AN EXTENSION OF PASCAL'S THEOREM*

BY

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INTRODUCTION

In 1825 the Académie Royale de Bruxelles proposed as a prize topic the extension of Pascal's theorem to space of three dimensions. The prize was won by Dandelin,† who showed that a skew hexagon formed of three lines from each regulus of a hyperboloid of revolution has the Pascal property that pairs of opposite planes meet on a plane, and the dual, or Brianchon property, that the lines joining pairs of opposite vertices meet in a point. Hesse‡ wrote several papers on the Dandelin skew hexagons, emphasizing the polar properties of the Pascal plane and the Brianchon point with respect to the quadric bearing the two reguli. Several of the older analytic geometries of three dimensions devote some space to the problem of the skew hexagon, as Salmon, and more notably Plücker§, who offered much original material. In a recent article, the present writer|| has discussed the skew hexagon from an elementary analytic approach.

The foregoing citations exhibit the idea that Pascal's theorem deals with six elements of a quadratic curve, and that the space extension offered will deal with six elements of a quadric surface. It happens that an extension of this sort is valid in space of three dimensions; but in S_n, where $n>3$, it is clear that a skew hexagon of six rulings of a hyperquadric must lie in an S_3 if it is to possess the Brianchon property. It seems, therefore, that the hexagon idea must be abandoned in the search for a valid extension of Pascal's theorem in S_n.

In the plane, a variant of Pascal's theorem affirms that if two triangles are in homology, then the six points of intersection of the sides of the one with the non-corresponding sides of the other lie upon a conic. From a consideration of the converses of this theorem and its dual it occurred in-

* Presented to the Society, April 6, 1928; received by the editors in October, 1928, and (revised) in April, 1929.

† Dandelin, *Mémoire sur l'hyperboloide de révolution et sur les hexagones de Pascal et de M. Brianchon*, Nouvelles Mémoires de l'Académie Royale de Bruxelles, vol. 3 (1826), pp. 1–14.

‡ Hesse, Werke, pp. 58, 651, 676.

§ Plücker, *System der Geometrie des Räumes*, §§87–92.

|| Rupp, *Stereographic projection of a quadric*, American Mathematical Monthly, vol. 25 (1928), pp. 415–421.

dependently to Chasles* and to Weddle† that the Pascal configuration might conveniently be considered as a property of a pair of triangles whose sides meet on a conic, and that the space extension would concern a pair of tetrahedra and a quadric; in the plane opposite sides of the triangles meet in a triple of points on a line, and hence these geometers reasoned that in space opposite faces of the tetrahedra would meet in a quadruple of lines on a regulus. They readily devised synthetic proofs of the theorem. Chasles made the pregnant observation that the twelve points common to the edges of a given tetrahedron and quadric could be arranged in several ways to define a second tetrahedron which would be effective in the theorem. Weddle went on to study various properties possessed by a pair of effective tetrahedra, and thereby discovered several properties later used by Schläfli in discussing Schläfli simplexes.

Closely allied to the theorem generalized by Chasles and by Weddle is another stating that if two triangles are polar reciprocal with respect to a conic, they are perspective, and hence the lines joining non-corresponding vertices touch a conic, and dually. Schläfli‡ discussed the analogous situation in S_n. His most important discovery was that two simplexes, or complete $(n+1)$-points, of S_n which are polar reciprocal with respect to a hyperquadric have what is now called the Schläfli property, i.e., the $n+1$ S_{n-2}'s of intersection of corresponding S_{n-1}'s (hyperplanes, faces) of the two simplexes lie in such a position that the lines of S_n which meet n of the S_{n-2}'s meet also the other. Following the suggestion of Berzolari,§ one now speaks of Schläfli simplexes, of a Schläfli set of lines or of S_{n-2}'s, or of lines or S_{n-2}'s in the position of Schläfli. Other contributions to the knowledge of the Schläfli situation are due to Brusotti‖ and the present writer.¶ The extension of Pascal's theorem which this paper presents has a close connection with the Schläfli situation.

It will be shown that it is possible to construct, from the points common to a hyperquadric and the edges of a simplex, a certain number of auxiliary simplexes, each of which may be paired with the given simplex to make a Schläfli pair; the ∞^{n-2} lines which meet the S_{n-2}'s of a Schläfli set thus

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* Chasles, *Aperçu Historique*, Note 32.

† Weddle, *On theorems in space analogous to those of Pascal and Brianchon in a plane*, Cambridge and Dublin Mathematical Journal, vol. 6 (1851), pp. 116–140.

‡ Schläfli, *Erweiterung des Sätzes, dass zwei polare Dreiecke perspektivisch liegen, auf eine beliebige Zahl von Dimensionen*, Journal für Mathematik, vol. 65 (1866), pp. 189–197.

§ Berzolari, *Sistemi di rette in posizione di Schläfli*, Rendiconti del Circolo Matematico di Palermo, vol. 20 (1905), pp. 229–248.

‖ Brusotti, Rendiconti del Circolo Matematico di Palermo, vol. 20 (1905), pp. 248–261.

¶ Rupp, (unpublished) Chicago (1928) doctoral thesis.

defined lie upon a variety of order and dimension $n-1$, denoted by the symbol V_{n-1}^{n-1}. On each V_{n-1}^{n-1} there are $(n-1)!$ families of generators; since in S_3 the families are called reguli, we shall use the term hyperregulus to refer to a set of ∞^{n-2} lines which are rulings of a V_{n-1}^{n-1}. The older literature about the variety appears to be scanty; Segre* twice mentions it casually in the Encyclopedia. Three papers on the variety have recently been published.†

I. The notation

The figure in S_n consisting of $n+1$ points, which do not lie in the same S_{n-1}, the $(n+1)n/2$ lines joining the points in pairs, the $(n+1)n(n-1)/6$ planes joining the points in triples, $\cdots$, and the $n+1$ S_{n-1}'s joining the points in n-tuples, is called a simplex. The points, lines, and hyperplanes (S_{n-1}'s) of a simplex are called its vertices, edges, and faces, respectively. Choose a given simplex F as the basis of a homogeneous coördinate system; the coördinates of the ith vertex P_i, where i runs from 0 to n, are all zero save at the ith place. The face of F that does not contain P_i is called the face opposite to P_i, and will be denoted by ρ_i; its equation is $x_i=0$.

A hypersurface (variety of dimension $n-1$) of the second order is the locus of points satisfying a quadratic equation; such a hypersurface is called a hyperquadric. Let there be given a hyperquadric of equation

$$(1)\qquad Q:\ \sum a_{ij}x_ix_j = 0 \qquad (i,j = 0,1,\cdots,n;a_{ij} = a_{ji}).$$

The edges of the fundamental simplex F meet the given hyperquadric Q in $2m$ piercing points P_{ij}, where $2m=n(n+1)$, and the points P_{ij}, P_{ji} lie on the line P_iP_j. To find the coördinates of these piercing points, solve the binary quadratic

$$(2)\qquad a_{ii}x_i^2 + 2a_{ij}x_ix_j + a_{jj}x_j^2 = 0.$$

It will be convenient to use the quantities defined by

$$(3)\qquad \begin{aligned} \Delta_{ij}^2 &= a_{ij}^2 - a_{ii}a_{jj},\\ b_{ii} = a_{ii}, \quad b_{ij} &= a_{ij} + \Delta_{ij} = b_{ji},\\ c_{ii} = a_{ii}, \quad c_{ij} &= a_{ij} - \Delta_{ij} = c_{ji}, \end{aligned}$$

in exhibiting the coördinates of P_{ij} and P_{ji} in the form

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* Segre, Encycklopädie der Mathematischen Wissenschaften, vol. III C7, p. 815 and p. 832.

† Wong, *On a certain system of* ∞^{r-2} *lines in* r*-space*, and *On the loci of the lines incident with* k $(r-2)$*-spaces in* S_r, Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 553–554, and pp. 715–717.

Rupp, *The equation of the* V_{n-1}^{n-1} *in* S_n, Bulletin of the American Mathematical Society, vol. 35 (1929), pp. 319–320.

(4)

	x_i	x_j	all other coördinates
P_{ij}	b_{ij}	$-b_{ii}$	0
P_{ij}	$-c_{jj}$	c_{ij}	0
P_{ji}	$-b_{jj}$	b_{ij}	0
P_{ji}	c_{ij}	$-c_{ii}$	0 .

Note that the coördinates of either piercing point on an edge of F may be expressed either in terms of b_{ij} or c_{ij}. In handling the first effective simplex which we shall set up, we shall use only the quantities b_{ij}; we shall find that the other effective simplexes can be found from the first, or standard one, by interchanging the elements in certain pairs of the points P_{ij} and P_{ji}. The corresponding change in the analytic work is accomplished by replacing the appropriate b_{ij} by c_{ij}. If the hyperquadric Q is a general one, the quantities b_{ij} and c_{ij} will all be different from zero.

The standard effective grouping of the $2m$ piercing points will be defined by the symbolic matrix

$$G_1 = (P_{ij}) \qquad (i \neq j). \tag{5}$$

The elements of the matrix (P_{ij}) are the piercing points P_{ij}; there are $n+1$ rows and columns, each containing n points. The points of the ith row share the first subscript i, and no pair of them share the second subscript. Geometrically, the points of the ith row lie one on each of the n edges of F which pass through the vertex P_i. Through the points of the ith row can be passed a unique S_{n-1}, which we call π_i and make correspond to the hyperplane ρ_i of F, which is the face of F opposite P_i.

The standard grouping G_1 accordingly defines $n+1$ hyperplanes π_i which constitute the faces of a simplex, T_1, said to be auxiliary to F, the fundamental simplex. The vertices of T_1 may be called the points R_i, R_i being opposite to π_i. The points R_i and P_i are said to be corresponding points.

Consider the problem of determining all the groups of the $2m$ piercing points which have the geometrical property of G_1. In choosing the points of the first row, we may take either of the piercing points on each of the n edges through the corresponding vertex, that is, there are 2^n possible ways to choose the first row. There are but 2^{n-1} ways to choose the second row, for on one of the edges there is but one available piercing point, the other having been already used. Proceeding in this fashion, we see that there are in all

$$2^n \times 2^{n-1} \times 2^{n-2} \times \cdots \times 2^2 \times 2 = 2^m$$

possible groupings of the $2m$ piercing points P_{ij} which have the same geometrical character that G_1 has.

In keeping account of the individual members of this set of 2^m effective groupings it is convenient to use a multiple index system; the general grouping of the set will be denoted by

$$G_\alpha = G_{\alpha_1\alpha_2\alpha_3\cdots\alpha_{n-1}},$$

where the general subscript α_r ranges through $1, 2, 3, \cdots$, to 2^{r+1} for all values of r from 2 to $n-1$ inclusive. The subscript α_1 ranges from 1 to 8. Each of the subscripts controls a certain sub-set of the points P_{ij}; the nature of this control is indicated in the following display:

	α_1	α_2	α_3	α_{n-1}
1	P_{10} $P_{20}\ P_{21}$	$P_{30}\ P_{31}\ P_{32}$	$P_{40}\ P_{41}\ P_{42}\ P_{43}$	$P_{n0}\ P_{n1}\cdots P_{n,n-1}$
2	P_{10} $P_{02}\ P_{21}$	$P_{03}\ P_{31}\ P_{32}$	$P_{04}\ P_{41}\ P_{42}\ P_{43}$	$P_{0n}\ P_{n1}\cdots P_{n,n-1}$
3	P_{10} $P_{20}\ P_{12}$	$P_{30}\ P_{13}\ P_{32}$	$P_{40}\ P_{14}\ P_{42}\ P_{43}$	$P_{n0}\ P_{1n}\cdots P_{n,n-1}$
4	P_{10} $P_{02}\ P_{12}$	$P_{30}\ P_{31}\ P_{23}$	$P_{40}\ P_{41}\ P_{24}\ P_{43}$	$\vdots$
5	P_{01} $P_{20}\ P_{21}$	$P_{03}\ P_{13}\ P_{32}$	$P_{40}\ P_{41}\ P_{42}\ P_{34}$	$\vdots$
6	P_{01} $P_{02}\ P_{21}$	$P_{03}\ P_{31}\ P_{23}$	$P_{04}\ P_{14}\ P_{42}\ P_{43}$	$\vdots$
7	P_{01} $P_{20}\ P_{12}$	$P_{30}\ P_{13}\ P_{23}$	$P_{04}\ P_{41}\ P_{24}\ P_{43}$	$\vdots$
8	P_{01} $P_{02}\ P_{12}$	$P_{03}\ P_{13}\ P_{23}$	$P_{04}\ P_{41}\ P_{42}\ P_{34}$	$\vdots$
$\vdots$			$\vdots$	$\vdots$
16			$P_{04}\ P_{14}\ P_{24}\ P_{34}$	$\vdots$
$\vdots$				$\vdots$
2^n				$P_{0n}\ P_{1n}\cdots P_{n-1,n}$.

To get the lower half of the matrix which is the complete display of $G_{\alpha_1\alpha_2\cdots\alpha_{n-1}}$, we adjoin the sets of points P_{ij} controlled by the various indices, and fill in the upper half of the matrix by inversive symmetry. As an example we give the upper left hand corner of the matrix $G_{38}\ldots$:

$$\begin{array}{rcccc} & * & P_{01} & P_{02} & P_{30} \\ \alpha_1 = 3 \rightarrow \left\{ \begin{array}{c} \\ \\ \end{array} \right. & \begin{array}{c} P_{10} \\ P_{20} \end{array} & \begin{array}{c} * \\ P_{12} \end{array} & \begin{array}{c} P_{21} \\ * \end{array} & \begin{array}{c} P_{31} \\ P_{32} \end{array} \\ \alpha_2 = 8 \rightarrow & P_{03} & P_{13} & P_{23} & * \; . \end{array}$$

Each of these groupings G_α will define, in the same manner that G_1 did, a simplex, T_α, auxiliary to the fundamental simplex F. The 2^m simplexes T_α are called the auxiliary simplexes of the extended Pascal configuration in S_n, or simply the simplexes of the Kf_n. The 2^m pairs of simplexes composed of the fundamental simplex F and one of the auxiliary simplexes will be called the pairs of Kf_n. It will be shown that the pairs of Kf_n are Schläfli pairs of simplexes, and hence each determines a hyperregulus; the 2^m hyperreguli will be called the hyperreguli of Kf_n; thus the symbol Kf_n means the total configuration of geometric elements associated with the extended Pascal theorem in S_n.

II. Theorems in S_n

THEOREM 1. *The intersections of corresponding faces (and the lines joining corresponding vertices) of the pairings of an extended Pascal configuration in S_n are linearly dependent.*

Consider first the pair of simplexes F and T_1, corresponding to the grouping G_1. By the use of (4), it is seen that the equations of the faces of T_1 are

$$\sum_j b_{ij}x_j = 0 \qquad (i,j = 0,1,\cdots,n). \tag{6}$$

The equations of the $n+1$ S_{n-2}'s of intersection with the corresponding faces of F may be written as

$$x_i = \sum_j b_{ij}x_j = 0. \tag{7}$$

The Plücker-Grassmann coördinates of these S_{n-2}'s are the two-rowed determinants formed from the matrices of coefficients in the equations (7). Each set of coördinates has exactly n elements that are not zero; they are shown in the following display:

$$
(8)\qquad
\begin{array}{cccccccccccccccc}
 & p_{01} & p_{02} & p_{03} & \cdots & p_{0n} & p_{12} & p_{13} & \cdots & p_{1n} & p_{23} & \cdots & p_{2n} & \cdots & & p_{n-1,n} \\
\pi_0\rho_0 & b_{01} & b_{02} & b_{03} & \cdots & b_{0n} & & & & & & & & & & \\
\pi_1\rho_1 & -b_{10} & & & & & b_{12} & b_{13} & \cdots & b_{1n} & & & & & & \\
\pi_2\rho_2 & & -b_{20} & & & & -b_{21} & & & & b_{23} & \cdots & b_{2n} & & & \\
\pi_3\rho_3 & & & -b_{30} & & & & -b_{31} & & & -b_{32} & & & \cdots & & \\
 & & \cdots & & \cdots & & & & & & & & & \cdots & & b_{n-1,n} \\
\pi_n\rho_n & & & & & -b_{n0} & & & & -b_{n1} & & & -b_{n2} & \cdots & & -b_{n,n-1}
\end{array}
$$

In each column there are two and only two elements, the sum of which is zero since $b_{ij}=b_{ji}$. The display shows that there is linear dependence between the sets of coördinates of the $n+1$ S_{n-2}'s of intersection of corresponding faces of the simplexes F and T_1, which is what is meant when it is said that the S_{n-2}'s themselves are linearly dependent.

By duality it follows that the lines joining corresponding vertices of the two simplexes are also linearly dependent.

Suppose now the members of certain pairs of piercing points are interchanged. The effect will be to replace the standard grouping G_1 by some particular one of the set G_α; if we know which pairs of piercing points are interchanged, we know which G_α is represented by the modified G_1. For example, suppose that P_{04} and P_{40} are interchanged; the standard grouping G_1 is replaced by $G_{112111\ldots1}$. If now we interchange b_{04} and c_{04} in the equations (7) and display (8), we have converted the proof that G_1 is an effective grouping in Theorem 1 into a proof that the modified grouping is also effective. Since all of the 2^m groupings G_α were obtained from the standard G_1 by such inversions of subscripts among the points P_{ij}, it follows that all the groupings G_α are effective in the theorem.

THEOREM 2. *The pairs of the extended Pascal configuration in S_n are pairs of Schläfli simplexes.*

This follows at once from the theorem* that a set of $n+1$ linearly dependent S_{n-2}'s in S_n are in the position of Schläfli. A quite different proof is based on the fact that the simplexes of a pair are polar reciprocal with respect to a hyperquadric, which, it will be remembered, was Schläfli's original point of departure. The form of equations (6) show that the hyperplanes π_i are the polar hyperplanes of the points P_i with respect to the hyperquadric of equation

$$
(9)\qquad \sum b_{ij}\, x_i\, x_j = 0,
$$

* Rupp, *A geometrical interpretation of linear dependence in S_n*, abstracted in the Bulletin of the American Mathematical Society, vol. 35 (1929), p. 171.

and accordingly that the points R_i are the poles of the faces ρ_i, that is, the simplexes F and T_1 are polar reciprocal with respect to the hyperquadric in question.

It is a fundamental property of Schläfli simplexes that all the lines which meet n of the S_{n-2}'s of intersection of corresponding faces meet also the remaining S_{n-2}. In the author's note on the V_{n-1}^{n-1} previously cited is given a method of obtaining the point equation of the variety made up of the lines meeting n given S_{n-2}'s of S_n. The use of that method here will give the equations of the varieties bearing the 2^m hyperreguli of the extended Pascal configuration.

By the use of the incidence criterion for line and S_{n-2}, as expressed by Grassmann-Plücker coördinates, the condition that x be the coördinates of a point such that the line joining x to a point of the S_{n-2} $\pi_0\rho_0$ meets also the $n-1$ S_{n-2}'s $\pi_i\rho_i$, where i runs from 1 to $n-1$, is found to be the equation

$$
\begin{vmatrix}
b_{01} & b_{02} & b_{03} & \cdots & b_{0,n-1} & b_{0n} \\
-\sum_{i}^{j\neq 1} b_{1j}x_j & b_{12}x_1 & b_{13}x_1 & \cdots & b_{1,n-1}x_1 & b_{1n}x_1 \\
b_{21}x_2 & -\sum_{i}^{j\neq 2} b_{2j}x_j & b_{23}x_3 & \cdots & b_{2,n-1}x_2 & b_{2n}x_2 \\
b_{31}x_3 & b_{32}x_3 & -\sum_{i}^{j\neq 3} b_{3j}x_j & \cdots & b_{3,n-1}x_3 & b_{3n}x_3 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
b_{n-1,1}x_{n-1} & b_{n-1,2}x_{n-1} & \cdot & \cdots & -\sum_{i}^{j\neq n-1} b_{n-1,j}x_j & b_{n-1,n}x_{n-1}
\end{vmatrix} = 0. \tag{10}
$$

It should be observed that two options were exercised in writing the foregoing equation, one in choosing the variable line through x to pass through a point of $\pi_0\rho_0$, and one in considering $\pi_n\rho_n$ as the last S_{n-2} of the Schläfli set. The consequence of the first choice is the absence of a variable in the top row of the determinant, and the consequence of the second choice is that the coördinate x_n appears only in the summations. A different ordering of the Schläfli set gives rise to a superficially different form of equation (10), but it is easy to prove the two forms are equivalent.

If we replace, in equation (10), certain of the symbols b_{ij} by the corresponding c_{ij}, the resulting equation will be that of the V_{n-1}^{n-1} bearing the hyperregulus associated with one of the groupings G_α.

THEOREM 3. *The section of the extended Pascal configuration associated with a given hyperquadric Q and fundamental simplex F by a space of the simplex F is itself an extended Pascal configuration plus certain residual flat spaces.*

Consider first the nature of the intersection of a hyperplane of F, say ρ_n, with the V_{n-1}^{n-1} given by equation (10); we shall call the V_{n-1}^{n-1}'s of the Kf_n simply the V_α, in which case the one given by equation (10) is V_1, for it corresponds to G_1. The equation of ρ_n is $x_n=0$; part of its intersection with V_1 is the S_{n-2} $\rho_n\pi_n$, as we may see by a manipulation of the equation of V_1. If the top row of the determinant in (10) is multiplied by x_0, and the other rows added to it, the new top row has the form

$$b_{1n}x_n \quad b_{2n}x_n \quad b_{3n}x_n \quad \cdots \quad b_{n-1,n}x_n \quad b_{0n}x_0 + b_{1n}x_1 + \cdots + b_{n-1,n}x_{n-1},$$

whose elements are identically zero if x satisfies the restrictions

$$x_n = \sum_i b_{in}x_i = 0,$$

that is, if x lies on $\rho_n\pi_n$. We remark that this incidentally furnishes an analytic proof of the theorem of Wong that the S_{n-2}'s defining a V_{n-1}^{n-1} lie on it; in this connection it may be said that the same theorem follows at once from the present writer's geometric interpretation of linear dependence in S_n, for, just as $n+2$ points which are linearly dependent lie in, and determine, a flat space, an S_n, so do $n+1$ linearly dependent S_{n-2}'s lie in, and determine, a curved manifold, a V_{n-1}^{n-1}.

To return to the discussion of Theorem 3, we have seen that one factor of the intersection of V_1 and ρ_n is an S_{n-2}; it remains to find the residual portion. Suppose a point of ρ_n is such that its coördinates annul the determinant which is the cofactor of $b_{n-1,\,n}x_{n-1}$ in the left member of (10); it will lie on a V_{n-2}^{n-2}, for it satisfies an equation of the same form as (10). We do not mean to say that the V_{n-2}^{n-2} and the S_{n-2} of intersection of V_1 with a hyperplane of F have no common points; they actually meet in a V_{n-3}^{n-2}. For our present purposes, nevertheless, it is the V_{n-2}^{n-2} and the S_{n-2} which are the important spaces of intersection.

We have seen that an S_{n-1} of F meets V_1 in a V_{n-2}^{n-2}. Consider now the intersection of the 2^n V_{n-1}^{n-1}'s whose symbols are

$$V_{11\cdots 11\alpha_{n-1}} \qquad (\alpha_{n-1} = 1,2,\cdots,2^n),$$

with the S_{n-1} ρ_n. They all share the V_{n-2}^{n-2} whose equation, in ρ_n, is obtained by setting the cofactor of $b_{n-1,\,n}x_{n-1}$ in the determinant in the left member of (10) equal to zero, because the index α_{n-1} by definition controls only the symbols b_{ni}, where i runs from 0, 1, to $n-1$, which appear exclusively, if we keep to the S_{n-1} ρ_n, in the last column of the determinant of (10). Varying the first $n-2$ indices of V_α is equivalent to replacing certain of the b_{ij} by the corresponding c_{ij}. Thus ρ_n meets the 2^m V_{n-1}^{n-1}'s which bear the Kf_n in 2^{n+1} V_{n-2}^{n-2}'s, and these V_{n-2}^{n-2}'s bear the Kf_{n-1} of the simplex $P_0P_1\cdots P_{n-1}$

set up by means of the hyperquadric which is the intersection of Q with ρ_n. The $S_{n-1}\,\rho_n$ also meets the V_{n-1}^{n-1}'s of the Kf_n in the S_{n-2}'s in which it meets its corresponding faces among the auxiliary simplexes $T_{11\ldots 11\alpha_{n-1}}$. We may clearly reduce the Kf_{n-1} to a Kf_{n-2} in a similar manner. Had we desired to ascertain the nature of the intersections of the Kf_n with another S_{n-1} of F, we could clearly have chosen the appropriate form of equation (10) and proceeded as above.

An example of the successive reduction of the Kf_n is given below where we show that the planes of a tetrahedron meet the 64 V_2^2's bearing the Kf_3 in a Kf_2 and certain residual spaces.

III. The extended Pascal configuration in the plane

For the case that n is 2, we consider a fundamental triangle F and conic Q. Paired with F are the triangles

$$T_{\alpha_1} \qquad (\alpha_1 = 1,2,\cdots,8),$$

each pairing of triangles possessing the property that the points of intersection of corresponding sides are linearly dependent, i.e., collinear. The line of collinearity is an axis of perspectivity for the pair of triangles, and also the ordinary Pascal line of the Pascal hexagon whose sides are the sides of the two triangles, arranged in proper order. The 2^m V_{n-1}^{n-1}'s here reduce to the eight lines V_{α_1}; the line V_{α_1} associated with an effective grouping G_{α_1} constitutes the entire class of lines meeting any two of the three linearly dependent Schläfli points of the pair of triangles, and as a member of such a class it has the Schläfli property of meeting the third point. The Kf_2 consists of these eight V_{α_1}; the reciprocals of the line coördinates of these are shown in the array below:

(11)

	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8
$1/u_1$	b_{12}	b_{12}	c_{12}	c_{12}	b_{12}	b_{12}	c_{12}	c_{12}
$1/u_2$	b_{20}	c_{20}	b_{20}	c_{20}	b_{20}	c_{20}	b_{20}	c_{20}
$1/u_3$	b_{01}	b_{01}	b_{01}	b_{01}	c_{01}	c_{01}	c_{01}	c_{01}.

We notice that the pairs of lines whose indices sum to nine intersect upon the line of equation

$$\sum_k\left(\frac{1}{b_{ij}}+\frac{1}{c_{ij}}\right)x_k = 0 \qquad (i,j,k \text{ a permutation of } 0,1,2), \tag{12}$$

an equation which may be rewritten, by the use of (3), in the form

$$\sum_k \frac{a_{ij}}{a_{ii}a_{jj}}x_k = 0. \tag{12'}$$

This line is a Steiner-Plücker line of the Pascal hexagon whose vertices are the six points P_{ij}; the three hexagons formed of the triangle pairs FT_1, T_1T_8, and T_8F, are readily seen to be three hexagons in the Steinerian relation, i.e., three vertices are fixed, and the remaining three permuted cyclically. It follows that the lines V_1 and V_8 meet in a Steiner point; since the same argument applies to all pairs of the Kf_2 whose indices sum to nine, the four points which define the line of equation (12) must be Steiner points, and hence their line of collinearity is a Steiner-Plücker line.

An independent check is found by using a table due to Cayley* which shows the nature of the intersection of the pairs of the 60 Pascal lines. In correlating the two notations, let the triangles F and T_1 be the hexagon denoted by the Cayley letters AE. The sides are arranged in the order $\rho_2\pi_0\rho_1\pi_2\rho_0\pi_1$, and the vertices in the order $P_{01}P_{02}P_{20}P_{21}P_{12}P_{10}$, or, for brevity, in the order 123456. If the symbols p, h, and g denote Pascal, Kirkman, and Steiner points respectively, Cayley's table can be correlated with the present paper as below:

Group	Hexagon	Cayley Letters	Nature of Intersection							
				V_1	V_2	V_3	V_4	V_5	V_6	V_7
1	123456	AE								
2	132456	EL	V_2	p						
3	123546	EF	V_3	p	h					
4	132546	EG	V_4	h	p	p				
5	162345	DE	V_5	p	h	h	g			
6	163245	EI	V_6	h	p	g	h	p		
7	162354	EM	V_7	h	g	p	h	p	h	
8	163254	EH	V_8	g	h	h	p	h	p	p

This display shows that the pairs V_1V_8, V_2V_7, V_3V_6, and V_4V_5 meet in Steiner points, but it does not show that the Steiner points thus garnered lie on a line.

THEOREM 4. *The Kf_2 determines a Steiner-Plücker line.*

In the plane, the Kf_2 is apparently a less rich configuration than the Hexagrammaticum Mysticum, for it has but 8 lines where the other has 60. Consider, however, a different approach to the general situation in S_n by taking, as did Schläfli, two simplexes polar reciprocal with respect to a hyperquadric. We have already observed that F and T_1 are polar reciprocal

* Cayley, *A notation for Pascal's theorem*, Quarterly Journal of Mathematics, vol. 9 (1868), pp. 268–274.

with respect to the hyperquadric given by equation (9). Let us take this hyperquadric as basic, and try to determine the hyperquadric Q whose matrix involves the numbers a_{ij}. We now define the set of 2^m points P_{ij} as the intersection of P_iP_j with π_i, and an analogous set of 2^m points R_{ij} as the intersection of R_iR_j with ρ_i; it is an easily proved property of Schläfli simplexes that the points P_{ij} and R_{ij} lie upon hyperquadrics, which are in general different, coinciding only when n is 2. In the plane there is associated with a fundamental triangle F by means of a conic Q a set of eight auxiliary triangles; associated with any one of these auxiliary triangles by means of the same conic Q is a set of eight more auxiliary triangles, which set partially overlaps the first. Indeed, the six points P_{ij} where the sides of F meet Q determine 15 lines, which may be grouped, by triples, into 15 distinct triangles. Any one of these 15 triangles pairs with eight others to make a Pascal hexagon; there are $(15\times 8)/(1\times 2)=60$ different Pascal hexagons. It is because the points R_{ij} coincide with the points P_{ij} in the plane case that there exists a configuration of more than 2^m lines. The richness of the usual Pascal configuration is from this point of view due to the coincidence of the hyperquadrics (in the plane case, conics) determined by the sets of points P_{ij} and R_{ij}.

The body of theorems about the Pascal configuration can be established from a consideration of the 15 triangles instead of the more usual approach by way of the 60 Pascal hexagons.

IV. The extended Pascal configuration in space

The edges of a fundamental tetrahedron F pierce a given quadric (1) in the 12 points P_{ij}, which are grouped in the 64 effective groupings $G_{\alpha_1\alpha_2}$ $(\alpha_1\alpha_2=1, 2, \cdots, 8)$, each of which groupings defines an auxiliary tetrahedron $T_{\alpha_1\alpha_2}$. The lines of intersection of corresponding faces of the fundamental tetrahedron F and an auxiliary tetrahedron $T_{\alpha_1\alpha_2}$ are linearly dependent, hence these four lines lie on a regulus. The Kf_3 is made up of the 64 conjugate reguli, that is, of the 64 one-parameter families of lines which meet the 64 sets of linearly dependent lines. A pairing of the Kf_3, as F and $T_{\alpha_1\alpha_2}$, is a Schläfli pair of tetrahedra, that is, such that all the lines meeting three of the lines of intersection of corresponding faces meet also the fourth.

The theorem of Chasles and Weddle asserts that the four lines of intersection of corresponding faces lie on a regulus. Chasles further remarked that there are several effective groupings of the 12 piercing points, but apparently he did not consider how many effective groupings there are, nor what incidence relations exist among the various reguli.

The V_{n-1}^{n-1} defined by a pairing is here a V_2^2, or quadric; it contains the regulus of lines which meet the four linearly dependent lines, and also the lines themselves.

THEOREM 5. *The fundamental tetrahedron F circumscribes each of the quadrics* $V_{\alpha_1\alpha_2}$.

In the derivation of $V_{\alpha_1\alpha_2}$ it appears that it contains the lines $\rho_i\pi_i$ $(i=0, 1, 2, 3)$, that is, it has a generator in each of the planes ρ_i, hence it is tangent to each of these planes.

THEOREM 6. *The* 128 *lines common to a face of F and the* 64 *quadrics* $V_{\alpha_1\alpha_2}$ *consist of two* Kf_2*'s, each line being counted eight times.*

The face ρ_3 has the equation $x_3=0$. It meets V_{11} in a conic of equation

$$0 = x_3 = \left(\sum_i^{012} b_{ij}b_{ik}x_i \right)\left(\sum_i^{012} b_{i3}x_i \right), \tag{13}$$

where i, j, k is a permutation of 0, 1, 2. The eight quadrics $V_{1\alpha_2}$ have equations differing from that of V_{11} only in the coefficients controlled by α_2, i.e., the coefficients b_{03}, b_{13}, and b_{23}, which appear segregated in one factor of (13). It follows that the eight quadrics $V_{1\alpha_2}$ share the ruling of equations

$$0 = x_3 = \sum_i^{012} b_{ij}b_{ik}x_i. \tag{14}$$

In like manner the eight quadrics $V_{2\alpha_2}$ share the ruling of equations

$$0 = x_3 = b_{01}c_{02}x_0 + b_{10}b_{12}x_1 + c_{20}b_{21}x_2.$$

The eight lines obtained by varying the first index constitute the Kf_2 of $P_0P_1P_2$ with respect to the conic cut from Q by $x_3=0$, as may be seen by comparing their equations with the display (11).

Octuples of quadrics sharing a common second index likewise have a common generator. The coördinates, in $x_3=0$, of the eight lines thus obtained are shown below:

(15)

	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8
u_1	b_{03}	c_{03}	b_{03}	b_{03}	c_{03}	c_{03}	b_{03}	c_{03}
u_2	b_{13}	b_{13}	c_{13}	b_{13}	c_{13}	b_{13}	c_{13}	c_{13}
u_3	b_{23}	b_{23}	b_{23}	c_{23}	b_{23}	c_{23}	c_{23}	c_{23}

The general similarity of (11) and (15) suggests that the second set of eight lines is the Kf_2 of $P_0P_1P_2$ with respect to some conic. To find this conic, solve (3) for a_{ij} in terms of b_{ij}, obtaining

$$a_{ii} = b_{ii}, \quad 2b_{ij}a_{ij} = b_{ij}^2 + b_{ii}b_{jj}.$$

We next consider how we might pass from the lines of (11) to the conic whose matrix involves the numbers a_{ij}, and use the same method to obtain the conic of matrix

$$(16) \qquad \begin{pmatrix} 2a_{00} & \beta_{23} + a_{00}a_{11}b_{23} & \beta_{13} + a_{00}a_{22}b_{13} \\ \beta_{23} + a_{00}a_{11}b_{23} & 2a_{11} & \beta_{03} + a_{11}a_{22}b_{23} \\ \beta_{13} + a_{00}a_{22}b_{13} & \beta_{03} + a_{11}a_{22}b_{03} & 2a_{22} \end{pmatrix},$$

where $\beta_{ij}b_{ij} = 1$, by means of which the lines of coördinates (15) are the Kf_2 of $P_0P_1P_2$. This conic appears to have no simple geometrical relation to the original quadric Q, whereas the corresponding conic associated with the first set of eight lines is the intersection of Q by $x_3 = 0$.

It is clear that ρ_3 is in no wise an exceptional face of F, hence the situation on it is duplicated on the other planes of the tetrahedron. This completes the proof of Theorem 6.

THEOREM 7. *A tetrahedron and a quadric determine two sets of Steiner-Plücker lines, each set lying on a regulus.*

On each face of the fundamental tetrahedron there are two Steiner-Plücker lines, one deriving from each of the Kf_2's. One Kf_2 consists of lines belonging to the reguli which form the Kf_3 of the tetrahedron and quadric; the four associated Steiner-Plücker lines have the equations

$$(17) \qquad 0 = x_i = \sum_j^{j\neq i} \frac{a_{km}}{a_{kk}a_{mm}} x_j \qquad (ijkm \text{ a permutation of } 0123),$$

equations which show by their symmetry that the four lines in question lie on a regulus. The same is true of the equations of the four Steiner-Plücker lines deriving from the Kf_2's whose lines lie on the conjugate reguli of the Kf_3; the equations are

$$(17') \qquad 0 = x_i = \sum_j^{j\neq i} (b_{ji} + c_{ji}) x_j .$$

V. THE EXTENDED PASCAL CONFIGURATION IN S_4

Here we have 1024 two-parameter families of lines meeting the quintuples of planes of intersection of corresponding hyperplanes in the pairings of a fundamental simplex F with the auxiliary simplexes $T_{\alpha_1\alpha_2\alpha_3}$ ($\alpha_1, \alpha_2 = 1, 2, \cdots,$ $8, \alpha_3 = 1, \cdots, 16$), these auxiliary simplexes being defined by the 1024 effective groupings $G_{\alpha_1\alpha_2\alpha_3}$ of the 20 points in which the edges of F pierce a given hyperquadric Q. The two-parameter families of lines, or hyperreguli, lie on a set of V_3^3's; the V_3^3 is a V_{n-1}^{n-1} which has been thoroughly studied by Segre* and Castelnuovo,† its co-discoverers, Berzolari,‡ and many others.

The 16 V_3^3's denoted by $V_{11\alpha_3}$ meet the S_3 whose equation is $x_4 = 0$ in the same quadric; its equation in ρ_4 is that of V_{11} in three-space. It follows that the 1024 V_3^3's meet ρ_4 in $64 V_2^2$'s, each counted 16 times;this is a special case of Theorem 3.

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* Segre, Atti, Accademia delle Scienze di Torino, vol. 22 (1887), pp. 791–801; Memorie, Accademia delle Scienze di Torino, (2), vol. 39 (1888), pp. 3–32.

† Castelnuovo, Atti, Istituto Veneto, (6), vol. 5 (1887), p. 1249, and vol. 6 (1888), p. 525.

‡ Berzolari, Rendiconti, Accademia dei Lincei, (5), vol. 25_1 (1917), pp. 29 and 102.

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