ON LINEAR UPPER SEMI-CONTINUOUS COLLECTIONS OF BOUNDED CONTINUA†

BY

WALLACE ALVIN WILSON

1. Introduction and notation. If $T=\{t\}$ denotes a closed set of points and with each point t there is associated a unique bounded continuum X(or X_t) in such a way that (a) $X_t \cdot X_t' = 0$ if $t \neq t'$, and (b) at each point $t=\tau$ of T the upper closed limit of X_t as $t \to \tau$ is a part of X_τ, we say that $X=f(t)$ is an upper semi-continuous function in T. The collection of continua $\{X\}$ is also known as an upper semi-continuous collection of continua. We say that $X=f(t)$ is a *minimal* upper semi-continuous function in T if there exists no upper semi-continuous function $Y=g(t)$ such that at every point t, $Y \subset X$ and at some point $Y \neq X$. Examples of this concept are given elsewhere.‡

The following notation will be convenient. If $X=f(t)$ in T and $M=\sum[X]$, we write $M=F(T)$. If T is a bounded continuum and $f(t)$ is upper semi-continuous, it is obvious that M is a bounded continuum; in this case we say that X is an *element* of M. If T is a simple arc ab, $M=F(ab)$ will be called a *generalized arc*, or simply an arc if no confusion is caused. This may be denoted by X_aX_b and the elements X_a and X_b will be called the ends. Likewise, $M-(X_a+X_b)$ is called a (*generalized*) *open arc* and denoted by $X_a^*X_b^*$. The meaning of $X_aX_b^*$ and $X_a^*X_b$ is apparent. The terms "upper limit" and "limit" of a system of continua are used in the sense of the closed limits of Hausdorff (*Grundzüge der Mengenlehre*, p. 236) as extended by L. S. Hill (*Properties of certain aggregate functions*, American Journal of Mathematics, vol. 49, pp. 420–421). If K is the upper limit of X_t as $t \to \tau$, we write $K = \limsup_{t\to\tau} X_t$; if the domain of definition T is a linear set and we restrict t to points at the right or left of τ, we write $R \limsup_{t\to\tau} X_t$ or $L \limsup_{t\to\tau} X_t$, respectively.

In the article referred to above various properties of upper semi-continuous functions were derived and in particular the following theorem was proved:

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‡ W. A. Wilson, *Some properties of upper semi-continuous collections of bounded continua*, Bulletin of the American Mathematical Society, vol. 34, pp. 599-606.

Let the aggregate $\{t\}$ be a simple arc ab, let $X=f(t)$ be a minimal upper semi-continuous function defined over ab, let $M=\sum[X]$ lie in a plane, and let no element of M separate X_a from X_b. Then the continuum M is irreducible between X_a and X_b.

The last two hypotheses are needed in the particular proof of the theorem that is given, but there is no evidence that they are essential for the validity of the conclusion. It is the purpose of this article to discuss this particular question and to give some properties of generalized arcs which lie in a plane and have one or more elements separating† the ends.

2. THEOREM. *Let $M=F(ab)$ be a generalized bounded arc in a plane Z. Let t_1, τ, and t_2 be points of ab, let $t_1<\tau<t_2$, and let X_τ separate $X_1=f(t_1)$ from $X_2=f(t_2)$. Then, if $t'<\tau<t''$, X_τ separates $X'=f(t')$ from $X''=f(t'')$.*

Let R and S be the components of $Z-X_\tau$ containing X_1 and X_2, respectively. Since $X_aX_\tau^*$ is connected and contains X_1 it lies in R. For a like reason $X_\tau^*X_b$ lies in S. But $X'\subset X_a X_\tau^*$ and $X''\subset X_\tau^*X_b$. Hence the theorem is proved.

Remarks. It should be noted that, if some sub-continuum K of M separates X_a from X_b, at least one element X has this property. (See reference under §1, p. 601.) As will be seen later, however, X need not be a part of K, nor K a part of X.

3. THEOREM. *Let $X_1=f(t_1)$ and $X_2=f(t_2)$ separate X_a from X_b and $t_1<t_2$. Then X_1 separates X_a from X_2, and X_2 separates X_1 from X_b.*

Let the components R_1 and S_1 of $Z-X_1$ contain X_a and X_b, respectively. Then $R_1\supset X_aX_1^*$ and $S_1\supset X_1^*X_b$. Let R_2 and S_2 be likewise defined for X_2. Then $R_2\supset X_aX_2^*$ and $S_2\supset X_2^*X_b$. Since $t_1<t_2$, $X_1\subset X_aX_2^*\subset R_2$ and $X_2\subset X_1^*X_b\subset S_1$. This proves the theorem.

As a consequence of this theorem, it follows that, if the elements of M which separate X_a and X_b are arranged in the order of the corresponding values of t, each such element may be said to lie within all those which follow it and to lie without all those which precede it, or vice versa.

4. THEOREM. *Let $M=F(ab)$ be a generalized bounded arc in a plane. Let t' be any point of ab for which $X'=f(t')$ separates X_a from X_b. Then the set of points $\{t'\}$ together with a and b form a closed set.*

Let this set be T' and let τ be a limiting point of T' different from a

† Throughout this article the term "separate" has reference to the plane in which the aggregates referred to lie.

and b. Then there is a sequence $\{t_n'\}$ of points of $T'-(a+b)$ converging to τ. Let $X_n'=f(t_n')$. Since $f(t)$ is upper semi-continuous, $X_\tau \supset \limsup_{n\to\infty} X_n'$. By hypothesis each X_n' separates X_a from X_b. Since $(X_a+X_b)\cdot X_\tau=0$, $\limsup_{n\to\infty} X_n'$ separates X_a from X_b.† Hence τ lies in T'. As $a+b \subset T'$, this shows T' is closed.

5. Theorem. *Let $X_t=f(t)$ be a minimal upper semi-continuous function defined over the interval ab. Let $M=F(ab)$ be bounded and lie in a plane. Let the set of points $T''=\{t'\}$ for which $X'=f(t')$ separates X_a from X_b be void or totally disconnected. Then M is irreducible between X_a and X_b.*

The case that T'' is void is merely the theorem cited in §1. In the alternative case set $T'=T''+a+b$. Then T' is a nowhere dense closed set and $ab-T'$ is a finite or enumerable set of open intervals whose end points are points of T'.

Let K be a sub-continuum of M irreducible between X_a and X_b. Let c^*d^* be any one of the open intervals whose sum is $ab-T'$. Set $g(t)=f(t)$ for $c<t<d$; $g(c)=R\limsup_{t\to c} f(t)$; and $g(d)=L\limsup_{t\to d} f(t)$. Then $g(t)$ is a minimal upper semi-continuous function in cd and it is clear that $G(cd)=\overline{X_c^*X_d^*}$. But for no t in c^*d^* does $g(t)$ separate $g(c)$ from $g(d)$ by the hypothesis regarding cd and §2. Then the theorem quoted in §1 shows that $\overline{X_c^*X_d^*}$ is irreducible between $g(c)$ and $g(d)$ and therefore between X_aX_c and X_dX_b. But K contains a sub-continuum irreducible between these arcs and $K\subset M$. Hence $K\supset \overline{X_c^*X_d^*}$.

Now set $Y_t=K\cdot X_t=k(t)$. We have just seen that for any point t in $ab-T'$, $Y_t=X_t$. Let t' be any point of T' except a or b. Since T' is nowhere dense, there are two univariant sequences of points $\{s_n\}$ and $\{t_n\}$ lying in $ab-T'$ and converging to t' such that $s_n<t'<t_n$; let M_n denote the arc of M whose ends are $f(s_n)$ and $f(t_n)$. Since $f(s_n)=k(s_n)$ and $f(t_n)=k(t_n)$, it is evident that $K\cdot M_n$ is a continuum. Obviously $X_{t'}=\prod_1^\infty[M_n]$. Then $Y_{t'}=K\cdot X_{t'}=\prod_1^\infty[K\cdot M_n]$, and is therefore a continuum. Similar reasoning shows that Y_a and Y_b are continua.

We now show that $Y_t=k(t)$ is upper semi-continuous. This needs a proof only for points t' of T', since $k(t)=f(t)$ in $ab-T'$. As $Y_t\subset K$ and $Y_t\subset X_t$, $\limsup_{t\to t'} Y_t \subset K\cdot\limsup_{t\to t'} X_t\subset K\cdot X_{t'}$. But $Y_{t'}=K\cdot X_{t'}$ by definition. Hence $\limsup_{t\to t'} Y_t\subset Y_{t'}$ everywhere.

Since $f(t)$ is a minimal upper semi-continuous function and $k(t)\subset f(t)$ everywhere, it follows that for every t, $Y_t=k(t)=f(t)=X_t$. Hence $K=M$ and the theorem is proved.

† For, if A and B are bounded continua; $\{X_n\}$ is a sequence of bounded continua, each of which separates A from B; $X=\limsup_{n\to\infty} X_n$; and $X\cdot(A+B)=0$; then X separates A from B.

Examples to which the theorem is applicable are to be found in the literature. Here is another. Let $a=0$ and $b=1$, and $T'=\{t'\}$ be any nowhere dense closed set in ab. For each t' in T' let $f(t')$ be a circumference whose center is X_0 and whose radius is t'; let $f(0)=X_0$ and $f(1)$ be a circumference with center X_0 and radius 1. Let t_1' and t_2' be any two points of T' which are the end points of one of the open intervals whose sum is $ab-T'$. In the ring bounded by $f(t_1')$ and $f(t_2')$ place a spiral approaching the bounding circumferences asymptotically and for any t between t_1' and t_2' meeting the circumference of radius t and center X_0 in a single point, which we take as $f(t)$. Do this for each open interval of $ab-T'$. Then $f(t)$ is a minimal upper semi-continuous function and $M=F(01)$ is irreducible between X_0 and X_1.

Let us now turn to the hypothesis that T' is totally disconnected. It is clear that, if $f(t)$ is merely upper semi-continuous in ab, there may be a sub-interval cd such that for every t in cd, X_t separates X_a from X_b. A set of concentric circumferences is an example, but in this case $f(t)$ would not be a *minimal* function. If it is impossible for $f(t)$ to have this quality and for T' at the same time to contain an interval, the hypothesis is redundant; on the other hand, the conclusion of the theorem may be true even if the hypothesis is not satisfied. Two examples bearing on these points will be given.

6. **Example I.**† Let K_i and L_i denote two simple closed plane curves having one common point x_i and let K_i-x_i lie within L_i. The union of K_i+L_i and the complementary domain whose frontier is this continuum we call a *crescent* and we denote it by C_i. The point x_i will be called a *cut point*.

Let R be a circular ring bounded by two concentric circumferences K and L, K being the smaller. It is easy to show that, if n is any integer, we can construct n crescents $C_1, C_2, \cdots, C_n$ in R which satisfy the following requirements. No two of the sets C_i have common points. $R-\sum[C_i]$ consists of $n+1$ rings (not necessarily circular) $R_0, R_1, \cdots, R_n$ whose frontiers are $K+K_1$, $L_1+K_2, \cdots, L_{n-1}+K_n$, L_n+L, respectively, and each of these rings has in common with every radius from the common center E of K and L a segment of uniform length w, which may be called the width of the ring. If E is taken for the pole of a system of polar coördinates, the vectorial angle θ of each cut point x_i is $2\pi i/n$. Finally this construction can be repeated in each of the rings $\{R_i\}$.

Set up a system of polar coördinates (r,θ), where θ is measured in terms

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† This is a modification of a continuum previously described by the author: *A curious irreducible continuum*, Bulletin of the American Mathematical Society, vol. 32, pp. 679–681.

of 2π as a unit. Let $0 \leqq t \leqq 1$. Let $X_0 = f(0)$ and $X_1 = f(1)$ be the circumferences $r=1$ and $r=2$, respectively, and let R_0 be the ring bounded by these. As described in the previous paragraph, construct in R_0 two crescents C_1 and C_2, whose frontiers are $X_{0,1} = K_1 + L_1$ and $X_{0,2} = K_2 + L_2$ and which divide R_0 into three rings $R_{0,0}$, $R_{0,1}$, and $R_{0,2}$, each of which has the width ϵ_1. Set $X_{0,1} = f(1/3)$ and $X_{0,2} = f(2/3)$ and let the cut points of C_1 and C_2 have θ equal to 1/2 and 1, respectively.

In $R_{0,0}$ construct four crescents $C_{0,1}$, $C_{0,2}$, $C_{0,3}$, and $C_{0,4}$, which divide $R_{0,0}$ into five rings $R_{0,0,0}$, $R_{0,0,1}$, $R_{0,0,2}$, $R_{0,0,3}$, $R_{0,0,4}$, each of width ϵ_2, and whose cut points have θ equal to 1/4, 1/2, 3/4, and 1, respectively. Let their frontiers be $X_{0,0,1} = f(1/15)$, $X_{0,0,2} = f(2/15)$, $X_{0,0,3} = f(3/15)$, and $X_{0,0,4} = f(4/15)$. In the same way treat $R_{0,1}$ and $R_{0,2}$, getting $f(6/15)$, $f(7/15)$, $f(8/15)$, $f(9/15)$, $f(11/15)$, $f(12/15)$, $f(13/15)$, and $f(14/15)$.

In each of the 15 rings thus formed construct 8 crescents whose frontiers are $f(1/135)$, $f(2/135)$, $\cdots$, $f(8/135)$; $f(10/135)$, $\cdots$, $f(17/135)$; $\cdots$, $f(134/135)$. In each of these 15 rings let the cut points have θ equal to 1/8, 1/4, 3/8, 1/2, 5/8, 3/4, 7/8, and 1.

Let this process be continued indefinitely; it is clear that the sequence $\epsilon_1, \epsilon_2, \cdots$ converges to zero. This defines $X = f(t)$ for any rational $t < 1$ of the form p/q, where p is relatively prime to q and q is one of the integers 3, $3 \cdot 5$, $3 \cdot 5 \cdot 9$, $3 \cdot 5 \cdot 9 \cdot 17$, etc. Any other t is the divisor of a decreasing sequence of intervals $d_1, d_2, \cdots$, of lengths 1/3, 1/15, 1/135, etc., and to each of these corresponds a definite ring of width ϵ_i lying between the crescents corresponding to the end points of the interval. Let the divisor of such a sequence of rings be $f(t)$; it is obviously a simple closed curve.

The construction insures that $f(t)$ is upper semi-continuous. It is obvious that any continuum P joining X_0 to X_1 contains all the cut points. As these are everywhere dense with respect to *θ in each ring*, they are everywhere dense in $M = \sum [X]$. Hence $P = M$ and M is irreducible between X_0 and X_1. Hence also $f(t)$ is a minimal function.

Thus we have an example where $f(t)$ is a minimal upper semi-continuous function defined over an interval ab and for every t between a and b the corresponding element separates $f(a)$ from $f(b)$. That is, the hypothesis in the theorem of §5 that the set of points T' be totally disconnected is not redundant and the conclusion of this theorem may be valid with this hypothesis omitted. We now proceed to construct an example which shows that this last situation does not always happen.

7. **Example II.** As a preliminary we state several facts regarding bounded continua which are either well known or easily demonstrated from known theorems.

(a) For the sake of brevity later a bounded plane continuum F which has precisely two complementary domains R and S and is the frontier of both of them will be called a continuum of type α. For such a continuum and for any $\epsilon > 0$, there exist simple polygons P and P' in R and S, respectively, which have these properties: $R+F$ lies in the interior of P'; $P+P' \subset V_\epsilon(F)$, $F \subset V_\epsilon(P)$, $F \subset V_\epsilon(P')$; the rings between P and F and between F and P' contain no circle of diameter ϵ; and the ring between P and P' contains no circle of diameter 2ϵ. (The notation $V_\epsilon(F)$ means the set of all points whose distances from F are less than ϵ.)

(b) It is possible to construct an indecomposable continuum K of type α such that there is no continuum C joining a point of R to one of S such that $C \cdot K$ is a proper sub-continuum of K. In other words $C \cdot K$ is always disconnected or identical with K. Such continua will be referred to later as of *type* β.†

(c) Let $\{F_i\}$ and $\{G_i\}$ be two sequences of continua of type α having the following properties. If $i < i'$, let F_i lie in the interior of $F_{i'}$, i.e., in the bounded complementary domain. Let every F_i be in the interior of every G_i. If $i < i'$, let $G_{i'}$ lie in the interior of G_i. For every $\epsilon > 0$ let there be an i_0 such that for every $i > i_0$, $F_i \subset V_\epsilon(G_i)$, $G_i \subset V_\epsilon(F_i)$, and the ring H_i whose frontier is $F_i + G_i$ contains no circle of diameter ϵ. Then, if $H = \prod_1^\infty [H_i]$, H is a continuum of type α separating every F_i from every G_i and $H = \lim_{i\to\infty} F_i = \lim_{i\to\infty} G_i$.

(d) Let P and P' be simple polygons, let P be in the interior of P', and let R be the ring between them. Then, for every $\epsilon > 0$, there is an integer n, such that there are n polygons $\{P_i\}$ dividing R into $n+1$ rings, such that no ring contains a circle of diameter ϵ, each ring is contained in an ϵ-vicinity of each of the polygons which bound it, and, if we set $P_0 = P$ and $P_{n+1} = P'$, every $P_i \subset V_\epsilon(P_{i+1})$ and every $P_{i+1} \subset V_\epsilon(P_i)$, $i = 0, 1, 2, \cdots, n$.

We are now ready to proceed with our construction. Let K be a continuum of type β, let P' and P'' be simple polygons, let P' be in the interior of K and K in the interior of P'', and let Γ be the ring between P' and P''.

Let C be a circumference of center E and radius 1. Let $0 < \epsilon_1 < 1/4$. Let $C_0 = E$ and $C_1, C_2, \cdots, C_{n_1} = C$ be a set of concentric circumferences,

† It is natural to infer that any indecomposable continuum of type α is necessarily of type β. That this is not true may be seen from the following modification of the continua of Wada, the idea for which is due to a suggestion by H. M. Gehman. Let C' and C'' be two circumferences, C' being within C'', and let L be a segment joining C' and C''. Let the simply connected region whose frontier is $C'+C''+L$ be the "island" of K. Yoneyama (Tôhoku Mathematical Journal, vol. 12, pp. 60–62), the interior of C' be the "lake of fresh water," and the exterior of C'' be the "sea." If "canals" are constructed as described by Yoneyama, the resulting indecomposable continuum is of type α, but not of type β. For the constituent containing L is accessible from both of the complementary domains.

each within the following and having a distance from the two adjacent ones less than ϵ_1. Put Γ in a homeomorphism with each of the n_1-1 circular rings thus determined and let K_i, $i=1, 2, \cdots, n_1-1$, be the image of K. Set $K_0=E$ and $K_{n_1}=C$. For each $i<n_1$, $K_i\subset V_{\eta_1}(K_{i+1})$ and $K_{i+1}\subset V_{\eta_1}(K_i)$, where $\eta_1=2\epsilon_1$. Moreover, the point $K_0=E$ and the continua $K_1, K_2, \cdots, K_{n_1-1}$ divide the interior of C into n_1 rings $R_0, R_1, R_2, \cdots, R_{n_1-1}$ no one of which contains a circle of diameter η_1. At the same time divide the interval $0\leqq t\leqq 1$ into n_1 equal intervals $D_0, D_1, D_2, \cdots, D_{n_1-1}$ by the points $c_1, c_2, \cdots, c_{n_1-1}$ and set $c_0=0$ and $c_{n_1}=1$. For each point $t=c_i$, set $f(c_i)=K_i$. We note that this also establishes a correspondence between the intervals $\{D_i\}$ and the rings $\{R_i\}$.

For some positive $\epsilon_2<\frac{1}{2}\epsilon_1$ and some integer n_2 there can be constructed, in each ring R_i, n_2 simple polygons dividing it into n_2+1 rings in such a manner that, if all the n_1n_2 polygons and the continua $\{K_i\}$ are taken in order starting from $K_0=E$, each polygon or continuum is contained in an ϵ_2-vicinity of the one preceding and the one following and no ring contains a circle of diameter ϵ_2. This is a consequence of (a) and (d), above. Each ring R_i contains n_2-1 rings bounded by polygons. Put Γ in a homeomorphism with each of these and let the images of K be $\{K_{i,j}\}$, $j=1, 2, \cdots, n_2-1$. Set $K_{i,0}=K_i$ and $K_{i,n_2}=K_{i+1}=K_{i+1,0}$. If $\eta_2=2\epsilon_2$, each $K_{i,j}$ is in an η_2-vicinity of $K_{i,j+1}$ and of $K_{i,j-1}$. Moreover, the n_1n_2 continua $\{K_{i,j}\}$, $i=0, 1, 2, \cdots, n_1-1$, $j=0, 1, 2, \cdots, n_2-1$ divide the interior of C into n_1n_2 rings $R_{0,0}, R_{0,1}, \cdots, R_{n_1-1,n_2-1}$ no one of which contains a circle of diameter η_2, and the frontier of each $R_{i,j}$ is $K_{i,j}+K_{i,j+1}$. At the same time divide each of the intervals D_i into n_2 equal intervals $\{D_{i,j}\}$, $j=0, 1, 2, \cdots, n_2-1$, by the points $c_{i,1}, c_{i,2}, \cdots, c_{i,n_2-1}$. For each point $c_{i,j}$ set $f(c_{i,j})=K_{i,j}$, $i=0, 1, 2, \cdots, n_1-1$, $j=0, 1, 2, \cdots, n_2-1$. This also establishes a correspondence between the intervals $\{D_{i,j}\}$ and the rings $\{R_{i,j}\}$.

Now take $\epsilon_3<\frac{1}{2}\epsilon_2$ and continue this process indefinitely. If t is one of the division points $c_{i,j,k}$, $c_{i,j,k,l}$, etc. set $f(t)$ equal to the corresponding $K_{i,j,k}$, $K_{i,j,k,l}$, etc. Any other point t is the divisor of a decreasing sequence of intervals D_i, $D_{i,j}$, $D_{i,j,k}$, etc., to which corresponds a decreasing sequence of rings R_i, $R_{i,j}$, $R_{i,j,k}$, etc., whose divisor is a continuum. This continuum we set equal to $f(t)$. It is clear from the construction that, if $M=\sum[f(t)]=F(01)$ and C' denotes the sum of C and its interior, $M=C'$.

Let us now consider the nature of $f(t)$. Since $\epsilon_n\rightarrow 0$, it follows from the construction and (c) above that $f(t)$ is not only upper semi-continuous, but continuous for each t not a division point c_i, $c_{i,j}$, $c_{i,j,k}$, etc. This is also true at the division points, for any such point, as $c_{i,j,k}$, can be regarded as the divisor of the sequence of intervals c_ic_{i+1}, $c_{i,j}c_{i,j+1}$, $c_{i,j,k-1}c_{i,j,k+1}$, $c_{i,j,k-1,n_4-1}$

$\cdot c_{i,j,k,1}$, etc., and the sequence of rings bounded by the pairs of continua corresponding to the end points of these intervals with their frontiers satisfy the hypotheses of (c). It also follows from (c) that, if t is different from 0 and 1, $f(t)$ separates $f(0)$ from $f(1)$.

Let $g(t) \subset f(t)$ and be upper semi-continuous in (01) and let $N = \sum [g(t)]$. The sets of continua K_i, $K_{i,j}$, $K_{i,j,k}$, etc. were constructed so as to be everywhere dense in M. Hence, if $M \neq N$, there is some K containing points not in N. Let this K be $f(\tau)$; then $g(\tau) \neq f(\tau)$. This gives us a continuum N joining the interior of $f(\tau)$ to the exterior and having in common with $f(\tau)$ a proper sub-continuum $g(\tau)$ of $f(\tau)$. This is impossible by (b) above since each K is of type β. Therefore $M = N$; i.e., $g(t) = f(t)$ at every point and $f(t)$ is a *minimal* upper semi-continuous function. As M fills a part of the plane, it is not an irreducible continuum.

8. **Remarks.** As already stated, the construction of Example II shows that, if $f(t)$ is a minimal upper semi-continuous function defined over an interval ab, $M = F(ab)$ need not be irreducible between X_a and X_b, even though M is a plane continuum. The example can be easily modified to give a similar construction in space. Whether such an example can be constructed in space without the use of cyclic continua is an open question. An example of a continuous decomposition of an irreducible continuum into elements none of which are points, communicated to the author by Dr. Bronislaw Knaster, makes it seem probable that a decomposition similar to that in Example II can be arrived at without the use of indecomposable continua. Finally, this example justifies the last sentence in §2.

YALE UNIVERSITY,
NEW HAVEN, CONN.