CONCERNING THE ARC-CURVES AND BASIC SETS OF A CONTINUOUS CURVE, SECOND PAPER*

BY

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I. Introduction

In an earlier paper‡ with the same title, we have defined and studied the properties of certain subsets of a continuous curve§ which we call the *arc-curves* of the continuous curve. In a recent paper, G. T. Whyburn|| has defined the *cyclic elements* of a continuous curve, and he has considered a continuous curve as composed of its cyclic elements and has given a large number of the properties of connected collections of cyclic elements. On examining the two papers it is found that arc-curves and connected collections of cyclic elements have many properties in common; and, in fact, in part II of the present paper we shall show that, although these two sets were defined very differently, every connected collection of cyclic elements of a continuous curve is an arc-curve of the continuous curve, and conversely, every arc-curve that contains more than one point is a collection of cyclic elements of the continuous curve. In part III we will develop some new theory concerning the basic sets of a continuous curve, which were defined in *Arc-curves, first paper*, and shall show the relation between the basic sets and the nodes of a continuous curve. In part IV we shall show that an irreducible basic set of a continuous curve resembles in its properties the set of all end points of the continuous curve.

All point sets considered in this paper are assumed to lie in a metric, separable, locally compact space.

Notation. We shall use the common notation of the theory of sets, such as $A+B$, $A-B$, $A\cdot B$, etc., in its usual meaning. If H is a point set, the symbol $\overline{H}$ denotes the point set consisting of the points of H together

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‡ *Concerning the arc-curves and basic sets of a continuous curve*, these Transactions, vol. 30 (1928), pp. 567–578. Hereafter in the present paper we will refer to this paper as *Arc-curves, first paper*.

§ A *continuous curve* is a point set which is closed, connected and connected im kleinen. Unless definitely stated, there is no assumption as to compactness.

|| *Concerning the structure of a continuous curve*, American Journal of Mathematics, vol. 50 (1928), pp. 167–194. Hereafter we will refer to this paper as *Structure*.

with all the limit points of H. If X and Y are point sets (either of which may, in particular, be a single point), the *distance* from X to Y, denoted by $d(X, Y)$, is the greatest lower bound of all the numbers $d(x, y)$, where x is a point of X, y is a point of Y and $d(x, y)$ denotes the distance from the point x to the point y. If xyz denotes an arc with end points x and z, the symbols $<xyz, xyz>, <xyz>$ denote $xyz-x$, $xyz-z$ and $xyz-x-z$ respectively.

II. The arc-curves and the cyclic elements

Definitions.* If K is a subset of the continuous curve M, the set of all points $[P]$, such that P lies on some arc of M whose end points belong to K, is called the *arc-curve of K with respect to M* and is denoted by $M(K)$. In case the set K consists of but a single point, the arc-curve of K with respect to M is defined to be this single point. A continuous curve M is said to be *cyclicly connected* if every two points of M lie together on some simple closed curve of M. A cyclicly connected continuous curve C which is a subset of a continuous curve M is said to be a *maximal cyclic curve* of M if and only if no cyclicly connected continuous curve which is a subset of M contains C as a proper subset. A subset E of a continuous curve M is said to be a *cyclic element* of M provided that E is either (a) a maximal cyclic curve of M, (b) a cut point† of M or (c) an end point‡ of M. A point set N is said to be a *simple cyclic chain* of M between two cyclic elements E_1 and E_2 of M provided that N is connected, contains E_1 and E_2, is the sum of the elements of some collection of cyclic elements of M, and furthermore no proper connected subset of N contains E_1 and E_2 and is the sum of the elements of such a collection.

THEOREM 1. *If M is a continuous curve and K is a subset of M containing more than one point, then the arc-curve $M(K)$ is a collection of cyclic elements of M. Conversely, if the connected subset H of M is a collection of cyclic elements of M, then there is a subset K of H such that H is the arc-curve $M(K)$.*

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* These definitions are given in *Arc-curves, first paper*, p. 568, and *Structure*, pp. 167–8.

† The point P of the connected set M is said to be a *non-cut point* or a *cut point* according as the set $M-P$ is connected or not. See R. L. Moore, *Concerning the cut points of continuous curves and other closed and connected point sets*, Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 101–106.

‡ The point P of a continuous curve M is said to be an *end point* of M provided that if P' is any other point of M and $P'P$ is any arc of M with end points P and P', then the set $M-P'P>$ contains no connected subset containing more than one point that contains the point P. See R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 358. For other definitions that are equivalent to that of Wilder, see H. M. Gehman, *Concerning end points of continuous curves and other continua*, these Transactions, vol. 30 (1928), pp. 63–84.

Let A and B be any two points of K and let α be any arc of M from A to B. Let P be any point of the set $<\alpha>$. If P is a cut point of M, then it is itself a cyclic element of M. If P is not a cut point of M, then by a result due to G. T. Whyburn* there is a maximal cyclic curve C having in common with α an arc β which contains P. Let X and Y denote the end points of β such that we have the order $AXYB$ on the arc α. If Q is any point of $C-X-Y$, it follows by a theorem due to the author† that there exists an arc γ of C with end points X and Y and containing the point Q. Then the arc γ together with the subarcs AX and YB of α is an arc of M with end points A and B. Hence every point Q of C belongs to the arc-curve $M(K)$. Thus if P is not a cut point of M, it belongs to a maximal cyclic curve of M which belongs to the arc-curve $M(K)$.

Now consider the point A, the case for B being similar. If A is an end point or a cut point of M, then it is itself a cyclic element of M. If the point A is neither, it belongs to some simple closed curve J which is a subset of M.‡ There is a maximal cyclic curve C containing the simple closed curve J.§ As A is a non-cut point of M, the curve C and the arc α have in common an arc β containing A.‖ Let X be the end point of β different from A. If Q is any point of $C-A$, there exists an arc γ of C with end points A and X and containing the point Q. Then the arc γ plus the subarc XB of α is an arc of M whose end points A and B belong to K. Hence every point Q of C belongs to the arc-curve $M(K)$.

We have shown that every point of the arc-curve $M(K)$ is a subset of

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* G. T. Whyburn, *Some properties of continuous curves*, Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 305–308. Whyburn states the result for two dimensions only, but with the use of certain results established in my paper *Concerning continuous curves in metric space*, to appear in the American Journal of Mathematics, it may be easily extended to the more general space considered here.

† See my paper *Continuous curves which are cyclicly connected*, Bulletin de l'Académie Polonaise des Sciences et des Lettres, 1928, pp. 127–142.

‡ See W. L. Ayres, *Concerning continuous curves and correspondences*, Annals of Mathematics, vol. 28 (1927), pp. 396–418, Theorem 3, and G. T. Whyburn, *Concerning continua in the plane*, these Transactions, vol. 29 (1927), pp. 369–400, Theorem 22. These results are extended to more general space in my paper *Concerning continuous curves in metric space*. In general this is true of all other references in this paper which state results for two dimensions only. When this is not the case mention will be made of the extension. Otherwise it will be understood that the extension is to be found in the above paper.

§ G. T. Whyburn, *Cyclicly connected continuous curves*, Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 31–38, Theorem 3.

‖ If P is a non-cut point of a continuous curve M belonging to a maximal cyclic curve C of M, and α is an arc of M containing P, then C and α have in common an arc β containing P, and if P is interior to the arc α it is an interior point of β.

The proof is sufficiently evident to be omitted.

some cyclic element of M that belongs to the arc-curve $M(K)$; or in other words, the arc-curve $M(K)$ is a collection of cyclic elements of M.

Conversely, we shall show that if H is a connected subset of M which is a collection of cyclic elements of M, then H contains a subset K such that H is the arc-curve $M(K)$. Let K be the set H. G. T. Whyburn* has shown that H contains every arc of M joining two points of H; hence the arc-curve $M(H)$ is a subset of H. But evidently H is a subset of the arc-curve $M(H)$. Therefore H is identical with the arc-curve $M(H)$. There are also many proper subsets K of H such that H is identical with the arc-curve $M(K)$.

THEOREM 2. *If A and B are distinct points of the continuous curve M, and K is the arc-curve $M(A+B)$, then there are two cyclic elements E_1 and E_2† of M containing A and B respectively such that K is a simple cyclic chain of M between the two cyclic elements E_1 and E_2. Conversely, if H is a simple cyclic chain of M between two cyclic elements E_1 and E_2 of M, then E_1 and E_2 contain points A and B respectively such that H is the arc-curve $M(A+B)$.*

By Theorem 1 there are two cyclic elements E_1 and E_2 of M that contain A and B respectively and belong to the arc-curve K. Also the set K is connected and is the sum of the elements of a collection of cyclic elements of M. There is a unique‡ simple cyclic chain H of M between E_1 and E_2 which is a continuous curve§ and is a subset of K.‖ If H is a proper subset of the continuous curve K,¶ let N be a component of $K-H$. If the component N of $K-H$ has more than one limit point in H, it is easy to see that M contains an arc whose end points belong to H and which lies except for its end points entirely in the set N. But this is impossible since H contains every arc of M joining two points of H.** Hence the set H contains just one limit point of N. Let P be this point. Then P is a cut point of the arc-curve K and the component N of $K-P$ contains neither A nor B. This is impossible by a theorem of the author.†† The assumption that H is a proper subset of K leads to an absurdity. Thus H is identical with K.

For the converse, if E_1 is a single point let this point be A. If E_1 is a

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* *Structure*, Theorem 8.

† The two elements E_1 and E_2 are not necessarily different.

‡ *Structure*, Theorem 3.

§ *Structure*, Theorem 4.

‖ *Structure*, Theorem 8.

¶ Whenever X is a closed point set, the arc-curve $M(X)$ is a continuous curve. See *Arc-curves, first paper*, Theorem 7.

** *Structure*, Theorem 8.

†† If P is a cut point of the arc-curve $M(K)$, every component of $M(K)-P$ contains a point of K. See *Arc-curves, first paper*, Theorem 9.

maximal cyclic curve of M, let A be any point of E_1 that is not a cut point of H. Only one point of E_1 is a cut point of H and thus A may be chosen as any other point of E_1. In a similar manner let us select a point B of the cyclic element E_2. Since every arc of M joining two points of H belongs to H, the arc-curve $M(A+B)$ is a subset of H; and in much the same manner as above we may show that it is not a proper subset. Then the arc-curve of $A+B$ with respect to M must be identical with the simple cyclic chain H.

In view of the relation between collections of cyclic elements and arc-curves shown in the preceding theorems, we have the following forms for three results of G. T. Whyburn.*

THEOREM 3. *If A and B are distinct points of the continuous curve M, K is the arc-curve $M(A+B)$ and G is a cyclic element of K that contains neither A nor B, then $K-G=K_1+K_2$, where K_1 and K_2 are mutually separated point sets, and* (1) *K_1 and K_2 are connected and contain A and B respectively,* (2) *the cyclic element G contains a point C such that K_1+G and K_2+G are the arc-curves of $A+C$ and $B+C$ respectively with respect to M.*

THEOREM 4. *If A and B are distinct points of the continuous curve M, the arc-curve H of $A+B$ with respect to M is not disconnected by the omission of any cyclic element of M which contains either A or B and belongs to H, or by any set of such cyclic elements.*

THEOREM 5. *Every arc-curve of the continuous curve M is uniformly connected im kleinen.*

Since every connected collection of cyclic elements is an arc-curve, and every arc-curve of two points is a simple cyclic chain, and the converse of each is true, it is to be expected that there is some overlap in the results of the two papers. For instance, Whyburn† proves that if H is a connected collection of cyclic elements of M, then (1) H is arc-wise connected, cyclicly chainwise connected‡ and contains every simple continuous arc of M which joins two points of H, (2) each point of $\overline{H}-H$ is a cyclic element, (3) if H is closed it is a continuous curve. Property (1) is contained in Theorem 3 of *Arc-curves, first paper*, which states that if K is any subset of M and α is an arc of M whose end points belong to $\overline{M(K)}$, then the arc-curve $M(K)$ contains every point of α except possibly the end points. In *Arc-curves*,

* *Structure*, Theorems 5, 6, 9.

† *Structure*, Theorems 8, 10, 11.

‡ A subset H of a continuous curve M is said to be *cyclicly chainwise connected* if every two cyclic elements of M which belong to H can be joined by a simple cyclic chain of M that is a subset of H. See *Structure*, p. 175.

first paper we have the result that every point of $\overline{M(K)}-M(K)$ is an end point of the continuous curve $\overline{M(K)}$. This corresponds closely to property (2). Property (3) is identical with Theorem 6 of *Arc-curves, first paper*.

Menger* has shown that a regular curve B is an acyclic continuous curve if and only if the common part of any two subcontinua of B is connected or vacuous. We may state a similar property of the arc-curves of any continuous curve.

THEOREM 6. *If K and H are two subsets of a continuous curve M, the set of common points of the arc-curves $M(H)$ and $M(K)$ is either vacuous or itself an arc-curve of M.*

Let N be the set of common points of the arc-curves $M(H)$ and $M(K)$. If N is vacuous or contains but a single point, our theorem is true, for a point is the arc-curve of that point with respect to M. Now let x and y be two distinct points of N, and let α be an arc of M with end points x and y. By a theorem of the author,† every point of an arc joining two points of an arc-curve belongs to the arc-curve. Hence every point of α belongs to both $M(K)$ and $M(H)$, and thus N contains the arc α. Thus every arc of M joining two points of N belongs to N. Hence N is its own arc-curve with respect to M.

Expressed in terms of cyclic elements, Theorem 6 is as follows: *The common part of any two connected collections of cyclic elements of the continuous curve M is either vacuous or itself a connected collection of cyclic elements of M.*

III. THE NODES AND BASIC SETS OF A CONTINUOUS CURVE

Definitions. The subset K of the continuous curve M is said to be a *basic set* of M if M is identical with the arc-curve $M(K)$. The subset K of M is said to be an *irreducible basic set* of M if K is a basic set of M but no proper subset of K is a basic set of M.‡ A subset E of a continuous curve M is said to be a *node* of M if E is an end point of M or is a maximal cyclic curve of M that contains at most one cut point of M.§ If H is a connected collection of cyclic elements of the continuous curve M, the subset E of H will be said to be a *node* or an *end point* of H provided it is a node or an end point of the continuous curve $\overline{H}$.

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* *Ueber reguläre Baumkurven*, Mathematische Annalen, vol. 96 (1926), p. 575.

† *Arc-curves, first paper*, Theorem 3.

‡ For these definitions, see *Arc-curves, first paper*, p. 575.

§ See *Structure*, p. 178.

THEOREM 7.* *In order that a cyclic element E of a continuous curve M should be a node of M it is necessary and sufficient that the set K should contain a point of E which is a non-cut point of M, for any arc-curve $M(K)$ that contains the set E.*

The condition is necessary. Let $M(K)$ be any arc-curve of M containing the set E. If E is an end point of M, no arc of M contains E as an interior point.† Hence, as E belongs to the set $M(K)$, it must be a point of the set K. As every end point of the continuous curve M is a non-cut point of M, our condition is satisfied. If E is a maximal cyclic curve of M containing no cut point of M, then $M=E$ and every point of K is a non-cut point of M and belongs to E. If E is a maximal cyclic curve of M containing just one cut point P of M, then either our condition is satisfied or $E-P$ contains no point of the set K. Suppose the latter holds. Then the set K is a subset of the set $M-(E-P)$. It is evident that

$$M = E + (M - (E - P)),$$

and that the sets E and $(M-(E-P))$ have just one point in common. Then the set $M-(E-P)$ contains every arc of M whose end points belong to it. As K is a subset of $M-(E-P)$, the arc-curve $M(K)$ is a subset of $M-(E-P)$. But this is impossible as the arc-curve $M(K)$ contains the set E.

The condition is sufficient. Suppose the cyclic element E is not a node of M. Then E is either a cut point or a maximal cyclic curve of M containing more than one cut point of M. If E is a cut point of M, let x and y be points of $M-E$ that do not belong to the same component of $M-E$. Then every arc of M with end points x and y contains E. Thus the arc-curve $M(x+y)$ contains E but the set $x+y$ belongs to $M-E$. This is contrary to our assumed condition. If E is a maximal cyclic curve of M containing more than one cut point of M, let x and y be two cut points of M that belong to E. As every point of E lies on some arc of E with end points x and y‡, the arc-curve $M(x+y)$ contains every point of E. But neither x nor y is a non-cut point of M, which contradicts our assumed condition. The assumption that E is not a node has led to a contradiction with our condition in both cases. Hence the condition is sufficient.

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* Compare this result with Theorem 13 of *Structure*.

† See G. T. Whyburn, *Concerning continua in the plane*, loc. cit., Theorem 12, and my paper *Concerning continuous curves and correspondences*, loc. cit., Theorem 4.

‡ See my paper *Continuous curves which are cyclicly connected*, loc. cit.

THEOREM 8. *In order that a subset K of a compact continuous curve M, which is not cyclicly connected, be a basic set of M it is necessary and sufficient that each node of M should contain at least one non-cut point of M that belongs to K.*

The condition is necessary by the preceding theorem. We shall show that the condition is also sufficient. Let P be any point of M. If P is an end point of M, then P belongs to K, for every end point is a node. If P is a cut point of M, let M_1 and M_2 be two components of $M-P$. If M_1+P is cyclicly connected, then M_1+P is a node of M and M_1 contains a point of K by the condition. If M_1+P is not cyclicly connected, then M_1+P contains two nodes of itself.* There are three cases to consider: (a) one of the nodes is an end point z of M_1+P distinct from P, (b) P is an end point of M_1+P and the other node is a maximal cyclic curve C of M_1+P, (c) both nodes are maximal cyclic curves of M_1+P. Under case (a) if there exists an arc of M containing the point z as an interior point, then this arc contains a subarc containing z as an interior point and belonging to M_1+P, for there is a neighborhood of z such that every point of M belonging to this neighborhood belongs to M_1+P. But, as z is an end point of M_1+P, no arc of M_1+P contains z as an interior point. Thus no arc of M has z as an interior point and z is an end point of M. Hence z belongs to K by our condition. In case (b) let Q be the cut point of M_1+P belonging to the node C. The point P does not belong to C since P is an end point of M_1+P and hence belongs to no simple closed curve of M_1+P. Then

$$M_1 + P - Q = (C - Q) + M_P,$$

where $C-Q$ and M_P are mutually separated sets and M_P contains the point P. As $M-(M_1+P)$ has just one limit point P in M_1+P, the sets $C-Q$ and $M-C$ are mutually separated. Then the set C is a node of M. Then $C-Q$ contains a point of K by our condition. In case (c) let C_1 and C_2 be two nodes of M_1+P which are maximal cyclic curves of M_1+P. If either C_1 or C_2 does not contain P, then, as in case (b), that maximal cyclic curve is a node of M and contains a point of K by the condition. If both C_1 and C_2 contain P, then P is a cut point of M_1+P as every point common to two maximal cyclic curves of a continuous curve is a cut point of that curve. But P is not a cut point of M_1+P, for the set M_1 is connected. Thus in any of the three cases we see that M_1 contains a point x of K. Similarly the component M_2 of $M-P$ contains a point y of K. Every arc of M whose end

* See *Structure*, Theorem 14.

points are x and y contains the point P. Hence every cut point of M belongs to the arc-curve $M(K)$.

If P is neither a cut point nor an end point of M, then P is a non-cut point of M which belongs to some simple closed curve J of M. Let C be the maximal cyclic curve of M containing the curve J. As M is not cyclicly connected the maximal cyclic curve C contains at least one cut point of M. If C contains just one cut point Q of M, then C is a node of M and contains a non-cut point y of M belonging to K. If this point is P, then the arc-curve $M(K)$ contains P. If not, let M_1 be a component of $M-Q$ not containing $C-Q$. As shown in the preceding paragraph, the component M_1 of $M-Q$ contains a point x of K. There exists an arc of C with end points Q and y and containing P, and there exists an arc with end points Q and x and lying wholly in M_1+Q. The sum of these two arcs is an arc of M containing P and whose end points belong to K. Hence P belongs to the arc-curve $M(K)$. If C contains more than one cut point of M, let Q_1 and Q_2 be two such points. Let M_1 and M_2 be components of $M-Q_1$ and $M-Q_2$ respectively that do not contain $C-Q$. Evidently M_1 and M_2 are mutually separated. By the argument of the preceding paragraph we see that M_1 and M_2 contain points x and y of K. There exists an arc of C with end points Q_1 and Q_2 and containing the point P, and there exist two arcs with end points x and Q_1 and y and Q_2 and lying in M_1+Q_1 and M_2+Q_2 respectively. The sum of the three arcs is an arc of M containing P and whose end points belong to K. Hence P belongs to the arc-curve $M(K)$.

COROLLARY 8A. *The set of all nodes of a compact continuous curve is a basic set of the continuous curve.*

With the use of the preceding theorem we may demonstrate the following three results.

THEOREM 9. *In order that a subset K of a compact continuous curve M, which is not cyclicly connected, be an irreducible basic set of M it is necessary and sufficient that* (1) *no point of K be a cut point of M, and* (2) *each point of K belong to some node of M and each node of M contain exactly one point of K.*

THEOREM 10. *The compact continuous curve M has a unique irreducible basic set if and only if every node of M is a point* (*i.e. every maximal cyclic curve of M contains at least two cut points of M*). *This unique irreducible basic set of M is the set of all end points of M.*

THEOREM 11. *The compact continuous curve M has a unique irreducible basic set if and only if the common part of every two basic sets of M is a basic set of M.*

THEOREM 12. *If x, y, z are three points of a continuous curve M, then either there is an arc of M containing all three points or there is a cyclic element C of M such that $M-C$ contains all three points x, y, z and no two of them lie in the same component of $M-C$.*

If the arc-curve $M(x+y)$ contains the point z, there is an arc of M with end points x and y and containing the point z. If not, then the component N of $M-M(x+y)$ containing z has just one limit point w in $M(x+y)$.* If the point w is either x or y, it is easy to see that the curve M contains an arc containing x, y and z. Suppose $x\neq w\neq y$. It is evident that neither x and z nor y and z can lie in the same component of M minus any subcontinuum of M containing w. If the point w is a cut point of the arc-curve $M(x+y)$, then w is a cut point of M† and is the desired cyclic element of M. If the point w is not a cut point of the arc-curve $M(x+y)$, it belongs to some simple closed curve J of $M(x+y)$, and let E be the maximal cyclic curve of M containing J. It is not difficult to see that the set E is a subset of the arc-curve $M(x+y)$. If E contains both x and y, then there exists an arc α of E with end points x and w and containing the point y. There exists an arc β with end points w and z and lying in N except for the point w. The set $\alpha+\beta$ is an arc of M containing x, y and z. If the curve E contains one of the two points x and y, but not both, we will suppose it contains x and not y. Let v be the limit point in E of the component H of $M(x+y)-E$ containing the point y. Since we have assumed that the point w was not a cut point of $M(x+y)$, the points w and v are distinct. There exists an arc γ with end points v and y and lying in $H+v$. There exists an arc η of E with end points w and v and containing the point x. Then $\beta+\gamma+\eta$ is an arc of M containing x, y and z. If the maximal cyclic curve E contains neither x nor y, since each point of E lies on some arc of M with end points x and y and no component of $M-E$ has more than one limit point in E, then x and y must belong to different components of $M-E$. Then E is the desired cyclic element of M.

COROLLARY 12A. *In order that every three points of a continuous curve M lie on some arc of M it is necessary and sufficient that $M-E$ should be connected or the sum of two connected sets, for each cyclic element E of M.*

THEOREM 13. *In order that a compact continuous curve M should have a basic set consisting of two points it is necessary and sufficient that every three points of M should lie on some arc of M.*

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* See *Arc-curves, first paper*, Theorem 10, part (2).

† *Arc-curves, first paper*, Theorem 9, part (2).

The condition is necessary. If M contains two points x and y such that $M=M(x+y)$, then, by Theorems 3 and 4, the set $M-E$ is connected or the sum of two connected sets for each cyclic element E of M. Thus, by Corollary 12A, it follows that every three points of M lie on some arc of M.

The condition is sufficient. If M is cyclicly connected, any two points of M form a basic set of M. If M is not cyclicly connected, by Theorem 1 and a result due to G. T. Whyburn,* the set M has at least two nodes of itself. And the continuous curve M cannot have more than two nodes, for if it did contain three nodes, then, as each node contains a non-cut point of M, there would be an arc of M containing three such points by the condition of the theorem. But it has been pointed out that no non-cut point of M which belongs to some node of M can be interior to an arc of M whose end points do not belong to this node. Hence M contains just two nodes, and let x and y be points of these two nodes which are non-cut points of M. Then $x+y$ is a basic set of M by Theorem 8.

COROLLARY 13A. *If the compact subcontinuum K of the continuous curve M is the sum of a collection of cyclic elements of M, then K is a simple cyclic chain of M if and only if every three points of K lie on some arc of M.*

In view of Theorem 2 we get the following form for a theorem of G. T. Whyburn.†

THEOREM 14. *If x and y are points of a compact continuous curve M, then there exist two points w and z of M such that the arc-curve $M(w+z)$ contains the arc-curve $M(x+y)$ and is not a proper subset of the arc-curve with respect to M of any two points of M.*

In the next theorem we will give an analogue to Theorem 14 for a more general arc-curve. Before doing this we will prove several lemmas.

LEMMA 15A. *A continuous curve M contains a countable set of points $x_1, y_1, x_2, y_2, x_3, y_3, \cdots$ such that* (1) *every point of M is either an end point of M or a point of one of the arc-curves $M(x_i+y_i)$,* (2) *for any positive integer n, the set $M(x_1+y_1)+M(x_2+y_2)+\cdots+M(x_n+y_n)$ is a continuous curve M_n,* (3) *M_n has only one point in common with the arc-curve $M(x_{n+1}+y_{n+1})$ and this is either the point x_{n+1} or the point y_{n+1},* (4) *for each positive number ϵ and each hypersphere S there exists an integer ρ such that for $n>\rho$, the set $M-M_n$ contains no component of diameter greater than ϵ which contains a point interior to S.*

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* See *Structure*, Theorem 14.

† See *Structure*, Theorem 15.

This lemma may be proved in much the same way as a theorem of R. L. Wilder of which it is an analogue.*

LEMMA 15B.† *If K is an arc-curve of the continuous curve M (or if K is a connected collection of cyclic elements of M), then there are only a countable number of nodes of K that are not nodes of M.*

Let G be the collection of all nodes of K that are not nodes of M. Since every cyclic element of K is a cyclic element of M and no continuous curve contains more than a countable number of maximal cyclic curves, all but a countable number of the nodes of the collection G are end points of K. Let G' be the collection of all elements of G that are end points of K. As each cyclic element of K is a cyclic element of M and no node in G is a node of M, each point of G' is a cut point of M. Then each point of G' belongs to one of the arc-curves $M(x_i+y_i)$. If any arc-curve $M(x_i+y_i)$ contains three points of G', then, by Theorem 13,‡ there is an arc α of $M(x_i+y_i)$ with two of these points as end points and containing the third. The arc α belongs to the arc-curve K since every arc-curve contains every arc of M whose end points belong to the arc-curve. But no end point of an arc-curve K is interior to an arc of K. Then no arc-curve $M(x_i+y_i)$ contains more than two points of the collection G'. Then the collection G' is countable. Hence the collection G must be countable.

In proving the last lemma we have established a somewhat more general result.

LEMMA 15C. *If K is an arc-curve of the continuous curve M, then there are only a countable number of nodes of K that are not end points of M.*

THEOREM 15. *If K is any subset of the compact continuous curve M, there exists a subset K' of M such that* (1) *the arc-curve $M(K')$ is closed and contains the arc-curve $M(K)$,* (2) *if K'' is any subset of M such that the arc-curve $M(K')$ is a subset of the arc-curve $M(K'')$ then every node of $M(K')$ is a node of $M(K'')$,* (3) *the set K' is countable unless $M(\overline{K})$ contains an uncountable number of end points of M,* (4) *if $M(\overline{K})$ has n nodes (n finite and greater than one) then K' contains just n points.*

If $M(\overline{K})$ is a point or is cyclicly connected, we may apply Theorem 14 and obtain a set $K'=w+z$ such that the arc-curve $M(w+z)$ contains the

* R. L. Wilder, *Concerning continuous curves*, loc. cit., Theorem 15.

† This lemma is closely related to a theorem of R. L. Moore. See his paper *Concerning the cut points of continuous curves and of other closed and connected point sets*, loc. cit., Theorem B*.

‡ Compactness was not used in proving the necessity of Theorem 13.

arc-curve $M(K)$, and is not a proper subset of any arc-curve of two points of M. It is not difficult to see that the set K' satisfies the conditions of the theorem.

If $M(\overline{K})$ is not cyclicly connected, let G be the set of all nodes of $M(\overline{K})$. Let E be any node of G. If E is a node of M, let P_e be a non-cut point of M which belongs to E. If E is not a node of M, then E is either a cut point of M or a maximal cyclic curve of M containing more than one cut point of M. In either case there is a component H_e of $M-M(\overline{K})$ that has just one limit point Q_e in E, and Q_e is not a cut point of $M(\overline{K})$. As in the proof of Theorem 8 we may show that H_e contains a node E' of M. Let $P_{e'}$ be a non-cut point of M which belongs to the node E'. Let K' be the set of all points P_e and all the points $P_{e'}$. We shall show that the set K' has the desired properties.

By Theorem 8 the set $[P_e]+[Q_e]$ is a basic set of $M(\overline{K})$. Hence if each point of this set belongs to the arc-curve $M(K')$, then every point of $M(\overline{K})$ belongs to $M(K')$ since every other point of $M(\overline{K})$ is on an arc whose end points belong to the set $[P_e]+[Q_e]$.* Since each point P_e belongs to K', it belongs to $M(K')$. Let q_e^1 be a point of $[Q_e]$ and let $P_{e'}^1$ be the corresponding point of $[P_{e'}]$ and H_e^1 be the component of $M-M(\overline{K})$ containing $P_{e'}^1$. Let P_e^2 be any point of $K'-P_{e'}^1$. Then $M-H_e^1$ contains P_e^2 and thus any arc of M from $P_{e'}^1$ to P_e^2 must contain the point Q_e^1. Hence the arc-curve $M(K')$ contains the arc-curve $M(\overline{K})$.

We shall now show that $M(K')$ is closed. If $M(K')$ is not closed, every point of $\overline{M(K')}-M(K')$ is a limit point of K'.† Let X be any point of $\overline{M(K')}-M(K')$. Either (1) the point X is a limit point of points of K' that belong to $M-M(\overline{K})$, or (2) it is a limit point of points of K' that belong to $M(\overline{K})$. Since $M(\overline{K})$ is a continuous curve‡ and is a subset of $M(K')$, case (2) is impossible. In case (1) the point X is a limit point of the subset $[P_{e'}]$ of K'. But no component of $M-M(\overline{K})$ contains more than one point of $[P_{e'}]$ and thus every limit point of $[P_{e'}]$ belongs to $M(\overline{K})$. But X cannot be such a point for $M(\overline{K})$ is a subset of $M(K')$ while X belongs to $\overline{M(K')}-M(K')$. Thus we obtain a contradiction in both cases and the set $M(K')$ must be closed.

Every point P_e of K' is a non-cut point of M which belongs to a node E of M and every point $P_{e'}$ of K' is a non-cut point of M that belongs to some node E' of M. It follows easily that $M(K')$ contains every point of all the nodes $[E]$ and $[E']$ of M. Then each of these sets is a node of $M(K')$ and of every arc-curve of M containing $M(K')$. This proves part (2).

* *Arc-curves, first paper*, Theorem 3.

† *Arc-curves, first paper*, Theorem 2.

‡ *Arc-curves, first paper*, Theorem 7.

From the definition of the set K' it is seen that the cardinal number of the set K' is the same as that of the set of nodes of the arc-curve $M(\overline{K})$, except when $M(\overline{K})$ is a point or is cyclicly connected. This establishes part (4) and, with Lemma 15C, establishes part (3).

COROLLARY 15A. *If the connected subset H of the compact continuous curve M is the sum of a collection of cyclic elements of M, then there exists a subcontinuum K of M such that* (1) *K contains H and is the sum of a collection of cyclic elements of M,* (2) *if N is any connected subset of M which contains K and is the sum of a collection of cyclic elements of M, then every node of K is a node of N,* (3) *the set of nodes of K has the same cardinal number as the set of nodes of $\overline{H}$, unless $\overline{H}$ is a point or is cyclicly connected, in which case K may have one or two nodes.*

IV. THE IRREDUCIBLE BASIC SETS AND THE SET OF END POINTS

In *Arc-curves, first paper* it was shown that every basic set of a continuous curve contains every end point of the continuous curve. However, a basic set may contain many other points, in fact it may be the entire continuous curve. The set of end points may or may not be a basic set of the curve. The condition under which this was true was given in Theorem 10. If the set of end points is a basic set, it is an irreducible basic set. An irreducible basic set of a continuous curve may contain a countable number of points that are not end points of the curve,* but in many respects an irreducible basic set is similar to the set of all end points of the continuous curve. No end point of a continuous curve is a cut point of the curve, and similarly no point of an irreducible basic set is a cut point of the curve. No end point of a continuous curve is interior to an arc of the curve. We have the somewhat analogous property that no point P of an irreducible basic set of a continuous curve M is interior to any arc of M, neither of whose end points belong to that node of M containing the point P. Karl Menger,† G. T. Whyburn‡ and the author§ have shown that the set of all end points

* By Theorem 9 there is a (1, 1) correspondence between the points of an irreducible basic set and the set of nodes in which they are contained. By Lemma 15C, all but a countable number of nodes are end points. Hence all but a countable number of the points of an irreducible basic set are end points of the continuous curve.

† K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), pp. 277–306. Menger defines the term *end point* in a different sense from that of R. L. Wilder, but the two definitions have been shown to be equivalent by H. M. Gehman. See his paper *Concerning end points* etc., loc. cit., Theorem 1.

‡ G. T. Whyburn, *Concerning continua in the plane*, loc. cit., Theorem 21.

§ W. L. Ayres, *Concerning continuous curves and correspondences*, loc. cit., Theorem 5.

of a continuous curve is totally disconnected.* It has been shown by the author† that if K is any subset of the set of end points of a continuous curve M, then $M-K$ is strongly connected.‡ We shall show that both of these properties hold for irreducible basic sets of a continuous curve.

THEOREM 16. *If M is a continuous curve, then every irreducible basic set of M is totally disconnected.*

Let K be an irreducible basic set of M and suppose that K contains a connected subset H containing more than one point. Since the set of end points of M is totally disconnected, the set H contains a point P which is not an end point of M. Since no point of K is a cut point of M,§ the point P belongs to some simple closed curve J of M and let C be the maximal cyclic curve of M containing J. By a theorem of G. T. Whyburn,‖ the common part of H and C is a connected set N. As P is not a cut point of M, the set N contains points other than the point P. Then the irreducible basic set K contains a connected set N containing more than one point and lying in a maximal cyclic curve C of M. But this is impossible for no maximal cyclic curve of a continuous curve contains more than two points of an irreducible basic set of the continuous curve by Theorem 9.

THEOREM 17. *If K is any subset of an irreducible basic set of a continuous curve M, which is not cyclicly connected, then $M-K$ is arc-wise connected.*

Let P_1 and P_2 be points of $M-K$. There are four possible cases: (1) both P_1 and P_2 belong to a maximal cyclic curve C of M which is a node, (2) neither P_1 nor P_2 belongs to a node of M containing a point of K, (3) both P_1 and P_2 belong to nodes of M containing points of K but not both to the same node, (4) either P_1 or P_2 belongs to a node of M containing a point of K but the other does not. In case (1), since M is not cyclicly connected, the curve C contains at most one point Q of K by Theorem 9. And as Q is not a cut point of C, there is an arc of $C-Q$ whose end points are P_1 and P_2. Since this arc belongs to $C-Q$, it belongs to $M-K$. In case (2) let α be any arc of M with end points P_1 and P_2. Every point of K is a non-cut point of M belonging to some node of M and no such point is interior to any arc of M whose end points do not belong to the node. Hence no point of K is an interior point of the arc α.

* A point set is said to be *totally disconnected* if it contains no connected subset containing more than one point.

† *Concerning continuous curves and correspondences*, loc. cit., Theorem 6.

‡ A point set K is said to be *strongly connected* if each pair of points of K belongs to some continuum which is a subset of K.

§ *Arc-curves, first paper*, Theorem 13.

‖ *Structure*, Theorem 30.

In case (3) let N_1 and N_2 be the nodes of M containing P_1 and P_2 respectively. Let Q_1 and Q_2 be the points of K and X_1 and X_2 be the cut points of M belonging to N_1 and N_2 respectively. Evidently $P_i \neq Q_i \neq X_i$ $(i=1, 2)$. Then, as $N_i - Q_i$ is connected, there exists an arc α_i of $N_i - Q_i$ from P_i to X_i. Let α_3 be any arc of M with end points X_1 and X_2. It is seen that $M - N_1 - N_2$ contains the set $<\alpha_3>$ and that no point of K belongs to the arc α_3. Then the set $\alpha_1 + \alpha_2 + \alpha_3$ is an arc of M with end points P_1 and P_2 and containing no point of K.

Case (4) is very similar to case (3) and need not be considered separately.

COROLLARY 17A. *If K is an irreducible basic set of a continuous curve M, then $M - K$ consists of a finite number of components.*

LEMMA 18A. *In order that a connected subset K of a continuous curve M be connected im kleinen at a point P of K it is necessary and sufficient that the set $H \cdot K$ should be connected im kleinen at the point P for each maximal cyclic curve H of M containing the point P.*

The condition is necessary. If P belongs to no maximal cyclic curve of M, the condition is satisfied vacuously. If not, let H be any maximal cyclic curve of M containing the point P, and let ϵ be any positive number. Since K is connected im kleinen at P, there exists a positive number δ_ϵ such that any point of K whose distance from P is less than δ_ϵ lies with P in a connected subset of K of diameter less than ϵ. Let Q be any point of $H \cdot K$ such that $d(P, Q) < \delta_\epsilon$ and let N be a connected subset of K of diameter less than ϵ containing both P and Q. As H is a maximal cyclic curve of M, the set $N \cdot H$ is connected. It is a subset of $H \cdot K$ of diameter less than ϵ and contains P and Q. Hence $H \cdot K$ is connected im kleinen at P.

The condition is sufficient. If ϵ is any positive number, there are only a finite number of components of $M - P$ of diameter greater than ϵ,* and let $N_1, N_2, N_3, \cdots, N_m$ denote these components. If P does not belong to a simple closed curve of $N_i + P$, then there is a point Q_i of N_i such that every point of the component L_i of $N_i + P - Q_i$ containing the point P is within a

* For two dimensions this result may be found in my paper *Note on a theorem concerning continuous curves*, Annals of Mathematics, (2), vol. 28 (1927), pp. 501–2. While this result does not hold in space of more than two dimensions, it may be shown that *if M is a continuous curve and N is a compact subcontinuum of M, ϵ is any positive number and η is any positive number less than ϵ, then only a finite number of the components C of $M - N$ that are of diameter greater than ϵ are such that the set of limit points of C in N is of diameter less than η.* The result of the paper mentioned above may be obtained from this theorem since if M lies in the plane and N is a continuous curve, we can show that there are only a finite number of components C of $M - N$ such that the set of limit points of C in N is of diameter greater than or equal to η.

distance ϵ of P.* It is easy to see that the set $K\cdot(L_i+Q_i)$ is connected. Let $\delta_{\epsilon i}=\frac{1}{2}d(P, N_i+P-L_i)$. Then every point of $K\cdot N_i$ whose distance from P is less than $\delta_{\epsilon i}$ belongs to $K\cdot(L_i+Q_i)$ and thus lies with P in a connected subset of K of diameter less than ϵ.

If P belongs to a simple closed curve of N_i+P, there is one (and only one) maximal cyclic curve C_i of N_i+P containing P. The set C_i is also a maximal cyclic curve of M. Hence by the condition there exists a positive number $\delta_{\epsilon i}'$ such that any point of $K\cdot C_i$ whose distance from P is less than $\delta_{\epsilon i}'$ lies with P in a connected subset of $K\cdot C_i$ every point of which is within a distance $\epsilon/2$ of P. Since only a finite number of the components of N_i+P-C_i are of diameter greater than a given positive number and each of them has just one limit point in C_i,† there exists a positive number $\delta_{\epsilon i}''$ such that if X is any point of N_i+P-C_i whose distance from P is less than $\delta_{\epsilon i}''$, then X belongs to a component of diameter less than $\epsilon/2$ and the limit point of this component in C_i is within a distance $\delta_{\epsilon i}'$ of P. Let $\delta_{\epsilon i}$ be the smaller of the numbers $\delta_{\epsilon i}'$ and $\delta_{\epsilon i}''$. Now let Q be any point of $K\cdot N_i$ whose distance from P is less than $\delta_{\epsilon i}$. If Q belongs to N_i+P-C_i, let L_i be the component of N_i+P-C_i containing Q and let Y_i be its limit point in C_i. The set $W_i=K\cdot L_i+Y_i$ is a connected subset of K of diameter less than $\epsilon/2$ and containing the point Y_i. If Q belongs to C_i, let W_i and Y_i be identical with the point Q. Since $d(P, Y_i)<\delta_{\epsilon i}'$, there is a connected subset V_i of $K\cdot C_i$ containing P and Y_i, every point of which is within a distance $\epsilon/2$ of P. Then every point Q of $K\cdot N_i$ whose distance from P is less than $\delta_{\epsilon i}$ lies with P in a connected subset W_i+V_i of $K\cdot N_i$ every point of which is within a distance ϵ of P.

If δ_ϵ is the smallest of the numbers $\delta_{\epsilon 1}, \delta_{\epsilon 2}, \delta_{\epsilon 3}, \cdots, \delta_{\epsilon m}$, it is easily seen that every point of K whose distance from P is less than δ_ϵ lies with P in a connected subset of K every point of which is within a distance ϵ of P.

THEOREM 18. *If K is any subset of an irreducible basic set of a continuous curve M, then $M-K$ is connected im kleinen.*

If M is cyclicly connected, then K has at most two points and $M-K$ is connected im kleinen since it is an open subset of M. If M is not cyclicly connected, by Theorem 17 the set $M-K$ is connected. Then, by the preceding lemma, the theorem is true if $(M-K)\cdot C$ is connected im kleinen for every maximal cyclic curve C of M. As M is not cyclicly connected, the maximal

* Any such point P is an end point of N_i+P by Theorem 3 of my paper *Concerning continuous curves and correspondences*, loc. cit., and thus the point Q_i exists. See H. M. Gehman, *Concerning end points* etc., loc. cit., especially property 7.

† If η is chosen as any positive number less than ϵ, then there is no component C of N_i+P-C_i such that the set of limit points of C in C_i is of diameter greater than η.

cyclic curve C contains at most one point of K. It is evident that any maximal cyclic curve or a maximal cyclic curve minus one point is connected im kleinen.

Whyburn* has defined the end elements of a continuous curve and has shown that in many respects they resemble the end points of a continuous curve. He has shown that if the continuous curve is neither cyclicly connected nor the sum of two cyclicly connected continuous curves, the set of end elements is totally disconnected in the sense that it contains no connected subset containing more than one end element. However the end-point properties mentioned in the two preceding theorems of this paper are not true for end elements, as is seen in the following example. Let M consist of the square with vertices $(0, 0)$, $(1, 0)$, $(1, -1)$ and $(0, -1)$ together with the circles with center $(1/n, r_n)$ and radius r_n where $n=1, 2, 3, \cdots$, and $r_n=1/(2n(n+1))$. In this example each circle is an end element of the continuous curve M. If K is the set of end elements, then $M-K$ consists of a countable infinity of components and is not connected im kleinen at the point $(0,0)$.

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* See *Structure*, p. 178 and Theorems 16–19.

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