ON LINEAR CONNECTIONS*

BY

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With the introduction of infinitesimal parallelism, by T. Levi-Civita† in 1917, and independently by J. A. Schouten‡ in 1918, tangent spaces began to play a leading rôle in differential geometry. The tangent space at a point, x, is the totality of all contravariant vectors, or differentials, associated with that point. By means of an affine connection§ the tangent spaces at any two points on a curve are related by an affine transformation, which will in general depend on the curve.

Linear connections of another kind were defined by R. König,|| who associated with each point of a given n-dimensional manifold a space of m dimensions. A linear connection arises in differential equations of the form¶

$$dZ^\alpha + Z^\beta \overset{\alpha}{L}_{\beta i} dx^i = 0, \tag{0.1}$$

by means of which the associated spaces at different points are related to each other, and which are said to define a linear displacement.

Even if $m=n$, a linear connection of the König type has nothing to do with an affine connection** unless we require explicitly that the associated space at each point is the tangent space of differentials at that point.

Schouten has proposed the use of linear connections in handling a scheme†† by which differential geometry is based on group theory, in the spirit of Klein's Erlanger Program. The associated spaces are to be the spaces, according to Klein, of some group, and are related through linear displacement

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† Rendiconti del Circolo Matematico di Palermo, vol. 42 (1917), pp. 173–205.

‡ Proceedings, Koninklijke Akademie van Wetenschappen, Amsterdam, vol. 21 (1918), pp. 607–613.

§ H. Weyl, Mathematische Zeitschrift, vol. 2 (1918), pp. 384–411. See also G. Hessenberg, Mathematische Annalen, vol. 78 (1918), p. 199.

¶ Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 28 (1919), pp. 213–228.

|| Greek letters, used as indices, will take on the values $1, \cdots, m$, and italic letters the values $1, \cdots, n$ ($m > n$ or $< n$).

** I mean by an affine connection any invariant with the transformation law

$$\bar{\Gamma}^i_{jk} = \left(\Gamma^a_{bc} \frac{\partial x^b}{\partial \bar{x}^j} \frac{\partial x^c}{\partial \bar{x}^k} + \frac{\partial^2 x^a}{\partial \bar{x}^j \partial \bar{x}^k}\right) \frac{\partial \bar{x}^i}{\partial x^a}.$$

†† Rendiconti del Circolo Matematico di Palermo, vol. 50 (1926), pp. 142–169. In particular Schouten has applied linear connections to the non-holonomic projective ($m=n+1$) and conformal ($m=n+2$) geometries.

by transformations of this group.* The general problem of imposing conditions upon the associated spaces, in order that they may be suitably related to the underlying manifold, has been discussed by Weyl† who solved the problem for the projective group.

Without touching on the questions which arise out of this scheme, there is a definite field for research in studying invariant properties of the differential equations (0.1) under transformations of the form (4.1). L. Schlesinger‡ has gone some distance in this direction, and we adopt this point of view in the present paper. Though most of our results refer to linear connections of the König type, they can all be interpreted in terms of affine connections and arbitrary n-uples. Given any affine connection we take $m=n$, and the associated spaces as the tangent spaces of differentials. In order to have a theory in which transformations of the form (4.1) are allowable, where (4.1a) is independant of (4.1b), we take

$$L^{\alpha}_{\beta i} = \gamma^{\alpha}_{\beta\sigma} u^{\sigma}_{i} \tag{0.2}$$

where u^{σ}_{i} are the covariant vectors of any n-uple, and $\gamma^{\alpha}_{\beta\sigma}$ are the scalar functions§ analogous to Ricci's coefficients of rotation. The equations (4.1a) will define a change over from one n-uple to another.

In §1 we give a geometrical proof of a theorem established by B. V. Williams|| and the author, in which we showed how to obtain an integrable connection which osculates (see §1 of this paper) a given linear connection. In §2 we prove a theorem about affine connections, which bears a formal re-

* This idea is mainly due to E. Cartan and is formulated by him in a paper (Bulletin de la Société Physico-Mathématique de Kazan, (2), vol. 3 (1927)), where he discusses Schouten's plan.

† Bulletin of the American Mathematical Society, vol. 35 (1929), pp. 716–725. Immediately preceding this, O. Veblen (Journal of the London Mathematical Society, vol. 4 (1929), pp. 140–160) had dealt with projective displacement from a different point of view. He showed how the space of projective vectors at any point, which plays the part of the associated space, is related to the space of differentials.

‡ Mathematische Annalen, vol. 99 (1928), pp. 413–434.

§ L. P. Eisenhart, *Non-Riemannian Geometry*, p. 47. These scalars are given by

$$\gamma^{\alpha}_{\beta\sigma} = u^{\alpha}_{i;j}\, v^{i}_{\beta} v^{j}_{\sigma},$$

where the semi-colon denotes covariant differentiation with respect to the affine connection, and v^{i}_{β} are the contravariant vectors of the n-uple. In his treatment of non-holonomic affine spaces Cartan (Annales de l'Ecole Normale Supérieure, 1923) uses n^2+n Pfaffian forms, ω^{α} and ω^{α}_{β}. The former give the coördinates of a point in each tangent space, and the latter define the affine connection. According to (0.2) these forms are given by

$$\omega^{\alpha} = u^{\alpha}_{i} dx^{i}, \qquad \omega^{\alpha}_{\beta} = L^{\alpha}_{\beta i} dx^{i}.$$

|| Annals of Mathematics, vol. 31 (1930), pp. 151–157. This paper will be referred to as T. L. C.

semblance to, but which differs essentially from either of those proved in T. L. C. We define a family of coördinate systems, which, like normal coördinate systems, have the property that each of them is uniquely determined by an affine connection, a point, and a given coördinate system. In §3 we show how the theorems of §§1 and 2 can be applied simultaneously to the theory of a linear connection together with an affine connection. In §4 we pass on to the study of invariants under transformations of the form (4.1). We prove a theorem for linear connections, and show that a similar theorem is true for symmetric affine connections. In the case of the latter this amounts to expressing

$$T^i_{\alpha[j;k_1;l_1;\cdots;k_p;l_p]}$$

in terms of T^i_α and the curvature tensor. In §5 we return to the study of a linear connection together with an affine connection, and show how a complete set of invariants may be obtained which are closely analogous to affine normal tensors. Dynamical systems with non-holonomic constraints provide a field of application for this theory, as we show briefly in §6. In §7 we apply the existing theory of Pfaffian forms to the equations for linear displacement, and show how the theorem in §1 is relevant to the study of integral subspaces.

As in T. L. C. we follow Schlesinger in his use of matrices. Instead of (0.1) we deal with the equations

$$dZ^\alpha_\beta + Z^\sigma_\beta L^\alpha_{\sigma i} dx^i = 0,$$

which we write as one equation

$$dZ + ZL_i dx^i = 0,$$

with a matrix for the unknown. This equation is completely integrable if, and only if, it is satisfied by a non-singular (i.e., with non-zero determinant) matrix $V(x)$. In this case we have

$$L_i = - V^{-1}V_{,i},$$

where we use the comma to denote partial differentiation.

1. **Osculating connections.** The necessary and sufficient conditions that the equation

$$dZ + ZL_i dx^i = 0$$

is completely integrable are that*

$$\tfrac{1}{2}R_{ij} = L_{[i,j]} + L_{[i}L_{j]} = 0.$$

* We follow J. A. Schouten in writing $p!A_{[i_1\ldots i_p]}$ for the alternating sum of the quantities $A_{i_1\ldots i_p}$.

In this case the connection L, and the displacement defined by this connection, are said to be integrable.

Williams and I showed that, given any linear connection L, and a coordinate system, x, there exists a unique integrable connection Γ such that

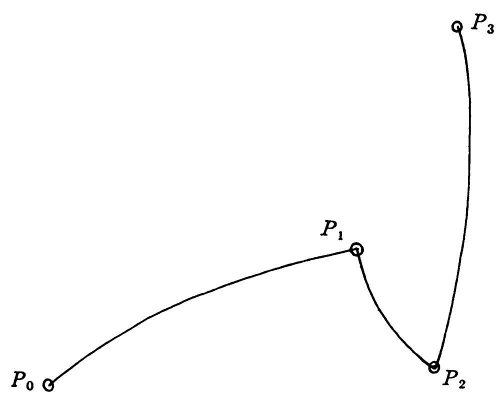
$$
\begin{aligned}
&\Gamma_n = L_n, \\
&\Gamma_{n-1} = L_{n-1} \quad \text{for} \quad x^n = 0, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\Gamma_p = L_p \quad \text{for} \quad x^{p+1} = \cdots = x^n = 0, \qquad (1.1)\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\Gamma_1 = L_1 \quad \text{for} \quad x^2 = \cdots = x^n = 0.
\end{aligned}
$$

We shall refer to this as Theorem C. Since the connection Γ is completely determined by these conditions we shall say that it osculates the connection L, in the manner described by (1.1). Notice that Γ is determined not by L alone, but by L together with the series of subspaces, each contained in the next, on which the components $\Gamma_1, \Gamma_2, \cdots$ agree with $L_1, L_2, \cdots$.

We shall give another proof of this theorem. Let $P_x \equiv (x^1, \cdots, x^n)$ be any point in the neighborhood of the point $P_0 = (0, \cdots, 0)$, and let

$$
\begin{aligned}
&P_1 \text{ be the point } (x^1, 0, \cdots, 0), \\
&P_2 \text{ `` `` `` } (x^1, x^2, 0, \cdots, 0), \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&P_r \text{ `` `` `` } (x^1, \cdots, x^r, 0, \cdots, 0), \qquad (1.2)\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&P_n \text{ `` `` `` } P_x \equiv (x^1, \cdots, x^n).
\end{aligned}
$$

There is a unique curve, $P_0P_1 \cdots P_x$, joining P_0 to each point P_x in the neighborhood of P_0. Each of these curves is analytic except at a finite number of points.



Let V_0 be any non-singular matrix. Then at each point, x, is defined a non-singular matrix, $V(x)$, by the linear displacement of V_0 from P_0 to P_x, along the curve $P_0P_1 \cdots P_x$. Along this curve we shall have

$$dV + VL_i dx^i = 0. \tag{1.3}$$

By repeated applications of the theorem that a solution of

$$\frac{dy^\alpha}{dt} = \phi^\alpha(t;\, a^1, \cdots, a^N),$$

where ϕ^α are analytic in all their arguments, is itself analytic in t and $a^1, \cdots, a^N$, we see that V is an analytic function of x. We can, therefore, define an integrable connection by the equations

$$\Gamma_i = -V^{-1}V_{,i}, \tag{1.4}$$

and, from (1.3) and (1.4), we have

$$(\Gamma_i - L_i)dx^i = 0 \tag{1.5}$$

along each curve $P_0P_1 \cdots P_x$. On the segment $P_{p-1}P_p$ we have

$$\Gamma_p = L_p \text{ for } x^{p+1} = \cdots = x^n = 0, \tag{1.6}$$

and by giving p the values* $1, \cdots, n$ we have the relations (1.1), and the theorem is established.

2. **A theorem on affine connections.** In this section we shall prove the following

THEOREM. *Let D be any affine connection (not necessarily symmetric). Then there exist coördinate systems in which the components, D^i_{jk}, of D satisfy the following conditions:*

$$\begin{aligned} &D^i_{nn} = 0; \\ &D^i_{\rho n-1} = 0, \ \rho \geqq n-1, \quad \text{for} \quad y^n = 0; \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &D^i_{\rho p} = 0, \ \rho \geqq p, \qquad \text{for} \ y^{p+1} = \cdots = y^n = 0; \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &D^i_{j1} = 0, \ j = 1, \cdots, n, \ \text{for} \quad y^2 = \cdots = y^n = 0. \end{aligned} \tag{2.1}$$

Any such coördinate system is uniquely determined by a point and n independent contravariant vectors associated with that point.

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* When $p=n$ we have simply $\Gamma_n = L_n$.

By taking these as the unit vectors tangent to the coördinate lines, in any coördinate system, we can associate, with each coördinate system and each point, a unique coördinate system in which (2.1) are satisfied. We shall thus have a class of coördinate systems, the totality of which will be an invariant of the affine connection D.

An essential difference between this theorem and those proved in T. L. C. is that it is concerned with the affine connection itself, and not with the affine connection together with a given system of curves and surfaces.

Let P_0 be any point in the space bearing the affine connection D. Let p_a be n independent contravariant vectors associated with P_0. A coördinate system may be constructed by the following procedure. The coördinates of P_0 are to be $(0, \cdots, 0)$. Let C_1 be the path which passes through P_0 in the direction determined by the vector p_1. Move the matrix (p_a^i) by parallel displacement along C_1 from P_0 to a point P_1. The coördinates of P_1 are to be $(y^1, 0, \cdots, 0)$. Let $v_a(y^1, 0, \cdots, 0)$ be the components of the vectors thus obtained, and let C_2 be the path through P_1 in the direction v_2. Then move the matrix v by parallel displacement along C_2 from P_1 to a point P_2, whose coördinates are to be $(y^1, y^2, 0, \cdots, 0)$. Repeating* this process we shall eventually reach a point P_n whose coördinates are to be $(y^1, \cdots, y^n)$.

The proof that this process gives an allowable coördinate system (i.e., a coördinate system obtained by an analytic transformation from a given coördinate system) is of the same nature as that required in §1.

Let H_{jk}^i be the components of D in a coördinate system x, in which the equations to C_1 are $x^i = \phi^i(y^1)$. The components of the n-uple v_a^i, at P_1, are given by those sets of solutions to

$$\frac{dx^i}{dy^1} + X^j H_{jk}^i \frac{d\phi^k}{dy^1} = 0 \tag{2.2}$$

which reduce to p_a^i for $y^1 = 0$. The equations to the path C_2 are given by those solutions, $\overset{i}{\phi_1}(y^1, t)$, to the equations

$$\frac{d^2x^i}{dt^2} + H_{jk}^i \frac{dx^j}{dt}\frac{dx^k}{dt} = 0.$$

which satisfy the initial conditions

$$\begin{aligned} \overset{i}{\phi_1}(y^1, 0) &= \phi^i(y^1), \\ \left(\frac{d\overset{i}{\phi_1}}{dt}\right)_{t=0} &= \overset{i}{v_2}(y^1, 0, \cdots, 0). \end{aligned} \tag{2.3}$$

* At the rth step we shall move the matrix along the path C_r, which passes through P_{r-1} in the direction v_r, from P_{r-1} to a point P_r, whose coördinates are to be $(y^1, \cdots, y^r, 0, \cdots, 0)$.

The components v_a^i at P_2 are given by the sets of solutions to the equations

$$\frac{dX^i}{dt} + X^j H^i_{jk}\frac{d\phi_1^k}{dt} = 0$$

which reduce to $v_a^i(y^1, 0, \cdots, 0)$ for $t=0$. If we put $y^2=t$ the coördinates of P_2, and the components of v_a^i at this point, will be given by

$$x^i = \phi_1^i(y^1, y^2), \qquad v_a^i = v_a^i(y^1, y^2, 0, \cdots, 0). \tag{2.4}$$

Since $\phi(y^1)$ and H^i_{jk} are analytic functions of y^1 and x respectively, ϕ_1^i and v_a^i, given by (2.4), will be analytic functions of y^1 and y^2. Proceeding in this way we shall eventually obtain, for the coördinates of P_n,

$$x^i = x^i(y^1, \cdots, y^n),$$

and for the components of v_a^i at this point,

$$v_a^i(y^1, \cdots, y^n),$$

where x^i and v_a^i are analytic functions of y. The process given above in descriptive terms does, therefore, give an allowable coördinate system, and an n-uple of contravariant vectors whose components are analytic functions of y.

From the construction for v_a^i it follows that

$$\begin{aligned} v_n^i &= \delta_n^i, \\ v_{n-1}^i &= \delta_{n-1}^i \text{ for } \quad y^n = 0, \\ &\cdots\cdots\cdots \\ v_p^i &= \delta_p^i \quad \text{for } \; y^{p+1} = \cdots = y^n = 0, \\ &\cdots\cdots\cdots \\ v_1^i &= \delta_1^i \quad \text{for } \quad y^2 = \cdots = y^n = 0. \end{aligned} \tag{2.5}$$

These equations may be concentrated into

$$v_\rho^i = \delta_\rho^i, \, \rho \geqq p, \text{ for } y^{p+1} = \cdots = y^n = 0, \, p = 1, \cdots, n.$$

Let

$$\overset{a}{u}_i \overset{i}{v}_b = \overset{a}{\delta}_b.$$

Then from (2.5) we have

$$\overset{a}{u}_\rho = \overset{a}{\delta}_\rho, \, \rho \geqq p \text{ for } y^{p+1} = \cdots = y^n = 0 \quad (p = 1, \cdots, n). \tag{2.6}$$

By an argument used in §1 we have

$$
(2.7)\qquad
\begin{aligned}
D^i_{jn} &= E^i_{jn},\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
D^i_{jp} &= E^i_{jp} \text{ for } y^{p+1} = \cdots = y^n = 0,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
D^i_{j1} &= E^i_{j1} \text{ for } y^2 = \cdots = y^n = 0,
\end{aligned}
$$

where $E^i_{jk} = v^i_a u^a_{j,k}$, and D^i_{jk} are the components, in y, of the given connection. From (2.6) we have

$$
(2.8)\qquad E^i_{\rho p} = 0,\ \rho \geqq p,\ y^{p+1} = \cdots = y^n = 0 \quad (p = 1, \cdots, n),
$$

which, combined with (2.7), give (2.1). The coördinate system is uniquely determined by the point P_0 and the matrix (p^i_a). The theorem is therefore established.

We shall give another proof that (2.1) hold, which will bring out their geometrical significance. The curves of the congruence defined by $v^i_n(=\delta^i_n)$ are paths. Hence

$$
D^i_{nn} = 0.
$$

The curves of the congruence defined by v_{n-1}, which lie in the hypersurface $y^n=0$, are paths. Since $v^i_{n-1}=\delta^i_{n-1}$ for $y^n=0$, we have

$$
D^i_{n-1,n-1} = 0 \text{ for } y^n = 0.
$$

But the vectors v_n are parallel at different points of these curves. Hence

$$
D^i_{n,n-1} = 0 \text{ for } y^n = 0.
$$

The remaining conditions may be obtained by a repetition of this argument.

In case D is symmetric all its components figure in the equations

$$
D^i_{\rho p} = 0,\ \rho \geqq p, \text{ for } y^{p+1} = \cdots = y^n = 0 \quad (p = 1, \cdots, n),
$$

which may be written

$$
(2.9)\qquad D^i_{pq} = 0 \text{ for } y^{s+1} = \cdots = y^n = 0,\ s = \min(p, q) \quad (p, q = 1, \cdots, n).
$$

3. **Linear connections together with affine connections.** The theorem proved in §1 belongs to the combined theory of a linear connection and an affine connection, for it refers to the connection L and the sub-spaces given in the coördinate system x by $x^2 = \cdots = x^n = 0$, $x^3 = \cdots = x^n = 0$, and so on.

These loci are flat sub-spaces, defined by a flat affine connection for which x is a cartesian coördinate system. If we are concerned with the general theory of a linear connection L, and an affine connection D, we can construct a coördinate system y and an n-uple v_a^i by the process given in §2. Theorem C, referred to the coördinate system y, will belong to the combined theory of L and D. In place of (1.1) we can write the relations

$$(3.1)\qquad (\Gamma_i - L_i)v_p^i = 0 \text{ for } y^{p+1} = \cdots = y^n = 0 \qquad (p = 1, \cdots, n).$$

The methods of §1 can be used to give other osculating integrable connections. The simplest of these is constructed by taking normal coördinates, y, for D at any point P_0, and considering the matrix function, V, given by the linear displacement of a non-singular matrix, V_0, from P_0 to any point y along the path joining these points. The equations giving V are

$$(3.2)\qquad dV + VL_i dx^i = 0,$$

or

$$(3.3)\qquad (\Delta_i - L_i)dy^i = 0,$$

where Δ is the integrable connection given by

$$(3.4)\qquad V_{,i} + V\Delta_i = 0.$$

Since y^i are normal coördinates, and (3.3) refer to displacement along paths through the origin, we have

$$(3.5)\qquad (\Delta_i - L_i)y^i = 0.$$

As in §1 the connection Δ is uniquely determined by this condition.

4. **Invariant theory.** In this section we take up the invariant theory of a linear connection under transformations of the form

$$(4.1)\qquad \begin{aligned} &\text{(a)} \qquad \bar{Z}^\alpha = Z^\beta p_\beta^\alpha, \\ &\text{(b)} \qquad \bar{x}^i = \bar{x}^i(x), \end{aligned}$$

where $\|p_\beta^\alpha\|$ is a non-singular matrix depending on x only. A coördinate system for the underlying manifold, together with a frame of reference in each of the associated spaces, will be called a *representation*; and a transformation of the form (4.1) will be called a change of representation. On this basis an invariant may be defined in terms of its transformation law* under changes of representation. We shall deal only with linear connections, and with tensors having m^2n^p components which obey the transformation law

* Schlesinger, loc. cit., p. 423.

$$\bar{T}^{\lambda}_{\beta(i)} P^{\alpha}_{\lambda} = P^{\lambda}_{\beta} T^{\alpha}_{\lambda(a)} \frac{\partial x^{(a)}}{\partial \bar{x}^{(i)}},$$

where the symbol (i) stands for any number of italic indices, and $P^{\lambda}_{\beta} p^{\alpha}_{\lambda} = \delta^{\alpha}_{\beta}$. The transformation law for a linear connection is given by

$$\bar{L}^{\alpha}_{\beta j} = (P^{\lambda}_{\beta,i} + P^{\mu}_{\beta} L^{\lambda}_{\mu i}) p^{\alpha}_{\lambda} \frac{\partial x^{i}}{\partial \bar{x}^{j}},$$

and we shall write these formulas*

$$(4.2)\quad \begin{aligned} &\text{(a)} \qquad \bar{T}_{(i)} P = P T_{(a)} \frac{\partial x^{(a)}}{\partial \bar{x}^{(i)}}, \\ &\text{(b)} \qquad \bar{L}_{j} P = (P_{,i} + P L_{i}) \frac{\partial x^{i}}{\partial \bar{x}^{j}} \cdot \end{aligned}$$

From (4.2b) we see that there exist representations in which all the components of an integrable connection vanish. For if Γ is an integrable connection, there will be a non-singular matrix-function V, such that

$$V_{,i} + V\Gamma_{i} = 0,$$

and the components of Γ will vanish in the representation given by

$$\begin{aligned} \bar{Z}^{\alpha} &= Z^{\beta} U^{\alpha}_{\beta}, \\ \bar{x}^{i} &= x^{i}, \end{aligned}$$

where $U = V^{-1}$. All representations in which the components of the connection vanish are related by equations of the form (4.1), where p is a constant matrix.

An operation analogous to covariant differentiation arises from the following considerations. Let V be a matrix which satisfies the equation

$$(4.3) \qquad dV + VL_{i} dx^{i} = 0$$

* Since the transformations (4.1a) and (4.1b) are independent, it might, for some purposes, be desirable to borrow from group-theory the notion of conjugacy. Two tensors K and H may be described as conjugate if there exists a non-singular matrix, V, such that

$$K_{(i)} V = V H_{(i)}.$$

The set of all tensors conjugate to a given tensor may be called the class of that tensor. Similarly two linear connections are in the same class if there exists a non-singular matrix, V, such that

$$M_{i} V = V_{i} + V L_{i}.$$

All tensors or linear connections belonging to the same class are seen to be equivalent under transformations of the form (4.1a).

along a curve C. It follows that

(4.4) $$dV^{-1} - L_i V^{-1} dx^i = 0.$$

Let $T_{(i)}$ be a given tensor and let

$$A_{(i)} = VT_{(i)}V^{-1}.$$

Differentiating along C we have, from (4.3) and (4.4),

$$dA_{(i)} = VT_{(i)/k}V^{-1},$$

where*

(4.5) $$T_{(i)/k} = T_{(i),k} + T_{(i)}L_k - L_k T_{(i)}.$$

Direct calculation shows that

(4.6) $$2T_{(i)/[j/k]} = T_{(i)}R_{jk} - R_{jk}T_{(i)}.$$

Let

(4.7) $$(TR)_s = R_{j_1k_1} \cdots R_{j_{s-1}k_{s-1}} T_{(i)} R_{j_sk_s} \cdots R_{j_pk_p},$$

and let an operator, α, be defined by the equation

$$\alpha(TR)_s = (TR)_{s+1}.$$

We can describe α as an operator which moves T one place to the right in any expression such as (4.7), without respect to particular values of s and p $(p > s)$. We can write (4.6) as

$$T_{(i)/[j/k]} = \tfrac{1}{2}(1 - \alpha)T_{(i)}R_{jk}.$$

In T. L. C. (p. 154) it was shown that

$$R_{[jk/l]} = 0.$$

Hence

$$T_{(i)/[j_1/k_1/j_2/k_2]} = \tfrac{1}{4}(1 - \alpha)^2 T_{(i)} R_{[j_1k_1} R_{j_2k_2]},$$

and, in general,

(4.8) $$T_{(i)/[j_1/k_1/\cdots/j_p/k_p]} = \frac{1}{2^p}(1 - \alpha)^p T_{(i)} R_{[j_1k_1} \cdots R_{j_pk_p]}.$$

* We cannot derive tensors from a given tensor by repeated applications of this operation, as there is no way of eliminating the second derivatives

$$\frac{\partial^2 x^i}{\partial \bar{x}^j \partial \bar{x}^k}.$$

Similar identities will occur in the theory of a symmetric affine connection. For if $m=n$ and L is an affine connection such that

$$L^{\alpha}_{\beta i} = L^{\alpha}_{i\beta},$$

we have

$$T^{\alpha}_{\beta[i;j]} = T^{\alpha}_{\beta[i/j]},$$

where the semi-colon denotes ordinary covariant differentiation, and $T^{\alpha}_{\beta i/j}$ means the same as before. The relation (4.8) was obtained by purely formal methods, and we have, therefore,

$$(4.9) \qquad T_{[i;j_1;k_1;\cdots;j_p;k_p]} = \frac{1}{2^p}(1-\alpha)^p T_{[i}R_{j_1k_1}\cdots R_{j_pk_p]}.$$

5. **Normal representations.** In the theory of a linear connection together with an affine connection, the comma on the right hand side of (4.5) can be taken to define covariant differentiation with respect to the latter. If, for example, C^i_{jk} are the components of the affine connection, we shall have*

$$T_{i/j} = \frac{\partial T_i}{\partial x^j} - T_s C^s_{ij} + T_i L_j - L_j T_i.$$

It will then be possible to obtain successive tensor invariants from a given tensor.

Let y be the normal coördinate system at a point q for the affine connection and the coördinate system, x, in some given representation. There will be representations in which the components of the integrable connection Δ, defined by (3.4) and (3.5), are zero. There is just one of these representations, the normal coördinate system, y, being retained throughout, which determines in the associated space at q the same frame of reference as the given representation. This is obtained by imposing the initial conditions

$$(V^{\alpha}_{\beta})_{y=0} = \delta^{\alpha}_{\beta},$$

in the equations (3.4), and may be called the *normal representation* at q for the linear connection together with the affine connection, and for the given representation. In this representation we have

$$(5.1) \qquad L_i y^i = 0,$$

and

* This is a simple application of a scheme introduced by A. W. Tucker in a paper which will shortly appear in the Annals of Mathematics.

(5.2) $$L_i = \sum_{p=1}^{\infty} \frac{1}{p!} H_{i\,k_1\cdots k_p} y^{k_1} \cdots y^{k_p},$$

where

(5.3) $$H_{i\,k_1\cdots k_p} = \left(\frac{\partial^p L_i}{\partial y^{k_1} \cdots \partial y^{k_p}}\right)_{y=0}.$$

From (5.1) and (5.2) it follows that

(5.4) $$H_{i\,k_1+\cdots+k_p} + H_{k_1 i\cdots k_p} + \cdots + H_{k_1\cdots k_p i} = 0.$$

Let $\overline{H}_{ik}, \overline{H}_{ik_1k_2}, \cdots, \overline{H}_{i(k)_p}, \cdots$ be the quantities obtained in the same way as $H_{i(k)_p}$, at the same point, but starting with a different representation. Just as in the affine theory, it follows that $H_{i(k)_p}$ and $\overline{H}_{i(k)_p}$ are related by the transformation law for a tensor. Hence a sequence of tensors,

$$H_{i\,k}, \cdots, H_{i\,k_1\cdots k_p}, \cdots,$$

analogous to affine normal tensors, is defined by the condition that the components of $H_{i(k)_p}$, at each point q, shall be given by (5.3). These, together with the normal tensors for the affine connection, constitute a complete set of invariants for the linear connection together with the affine connection. This may be proved by methods similar to those used in proving the analogous theorem for affine connections.*

6. **Application to dynamics.** The mathematical machinery used by G. Vranceanu† in his treatment of dynamical systems with non-holonomic constraints, may be regarded as the combined theory of a linear connection and a Riemannian metric. The metric $g_{ij}dx^i dx^j$ represents the kinetic energy, and the constraints can be represented by m unit orthogonal vectors $\xi_1, \cdots, \xi_m$ $(m \leqq n)$. If $\gamma^{\alpha}_{\beta\sigma}$ are the rotation functions, given by

$$\overset{\alpha}{\gamma}_{\beta\sigma} = \xi_{\alpha i}\overset{i}{\xi}_{\beta;\sigma},$$

where $\xi_{\beta;\sigma}$ is the intrinsic derivative of ξ_β, a linear connection is defined by

$$\overset{\alpha}{L}_{\beta i} = \overset{\alpha}{\gamma}_{\beta\sigma}\xi_{\sigma i}.$$

We should limit (4.1a) to orthogonal transformations by imposing the condition

$$\overset{\sigma}{p}_{\alpha}\overset{\sigma}{p}_{\beta} = \delta_{\alpha\beta}.$$

* T. Y. Thomas, Mathematische Zeitschrift, vol. 25 (1926), pp. 723–733. Thomas was considering a special type of affine connection, but the method is general.

† Comptes Rendus, vol. 183 (1926), p. 852, also p. 1083.

The associated space at each point can be identified with the sub-space spanned in the tangent space by the vectors ξ_α, but the essential feature which distinguishes the theory of a linear connection from that of an affine connection is retained: namely that the frame of reference may be changed in each associated space independently of coördinate transformations.

7. **Integral sub-spaces.** In this section we shall show how some of the general ideas in the theory of Pfaffian forms* can be interpreted in terms of linear displacement, when considering the equations

$$dZ^\alpha + Z^\beta L^{\alpha}_{\beta i} dx^i = 0. \tag{7.1}$$

It will be convenient to say that a set of numbers $(x^1, \cdots, x^n; Z^1, \cdots, Z^m)$ determine, on the one hand a point x in the underlying manifold V_n, together with a point Z in the linear space associated with x, and on the other hand a point in a space of $m+n$ dimensions, which we shall denote by S_{n+m}. We shall discuss some of the simpler properties of the integral sub-spaces, in S_{n+m}, of the equations (7.1).

Let R_{ij} be the curvature tensor derived from the linear connection L. We shall say that any two vectors ξ and η which satisfy the condition

$$R_{ij}\xi^i\eta^j = 0 \tag{7.2}$$

are in involution† with respect to L. Any two vectors linearly dependent on ξ and η will also satisfy (5.2). Let a set of vectors $\xi_1, \cdots, \xi_p$, such that

$$\overset{i}{\xi}_{\lambda,j}\overset{j}{\xi}_{\mu} - \overset{i}{\xi}_{\mu,j}\overset{j}{\xi}_{\lambda} = \overset{\nu}{c}_{\lambda\mu}\overset{i}{\xi}_{\nu} \qquad (\nu = 1, \cdots, p), \tag{7.3}$$

be mutually in involution with respect to L. In virtue of (7.3) we can find a set of vectors $X_1, \cdots, X_p$, linearly dependent on $\xi_1, \cdots, \xi_p$, and such that the equations

$$\frac{\partial x^i}{\partial t^\lambda} = X^i_\lambda \qquad (\lambda = 1, \cdots, p) \tag{7.4}$$

are completely integrable. The vectors $X_1, \cdots, X_p$ will, therefore, define a congruence‡ of p-spaces given by

---

* All these ideas are to be found in Goursat's *Leçons sur le Problème de Pfaff*, especially in chapters VI and VIII. The latter chapter is mainly an exposition of Cartan's work.

† This is not the same as saying that ξ and η are in involution with respect to the equations (7.1), the conditions for which are

$$Z^\beta R^{\alpha}_{\beta ij}\xi^i\eta^j = 0.$$

We require that ξ^i and η^i shall not depend on Z, in which case these equations imply (7.2).

‡ By a congruence we mean a family of p-spaces such that one and only one passes through each point of some given n-cell in V_n.

$$x^i - x_0^i = x^i(t^1, \cdots, t^p; x_0), \tag{7.5}$$

where $x^i(t^1, \cdots, t^p; x_0)$ satisfy (7.4). Since the vectors $X_1, \cdots, X_p$ are mutually in involution with respect to L, we shall have

$$R_{ij}\frac{\partial x^i}{\partial t^\lambda}\frac{\partial x^j}{\partial t^\mu} = 0, \tag{7.6}$$

and so the equations

$$\frac{\partial Z^\alpha}{\partial t^\lambda} + \overset{\beta}{Z} \overset{\alpha}{L}_{\beta i}\frac{\partial x^i}{\partial t^\lambda} = 0 \qquad (\lambda = 1, \cdots, p) \tag{7.7}$$

will be completely integrable. On each p-space of the congruence given by (7.5) the connection L determines, therefore, an integrable displacement. Any solution to (7.7) is of the form

$$\overset{\alpha}{Z} = \overset{\beta}{Z_0}\overset{\alpha}{\phi_\beta}(t^1, \cdots, t^p; x_0),$$

where $\overset{\alpha}{Z_0}$ are arbitrary constants.

In terms of the space S_{n+m} we say that the equations

$$\begin{aligned} \overset{\alpha}{Z} &= \overset{\beta}{Z_0}\overset{\alpha}{\phi_\beta}(t; x_0), \\ x^i &= x_0^i + x^i(t; x_0) \end{aligned} \tag{7.8}$$

define a congruence of integral p-spaces (in S_{n+m}) with respect to the equations (7.1). Such a family of integrals is called *generic*, since (7.4) and (7.7) are completely integrable at a "typical point" of S_{n+m}.

It may happen that there are singular* integral sub-spaces in S_{n+m}. Singular integrals arise in any of the three following cases:

(1) The equations (7.6) are satisfied by a complete system of vectors $X_1, \cdots, X_p$, but only on some sub-space of V_n (i.e. subject to certain conditions, $\phi(x)=0, \psi(x)=0, \cdots$).

(2) The equations (7.4) admit solutions, but are not completely integrable.

(3) The equations (7.7) admit solutions, but are not completely integrable. In the third case let (7.7) admit a complete set of q independent solutions $U_1, \cdots, U_q$, $q<m$. Then a^sU_s, $s=1, \cdots, q$, where a^s are constants, will also be a solution, and so, for $x=$const., the totality of solutions to (7.7) will be the linear space spanned by $U_1, \cdots, U_q$. In terms of the linear connection L we have an integrable displacement of linear q-spaces in

* An integral sub-space is called singular if it does not belong to a congruence, but to a family which is entirely contained in some sub-space of higher dimensionality.

the associated spaces.* Since the matrix (U_s^α), $\alpha=1, \cdots, m$, $s=1, \cdots, q$, is of rank q we may assume that the determinant $|U_s^t|$, $s, t=1, \cdots, q$, does not vanish. Apply the change of representation given by

$$Z^\alpha = \bar{Z}^s U_s^\alpha + \bar{Z}^\rho \delta_\rho^\alpha \quad (s = 1, \cdots, q, \rho = q+1, \cdots, m),$$
$$\bar{x}^i = x^i.$$

Then the linear q-spaces in question are given, in the new representation, by $\bar{Z}^\rho=0$. Since the equations (7.1) are invariant in form under all changes of representation, it follows that

$$L_{si}^\rho dx^i = 0 \tag{7.9}$$

for values of dx tangent to any sub-space on which (7.7) admit the solutions $U_1, \cdots, U_q$.

In the remainder of this section we shall suppose that (7.7) either admit no solutions, or else are completely integrable, in which case the vectors $X_1, \cdots, X_p$ are mutually in involution with respect to L. Singular integrals will occur, therefore, only in the event of (1) or (2) arising, and we shall combine these into the case where V_n admits a family of sub-spaces, on each of which L defines an integrable displacement, and which, in a suitable coordinate system, are defined by equations of the form

$$x^{p+1} = c^{p+1}, \cdots, x^q = c^q, \; x^{q+1} = \cdots = x^n = 0, \qquad n > q > p, \tag{7.10}$$

where $c^{p+1}, \cdots, c^q$ are arbitrary constants. If $q=n$ this family is a congruence, and the corresponding integrals generic. We shall show how Theorem C is relevant to this simplified theory of integral sub-spaces, and to the study of the characteristics. A vector ξ will be described as a characteristic of the connection L if it is in involution, with respect to L, with every other vector. The necessary and sufficient conditions for this to be the case are that†

$$R_{ij}\xi^j = 0. \tag{7.11}$$

Let ξ_ρ, $\rho=p+1, \cdots, n$, be a complete set of solutions to these equations. Differentiating

$$R_{ij}\xi_\rho^j = 0,$$

* If L were an affine connection $U_1, \cdots, U_q$ would be parallel fields of contravariant vectors, defined on some sub-space in V_n.

† A vector ξ is a characteristic for the Pfaffian system (7.1) if

$$Z^\beta R_{\beta ij}^\alpha \xi^j = 0,$$

and we can only deduce (7.11) from these equations when ξ^i are independent of Z.

we can write the result in the forms

$$
\begin{aligned}
R_{ij/k}\xi^j_\rho + R_{ij}\xi^j_{\rho,k} &= 0,\\
R_{ik/j}\xi^k_\sigma + R_{ik}\xi^k_{\sigma,j} &= 0,\\
R_{jk/i}\xi^j_\rho + R_{jk}\xi^j_{\rho,i} &= 0,\\
R_{jk/i}\xi^k_\sigma + R_{jk}\xi^k_{\sigma,i} &= 0.
\end{aligned}
$$

Multiplying these equations by ξ^k_σ, $-\xi^j_\rho$, $\frac{1}{2}\xi^k_\sigma$, and $\frac{1}{2}\xi^j_\rho$ respectively, adding them together, and taking into account the relations $R_{[ij/k]}=0$ and (7.11), we obtain

$$R_{ij}\xi^j_{[\rho,\sigma]} = 0,$$

where

$$\xi^i_{\rho,\sigma} = \xi^i_{\rho,j}\xi^j_\sigma.$$

Since ξ_ρ form a complete set of solutions to (7.11) we have,* from these equations,

$$\xi^i_{[\rho,\sigma]} = C^\tau_{\rho\sigma}\xi^i_\tau \qquad (\tau = p+1, \cdots, n). \tag{7.12}$$

There exists, therefore, a coördinate system x, in which δ^i_ρ form a complete set of solutions to (7.11), and in this coördinate system,

$$R_{i\rho} = 0. \tag{7.13}$$

The vectors δ^i_ρ define the congruence of $(n-p)$-spaces given by

$$x^1 = c^1, \cdots, x^p = c^p,$$

where $c^1, \cdots, c^p$ are arbitrary constants. This will be called the congruence of characteristic sub-spaces for the connection L. Applying Theorem C to the connection L, in the coördinate system x, and choosing a representation in which the components of the integrable connection Γ vanish, we have

$$
(7.14)\qquad
\begin{array}{llll}
(a_n) & L_n = 0; & & \\
(a_{n-1}) & L_{n-1} = 0 & \text{for} & x^n = 0; \\
\cdot & \cdot\ \cdot\ \cdot & & \cdot\ \cdot\ \cdot \\
(a_p) & L_p = 0 & \text{for} & x^{p+1} = \cdots = x^n = 0; \\
\cdot & \cdot\ \cdot\ \cdot & & \cdot\ \cdot\ \cdot \\
(a_1) & L_1 = 0 & \text{for} & x^2 = \cdots = x^n = 0.
\end{array}
$$

* The linear elements, in S_{n+m}, given by $(\xi^i_\rho, -Z^\beta L^\alpha_{\beta i}\xi^i_\rho)$, do not necessarily define the characteristic manifolds (in S_{n+m}) for the equations (7.1). The relations (7.12) do not, therefore, follow from the general theory of Pfaffian systems.

From (7.13) and ($7.14a_n$) we have, if $p<n$,

$$L_{i,n} = 0.$$

The condition $x_n=0$ may, therefore, be discarded in all the relations (7.14). We have, therefore, $L_{n-1}=0$, and from (7.13), if $p<n-1$,

$$L_{i,n-1} = 0.$$

Hence the condition $x^{n-1}=0$ may also be discarded in (7.14). Repeating this argument we can replace (7.14) by the relations

$$\begin{aligned} &L_\rho = 0, \ \rho = p+1, \cdots, n; \\ &L_p = 0; \\ &L_{p-1} = 0 \quad \text{for} \quad x^p = 0; \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &L_1 = 0 \quad \text{for} \quad x^2 = \cdots = x^p = 0, \end{aligned} \tag{7.15}$$

with the condition $L_{i,\rho}=0$, $\rho=p+1, \cdots, n$. The study of the connection L may, therefore, be confined to its behavior on the sub-space V_p, of V_n, given by $x^{p+1}= \cdots =x^n=0$. The number p may be called the class of the connection* L.

Let us suppose the class to be n, and consider the equations (7.14). The equations ($7.14a_n$) assert that the connection L defines an integrable displacement along the curves of parameter x^n. If L defines an integrable displacement on each surface of the congruence given by

$$x^a = c^a \qquad (a = 1, \cdots, n-2),$$

where c^a are arbitrary constants, we can discard the condition $x^n=0$ in ($7.14a_{n-1}$), and conversely. In this case the surfaces in S_{n+m}, given by

$$x^1 = c^1, \cdots, x^{n-2} = c^{n-2}, Z^1 = A^1, \cdots, Z^m = A^m,$$

where c and A are arbitrary constants, belong to a generic family of integrals with respect to the equations (7.1).

In order that the surfaces in S_{n+m} given by

$$x^1 = c^1, \cdots, x^{p-1} = c^{p-1}, Z^\alpha = A^\alpha$$

---

* If p is the class of a linear connection Γ, the tensor (see §4) whose components are

$$R_{[j_1k_1} \cdots R_{j_qk_q]}$$

will vanish for $2q>p$. This follows from the existence of coördinate systems in which $R_{i\rho}=0$, for $\rho>p$.

shall be a generic family of integrals, it is necessary and sufficient that $L_p = \cdots = L_n = 0$.

Starting from the other end, the sub-space in S_{n+m}, given by

$$x^{p+1} = \cdots = x^n = 0, \quad Z^\alpha = A^\alpha,$$

will be a singular integral if, and only if, $L_1 = \cdots = L_p = 0$ for $x^{p+1} = \cdots = x_n = 0$. The connection L will then define an integrable displacement over the p-space in V_n, given by $x^{p+1} = \cdots = x^n = 0$.

In general let a_q^p, $n \geqq q > p$, stand for the condition $x^q = 0$ which is imposed in $(7.14a_p)$. If the connection L is such that any given set of these conditions, $a_{q_1}^{p_1}, \cdots, a_{q_s}^{p_s}$, can be discarded, there will be a family of integrals, whose equations will be apparent from (7.14). If, for example, the conditions a_{n-2}^{n-1}, a_{n-2}^{n} are unnecessary, the surfaces in S_{n+m} given by

$$x^1 = c^1, \cdots, x^{n-3} = c^{n-3}, x^{n-1} = c^{n-1}, Z^\alpha = A^\alpha,$$

will be generic integrals, and those given by

$$x^1 = c^1, \cdots, x^{n-3} = c^{n-3}, x^n = 0, Z^\alpha = A^\alpha$$

singular integrals.

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