Since the leading coefficients in the six expansions of (17), (18) and (19) are all $\neq 0$, the sections $\bar{O}_{xy}$, $\bar{O}_{xz}$ and $\bar{O}_{yz}$ are of the same types as the respective projections O_{xy}, O_{xz} and O_{yz}, as given by (13), §10. Hence everything stated in §§10–13 about the categories, classes and species of the three projections applies equally well to the three sections.

15. Let $\bar{\rho}_{xy}$, $\bar{\rho}_{xz}$ and $\bar{\rho}_{yz}$ be the respective radii of curvature of the three sections at the point O. Their values are easily found to be the following:

$$(20)\qquad \bar{\rho}_{xy} = \begin{Bmatrix} 0 \\ a_0 \\ \infty \end{Bmatrix} \text{ according as } \alpha \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \beta,$$

where

$$(20')\qquad a_0 = \frac{(\alpha+\gamma)^2a^2}{2\gamma(2\alpha+\gamma)b};$$

$$(21)\qquad \bar{\rho}_{xz} = \begin{Bmatrix} 0 \\ a_1 \\ \infty \end{Bmatrix} \text{ according as } \alpha \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \beta+\gamma,$$

where

$$(21')\qquad a_1 = -\frac{\beta^2a^2}{2\gamma(2\beta+\gamma)c};$$

$$(22)\qquad \bar{\rho}_{yz} = \begin{Bmatrix} 0 \\ a_2 \\ \infty \end{Bmatrix} \text{ according as } \alpha+\beta \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \gamma,$$

where

$$(22')\qquad a_2 = -\frac{\beta^2b^2}{2\alpha(\alpha+2\beta)c}.$$

Comparing (20) and (20′) with (12_1) and (11_1), we see that when a point is of type (α, α, γ) then

$$(23)\qquad \bar{\rho}_{xy} = \frac{(\alpha+\gamma)^2}{\gamma(2\alpha+\gamma)}\rho \neq 0,$$

and therefore

$$\bar{\rho}_{xy} > \rho\ (= \rho_{xy} = a^2/(2b)), \qquad \text{and}$$

$$(24) \qquad \bar{\rho}_{xy} \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} \frac{4}{3}\rho, \text{ according as } \gamma \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} \alpha.$$

In other words *if the radius of curvature ρ of a curve C at a point O is finite and different from zero, then the radius of curvature $\bar{\rho}_{xy}$ of the section of the tangent surface of C by the osculator at O is always finite and greater than the radius of curvature ρ_{xy} $(=\rho)$ of the projection of C on the osculator, and is greater than, equal to, or less than* $(4/3)\rho$, *according as the radius of torsion τ of C at O is zero, finite and different from zero, or infinite.* The osculating circle of the section and that of the projection lie on the same side of the tangent.

16. By a similar comparison of (21) and (21′) with (14_1) and (15_1), we see that when $\alpha=\beta+\gamma$, then

$$(25) \qquad \bar{\rho}_{xz} = -\frac{\beta^2}{\gamma(2\beta+\gamma)}\rho_{xz} \neq 0\,;$$

hence $|\bar{\rho}_{xz}| \neq \rho_{xz}$ and

$$|\bar{\rho}_{xz}| \left\{ \begin{matrix} > \\ < \end{matrix} \right\} \rho_{xz} \text{ according as } \beta \left\{ \begin{matrix} > \\ < \end{matrix} \right\} (1+2^{1/2})\gamma\,;$$

in particular if $\beta=\gamma$, $\bar{\rho}_{xz}=-\rho_{xz}/3$.

That is, *at a singular point O of type $(\beta+\gamma, \beta, \gamma)$, the radius of curvature $|\bar{\rho}_{xz}|$ of the section of the tangent surface by the rectifying plane is never equal to, but is always a rational multiple of, the radius of curvature ρ_{xz} of the projection of the curve on the rectifying plane, and is greater or less than ρ_{xz}, according as β is greater or less than* $(1+2^{1/2})\gamma$. The osculating circle of the section and that of the projection lie on opposite sides of the tangent.

In particular, if the point is of type $(2\gamma, \gamma, \gamma)$, then $\bar{\rho}_{xz}=-\rho_{xz}/3$.

Finally, comparing (22) and (22′) with (14_2) and (15_2), we see that if $\alpha+\beta=\gamma$, then

$$(26) \qquad \bar{\rho}_{yz} = -\frac{\beta^2}{\alpha(\alpha+2\beta)}\rho_{yz} \neq 0\,;$$

hence $|\bar{\rho}_{yz}| \neq \rho_{yz}$, and

$$|\bar{\rho}_{yz}| \left\{ \begin{matrix} > \\ < \end{matrix} \right\} \rho_{yz} \text{ according as } \beta \left\{ \begin{matrix} > \\ < \end{matrix} \right\} (1+2^{1/2})\alpha\,;$$

in particular, if $\beta=\alpha$, $\bar{\rho}_{yz}=-\rho_{yz}/3$.

That is, *at a singular point of type $(\alpha, \beta, \alpha+\beta)$ the radius of curvature $|\bar{\rho}_{yz}|$ of the section by the normal plane is never equal to, but is always a rational* *multiple of, the radius of curvature ρ_{yz} of the projection on the normal plane, and is greater or less than ρ_{yz}, according as β is greater or less than* $(1+2^{1/2})\alpha$. The osculating circle of the section and that of the projection lie on opposite sides of their common tangent (the principal normal of the original curve).

In particular, if the point is of type $(\alpha, \alpha, 2\alpha)$, then $\bar{\rho}_{yz} = -\rho_{yz}/3$.

SPHERICAL AND QUASI-SPHERICAL CURVATURE

17. The dual counterpart of the osculating sphere of a twisted curve C at a point O we shall call its quasi-osculating sphere* at O. Its center is the corresponding point on the edge of regression of the rectifying developable and its radius is the distance from that point to the corresponding tangent to C. Since the radius R_1 of the osculating sphere has been called the radius of *spherical curvature* of C, we shall call the radius R_2 of the quasi-osculating sphere the radius of *quasi-spherical curvature* of C. By means of the known formulas

$$R_1^2 = \rho^2 + \left(\frac{\rho'}{\eta'}\right)^2, \qquad R_2 = \left|\frac{s'}{\sigma'}\right|, \tag{27}$$

we proceed to find the values of R_1 and R_2 at a point (α, β, γ).

18. Differentiating (8_1) and (8_3), §6, we obtain

$$\begin{aligned} \rho' &= \frac{\alpha^2(\alpha - \beta)a^2}{\beta(\alpha+\beta)b} t^{\alpha-\beta-1} + \cdots, \\ \sigma' &= \frac{\alpha\gamma(\beta^2 - \gamma^2)(\alpha+\beta+\gamma)ac}{\beta^2(\alpha+\beta)^2 b^2} t^{\gamma-\beta-1} + \cdots, \end{aligned} \tag{28}$$

where the initial coefficients will vanish under the respective conditions $\alpha=\beta$, $\beta=\gamma$. Some of the succeeding coefficients may also vanish, and we have the following:

$$\begin{aligned} &\text{if} \quad \alpha = \beta, \quad \rho' = a_i t^i + \cdots \qquad (i \geqq 0,\ a_i \neq 0); \\ &\text{if} \quad \beta = \gamma, \quad \sigma' = b_k t^k + \cdots \qquad (k \geqq 0,\ b_k \neq 0). \end{aligned} \tag{29}$$

From (28_1), (29_1) and (9_3) we see that

$$\begin{aligned} &\text{if} \quad \alpha \neq \beta, \quad \rho'/\eta' = \frac{\alpha^2(\beta-\alpha)a^2}{\gamma(\beta+\gamma)(\alpha+\beta+\gamma)c} t^{\alpha-\beta-\gamma} + \cdots, \\ &\text{if} \quad \alpha = \beta, \quad \rho'/\eta' = -\frac{2\alpha^2 b a_i}{\gamma(\alpha+\gamma)(2\alpha+\gamma)c} t^{i+1-\gamma} + \cdots. \end{aligned} \tag{30}$$

* Quarterly Journal of Mathematics, vol. 46 (1915), pp. 364–366; Mathematische Annalen, vol. 101 (1929), p. 2; G. Loria, *Curve Sghembe Speciali*, vol. 1, 1921, p. 6; Th. Schmid, *Darstellende Geometrie*, vol. 1, 1919, p. 164.

For the sake of brevity we put

$$
\begin{aligned}
g_1 &= \left| \frac{\alpha^2(\beta - \alpha)a^2}{\gamma(\beta+\gamma)(\alpha+\beta+\gamma)c} \right|, \\
h_1 &= \left| \frac{2\alpha^2 b a_i}{\gamma(\alpha+\gamma)(2\alpha+\gamma)c} \right|,
\end{aligned} \tag{31}
$$

so that (30) becomes the following:

$$
\begin{aligned}
&\text{if} \quad \alpha \neq \beta, \quad \rho'/\eta' = \pm\, g_1 t^{\alpha-\beta-\gamma} + \cdots, \\
&\text{if} \quad \alpha = \beta, \quad \rho'/\eta' = \pm\, h_1 t^{i+1-\gamma} + \cdots.
\end{aligned} \tag{32}
$$

19. Comparing (27_1), (8_1) and (32_1) we see that if $\alpha \neq \beta$, the significant term in the expression for R_1^2 is $(\rho'/\eta')^2$. Hence

$$
\text{if} \quad \alpha \neq \beta, \; R_1 = \left| g_1 t^{\alpha-\beta-\gamma} + \cdots \right|. \tag{33}
$$

On the other hand if $\alpha = \beta$, so that (8_1) reduces to

$$
\rho = a^2/(2b) + \cdots, \tag{34}
$$

then by comparing (32_2) and (34) we see that the significant term depends on whether $i+1-\gamma$ is negative, zero or positive. Hence

$$
\text{if } \alpha = \beta \text{ and } \begin{cases} i+1 < \gamma, & \text{then } R_1 = \left| h_1 t^{-(\gamma-i-1)} + \cdots \right|, \\ i+1 = \gamma, & \text{then } R_1 = (h_1^2 + a^4/(4b^2))^{1/2} + \cdots, \\ i+1 > \gamma, & \text{then } R_1 = a^2/(2b) + \cdots. \end{cases} \tag{35}
$$

Equations (33) and (35) give the value of R_1 at a non-singular point in the neighborhood of the singular point O. Passing to the limit by letting $t \to 0$, we find for the value of R_1 at O the following cases:

$$
\begin{aligned}
&\text{(a)} \quad \text{If } \alpha > \beta + \gamma, \quad R_1 = 0. \\
&\text{(b)} \quad \text{If } \alpha = \beta + \gamma, \quad R_1 = g_1 = a^2/(2c). \\
&\text{(c)} \quad \text{If } \alpha < \beta + \gamma \text{ and } \alpha \neq \beta, \quad R_1 = \infty. \\
&\text{(d)} \quad \text{If } \alpha = \beta \text{ and } i+1 > \gamma, \quad R_1 = a^2/(2b). \\
&\text{(e)} \quad \text{If } \alpha = \beta \text{ and } i+1 = \gamma, \quad R_1 = (h_1^2 + a^4/(4b^2))^{1/2}. \\
&\text{(f)} \quad \text{If } \alpha = \beta \text{ and } i+1 < \gamma, \quad R_1 = \infty.
\end{aligned} \tag{36}
$$

In cases (b), (d) and (e), in view of (31), R_1 is finite and $\neq 0$. If $\gamma = 1$, case (f) does not occur.

20. Turning now to R_2, given by (27_2), we soon find that by using (9_1), (28_2) and (29_2) and putting

$$(37)\qquad g_2 = \left|\frac{\beta^2(\alpha+\beta)^2b^2}{\gamma(\beta^2-\gamma^2)(\alpha+\beta+\gamma)c}\right|, \qquad h_2 = \left|\frac{\alpha a}{b_k}\right|,$$

we obtain the following expansions of R_2:

$$(38)\qquad \text{if}\quad \beta\neq\gamma,\quad R_2 = \left|\, g_2 t^{\alpha+\beta-\gamma}+\cdots\right|; \qquad \text{if}\quad \beta=\gamma,\quad R_2 = \left|\, h_2 t^{\alpha-k-1}+\cdots\right|.$$

Again passing to the limit as $t\to 0$, we find for the value of R_2 at the singular point O the following cases:

(39)

(a) If $\alpha+\beta<\gamma$, $R_2=\infty$.

(b) If $\alpha+\beta=\gamma$, $R_2=g_2=\beta^2b^2/(2\alpha(\alpha+2\beta)c)$.

(c) If $\alpha+\beta>\gamma$, and $\beta\neq\gamma$, $R_2=0$.

(d) If $\beta=\gamma$ and $k+1>\alpha$, $R_2=\infty$.

(e) If $\beta=\gamma$ and $k+1=\alpha$, $R_2=h_2$.

(f) If $\beta=\gamma$ and $k+1<\alpha$, $R_2=0$.

In cases (b) and (e), R_2 is finite and $\neq 0$. If $\alpha=1$, case (f) does not occur.

21. From (36) and (39) we easily derive a number of interesting consequences. In view of (14), (21), (22) and (12) we see that R_1 is closely connected with ρ_{xz}, $\bar\rho_{xz}$ and ρ, and that R_2 is connected with ρ_{yz}, $\bar\rho_{yz}$ and σ. Indeed:

The radius of spherical curvature R_1 at a singular point O is zero, infinite or neither, according as the radii of curvature ρ_{xz} and $\bar\rho_{xz}$ (of the projection on, and the section by, the rectifying plane at O) are zero, infinite or neither, provided that when ρ_{xz} and $\bar\rho_{xz}$ are infinite, the polar line at O either coincides with the binormal or lies at infinity; ρ_{xz} and $\bar\rho_{xz}$ are zero, if, and only if, R_1 is zero.

If ρ is finite and $\neq 0$ and if the osculator is simple ($\gamma=1$), then R_1 is finite and $\neq 0$.

The radius of quasi-spherical curvature R_2 at a singular point O is zero, infinite or neither, according as the radii of curvature ρ_{yz} and $\bar\rho_{yz}$ (of the projection on, and the section by, the normal plane at O) are zero, infinite or neither, provided that when ρ_{yz} and $\bar\rho_{yz}$ are zero, the rectifying line at O coincides either with the binormal or with the tangent; ρ_{yz} and $\bar\rho_{yz}$ are infinite, if and only if R_2 is infinite and σ is zero.

If σ is finite and $\neq 0$ and if the point is simple ($\alpha=1$), then R_2 cannot vanish.

If ρ_{zz} and $\bar{\rho}_{zz}$ are finite and $\neq 0$ $(\alpha=\beta+\gamma)$, we can say more. For by comparing (36), case (b), with (14_1) and (15_1) we see that R_1 is not only finite and $\neq 0$, but is actually equal to ρ_{zz}; also since $\rho=0$ $(\alpha>\beta)$, the polar line coincides with the binormal.

Hence *if the curvature of the projection of the curve on the rectifying plane is finite and different from zero* $(\alpha=\beta+\gamma)$, *then the osculating circle of this projection is a great circle of the corresponding osculating sphere of the original curve.* This sphere will lie on the positive side $(z>0)$ of the osculator.

If ρ_{yz} and $\bar{\rho}_{yz}$ are finite and $\neq 0$ $(\alpha+\beta=\gamma)$, we can say more. For by comparing (39), case (b), with (22) and (22′) we see that R_2 is not only finite and $\neq 0$, but is actually equal to $|\bar{\rho}_{yz}|$; also since $\sigma=0(\gamma>\beta)$, the rectifying line coincides with the binormal.

If the curvature of the section of the tangent surface by the normal plane is finite and different from zero $(\alpha+\beta=\gamma)$, *then the osculating circle of this section is a great circle of the corresponding quasi-osculating sphere of the original curve.* This sphere will lie on the negative side $(z<0)$ of the osculator.

22. If $\alpha=\beta+\gamma$, either all three of the integers α, β, γ are even or just one is even, so that by §4 the point belongs to category 1, 2, 3 or 4. Hence, by §§12 and 15, we see that if the negative branch of the curve lies in one of the octants 5, 6, 7 or 8,

and if $\rho=0$ $(\alpha>\beta)$, then R_1, ρ_{zz}, $\bar{\rho}_{zz}$, ρ_{yz}, and $\bar{\rho}_{yz}$ are all $=0$ or ∞.

and if $\sigma=0$ $(\gamma>\beta)$, then R_2, ρ_{zz}, $\bar{\rho}_{zz}$, ρ_{yz} and $\bar{\rho}_{yz}$ are all $=0$ or ∞.

In other words *if the two branches of the curve at a point O lie on opposite sides of the osculating plane,*

and if the polar line at O coincides with the binormal, then no one of the five variables ρ_{zz}, $\bar{\rho}_{zz}$, ρ_{yz}, $\bar{\rho}_{yz}$, R_1 *can be finite and different from zero.*

If R_1 is finite and $\neq 0$, either $\alpha=\beta+\gamma$ or $\alpha=\beta$. By §8 this can happen only for points of categories 1, 2, 3, 4, 7 or 8.

Hence *if the negative branch of the curve lies in the* 5*th or* 6*th octant, no one of the four variables* ρ, ρ_{zz}, $\bar{\rho}_{zz}$, R_1 *can be finite and different from zero.*

and if the rectifying line at O coincides with the binormal, then no one of the five variables ρ_{zz}, $\bar{\rho}_{zz}$, ρ_{yz}, $\bar{\rho}_{yz}$, R_2 *can be finite and different from zero.*

If R_2 is finite and $\neq 0$, either $\alpha+\beta=\gamma$ or $\beta=\gamma$. By §8 this can happen only for points of categories 1, 2, 3, 4, 5 or 8.

Hence *if the negative branch of the curve lies in the* 6*th or* 7*th octant, no one of the four variables* σ, ρ_{yz}, $\bar{\rho}_{yz}$, R_2 *can be finite and different from zero.*

An immediate consequence is the remarkable fact that *if the negative branch lies in the* 6*th octant* $[+1, -1, -1]$, *no one of the eight variables* ρ, σ, ρ_{xz}, $\bar{\rho}_{xz}$, ρ_{yz}, $\bar{\rho}_{yz}$, R_1, R_2 *can be finite and different from zero.* The simplest example of this is a point of type (2, 1, 2).

23. Finally we shall exhibit in the following table some of our results as applied to a number of the simpler types of singular points, together with the type (1, 1, 1) of non-singular points. Here a, b, c, c_1, and h_2 are all finite and $\neq 0$; a, b, c are the coefficients in the equations (5) of §3; h_2 is given by (37_2) of §20; and c_1 is the value of R_1 when it may come under case (d) or case (e) of (36), §19.

The table includes the twenty-seven types for which α, β, γ are $\leqq 3$ and a few others. Wherever three types are grouped together, the first and third are dual and the second is self-dual. All of the types listed belong to distinct species except (1, 1, 1) and (3, 3, 3); the latter is the simplest type of singular point that looks exactly like a non-singular point.

	$\alpha\beta\gamma$	α, $\alpha+\beta$, $\alpha+\beta+\gamma$	Octant	ρ	τ	σ	ρ_{xz}	$\bar{\rho}_{xz}$	R_1	$\bar{\rho}_{xy}$	ρ_{yz}	$\bar{\rho}_{yz}$	R_2
1	111	123	8	$\frac{a^2}{2b}$	$-\frac{ab}{3c}$	$-\frac{3ac}{2b^2}$	∞	∞	c_1	$\frac{2a^2}{3b}$	0	0	h_2 or ∞
2	112	124	2	$\frac{a^2}{2b}$	∞	0	∞	∞	c_1 or ∞	$\frac{9a^2}{16b}$	$\frac{b^2}{2c}$	$-\frac{b^2}{6c}$	$\frac{b^2}{6c}$
3	121	134	3	∞	$-\frac{ab}{2c}$	∞	∞	∞	∞	∞	0	0	0
4	211	234	4	0	0	$-\frac{16ac}{9b^2}$	$\frac{a^2}{2c}$	$-\frac{a^2}{6c}$	$\frac{a^2}{2c}$	0	0	0	0, h_2 or ∞
5	122	136	5	∞	∞	$-\frac{10ac}{9b^2}$	∞	∞	∞	∞	0	0	h_2 or ∞
6	212	235	6	0	$-\frac{ab}{5c}$	0	∞	∞	∞	0	0	0	0
7	221	245	7	$\frac{a^2}{2b}$	0	∞	∞	∞	c_1	$\frac{9a^2}{10b}$	0	0	0
8	222	246	1	$\frac{a^2}{2b}$	$-\frac{ab}{3c}$	$-\frac{3ac}{2b^2}$	∞	∞	c_1 or ∞	$\frac{2a^2}{3b}$	0	0	0, h_2 or ∞
9	123	136	3	∞	∞	0	∞	∞	∞	∞	$\frac{b^2}{2c}$	$-\frac{2b^2}{5c}$	$\frac{2b^2}{5c}$
10	132	146	2	∞	∞	∞	∞	∞	∞	∞	0	0	0
11	213	236	4	0	∞	0	∞	∞	∞	0	$\frac{b^2}{2c}$	$-\frac{b^2}{16c}$	$\frac{b^2}{16c}$
12	231	256	4	∞	0	∞	∞	∞	∞	∞	0	0	0
13	312	346	2	0	0	0	$\frac{a^2}{2c}$	$-\frac{a^2}{16c}$	$\frac{a^2}{2c}$	0	0	0	0
14	321	356	3	0	0	∞	$\frac{a^2}{2c}$	$-\frac{2a^2}{5c}$	$\frac{a^2}{2c}$	0	0	0	0

	$\alpha\beta\gamma$	α, $\alpha+\beta$, $\alpha+\beta+\gamma$	Octant	ρ	τ	σ	ρ_{xz}	ρ_{xz}	R_1	ρ_{xz}	ρ_{yz}	ρ_{yz}	R_2
15	113	125	8	$\frac{a^2}{2b}$	∞	0	∞	∞	c_1 or ∞	$\frac{8a^2}{15b}$	∞	∞	∞
16	131	145	8	∞	$-\frac{3ab}{5c}$	∞	∞	∞	∞	∞	0	0	0
17	311	345	8	0	0	$-\frac{15ac}{8b^2}$	0	0	0	0	0	0	0, h_2 or ∞
18	133	147	8	∞	∞	$-\frac{7ac}{8b^2}$	∞	∞	∞	∞	0	0	h_2 or ∞
19	313	347	8	0	$-\frac{ab}{7c}$	0	∞	∞	∞	0	0	0	0
20	331	367	8	$\frac{a^2}{2b}$	0	∞	∞	∞	c_1	$\frac{8a^2}{7b}$	0	0	0
21	223	247	7	$\frac{a^2}{2b}$	∞	0	∞	∞	c_1 or ∞	$\frac{25a^2}{42b}$	0	0	0
22	232	257	6	∞	$-\frac{3ab}{7c}$	∞	∞	∞	∞	∞	0	0	0
23	322	357	5	0	0	$-\frac{42ac}{25b^2}$	∞	∞	∞	0	0	0	0, h_2 or ∞
24	233	258	4	∞	∞	$-\frac{32ac}{25b^2}$	∞	∞	∞	∞	0	0	0, h_2 or ∞
25	323	358	3	0	$-\frac{ab}{4c}$	0	∞	∞	∞	0	0	0	0
26	332	368	2	$\frac{a^2}{2b}$	0	∞	∞	∞	c_1 or ∞	$\frac{25a^2}{32b}$	0	0	0
27	333	369	8	$\frac{a^2}{2b}$	$-\frac{ab}{3c}$	$-\frac{3ac}{2b^2}$	∞	∞	c_1 or ∞	$\frac{2a^2}{3b}$	0	0	0, h_2 or ∞
28	124	137	5	∞	∞	0	∞	∞	∞	∞	∞	∞	∞
29	142	157	5	∞	∞	∞	∞	∞	∞	∞	0	0	0
30	214	237	6	0	∞	0	∞	∞	∞	0	∞	∞	∞
31	241	267	7	∞	0	∞	∞	∞	∞	∞	0	0	0
32	412	457	6	0	0	0	0	0	0	0	0	0	0
33	421	467	7	0	0	∞	0	0	0	0	0	0	0
34	224	248	1	$\frac{a^2}{2b}$	∞	0	∞	∞	c_1 or ∞	$\frac{9a^2}{16b}$	$\frac{b^2}{2c}$	$-\frac{b^2}{6c}$	$\frac{b^2}{6c}$
35	242	268	1	∞	$-\frac{ab}{2c}$	∞	∞	∞	∞	∞	0	0	0
36	422	468	1	0	0	$-\frac{16ac}{9b^2}$	$\frac{a^2}{2c}$	$-\frac{a^2}{6c}$	$\frac{a^2}{2c}$	0	0	0	0, h_2 or ∞

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