# NON-SEPARATED CUTTINGS OF CONNECTED POINT SETS*

BY

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1. We shall consider a connected, metric and separable space which we denote by $M$. A subset $X$ of $M$ is called a cutting of $M$ provided that the complement $M-X$ of $X$ is not connected and hence is the sum of two mutually separated sets $M_1(X)$ and $M_2(X)$; $X$ is said to separate two points or point sets $A$ and $B$ in $M$ when the sets $M_1(X)$ and $M_2(X)$ can be so chosen that $M_1(X) \supset A$ and $M_2(X) \supset B$, and is said to separate a single set $N$ in $M$ when $M_1(X)$ and $M_2(X)$ can be chosen so that $N \cdot M_1(X) \neq 0 \neq N \cdot M_2(X)$.

A collection $G$ of subsets of $M$ will be called *non-separated* provided that the elements of $G$ are mutually exclusive and no element of $G$ separates any other element of $G$ in $M$.

A subset $P$ of $M$ is said to have the potential order $\alpha$ in $M$ relative to a given collection $G$ of subsets of $M$ provided that $\alpha$ is the least cardinal number such that there exists a monotonic decreasing sequence $[U_i]$ of neighborhoods of $P$ such that $P = \prod_1^\infty \overline{U}_i$ and such that for each $i$, the boundary $F(U_i)$ of $U_i$ is a subset of the sum of $\alpha$ of the sets of the collection $G$.

In this paper we shall show, first, that if $G$ is any uncountable non-separated collection of cuttings of $M$ then *all save a countable number of the elements of $G$ have the potential order 2 in $M$ relative to $G$.* Now obviously if the elements of any collection $G$ of mutually exclusive cuttings of $M$ are connected or if they reduce to single points, then the collection $G$ is non-separated. And since for the case where $M$ is compact, the potential order of a point of $M$ is the same as its order in the Menger-Urysohn sense, our theorem yields as corollaries many important known results concerning the cut points and connected cuttings of connected sets and of continua; for example: (1) the theorem of Wazewski-Menger‡ that the ramification points of any acyclic continuous curve are countable, (2) the theorem of Kuratowski and Zarankiewicz§ that the set of all points of any connected set $M$ whose complement in $M$ is neither connected nor the sum of two connected point

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‡ See Wazewski, Annales de la Société Polonaise de Mathématique, vol. 2 (1923), p. 49; and Menger, Fundamenta Mathematicae, vol. 10 (1927), p. 108.

§ Bulletin of the American Mathematical Society, vol. 33 (1927), p. 571.

sets is countable; (3) the theorem of the author* that all save a countable number of the cut points of any continuum are points of order 2 of $M$ in the Menger-Urysohn sense; and (4) other results concerning cuttings due to Zarankiewicz† and to the author.‡

Second, with the aid of this theorem we shall show that if the space $M$ contains an uncountable non-separated collection $G$ of cuttings, then there exists an upper semi-continuous collection $S$ of elements such that all save a countable number of the sets of $G$ are elements of $S$ and such that every two elements of $S$ may be separated in $M$ by some third element. In case $M$ is compact, the decomposition space $S$ is an acyclic continuous curve.

Finally, we shall prove an existence theorem to the effect that *every locally connected space $M$ contains an uncountable non-separated collection of cuttings.* Therefore, the above mentioned decomposition is always realisable for locally connected sets $M$, and notably for the case where $M$ is a continuous curve, this decomposition gives rise to a decomposition space which is a non-degenerate acyclic continuous curve.

2. **Preliminary lemmas.** Let $X$ and $Y$ be any two cuttings of $M$ and set

$$M - X = M_1(X) + M_2(X), \tag{i}$$

$$M - Y = M_1(Y) + M_2(Y), \tag{ii}$$

representing decompositions of $M-X$ and $M-Y$ respectively into mutually separated sets. Then if $i, j, r$, and $s$ are positive integers such that $i+j=3=r+s$, it follows immediately that the following equation is valid:

$$M = M_i(X) + M_r(Y) + M_j(X)\cdot M_s(Y) + X + Y. \tag{2.1}$$

With the aid of this equation, we deduce at once the result

(2.2) *If neither of the sets $X$ and $Y$ separates the other, we may choose the indices $i$ and $r$ such that*

$$X \subset M_r(Y) \text{ and } Y \subset M_i(X); \tag{a}$$

*and these relations imply also the relations*

$$M_j(X)\cdot M_s(Y) = 0,\ M_j(X) + X \subset M_r(Y), \text{ and } M_s(Y) + Y \subset M_i(X). \tag{b}$$

Clearly this is the case, because by virtue of the relations (a) we may omit the last two terms in equation (2.1); and since $M$ is connected, the term $M_j(X)\cdot M_s(Y)$ must vanish. This fact gives at once the remaining two relations (b).

* These Transactions, vol. 30 (1928), p. 606.

† See Fundamenta Mathematicae, vol. 12 (1928), pp. 119–125.

‡ See Bulletin of the American Mathematical Society, vol. 35 (1929), pp. 87–104.

Now let $G$ be any non-separated collection of cuttings of $M$ and let $E(a, b)$ be the collection of all those elements of $G$ which separate two given points $a$ and $b$ in $M$. Let $X$ and $Y$ be any two elements of $E(a, b)$ and let the indices in (i) and (ii) be chosen so that

(iii) $$M_1(X)\cdot M_1(Y) \supset a \text{ and } M_2(X)\cdot M_2(Y) \supset b.$$

The element $X$ of $E(a, b)$ will be said to precede the element $Y$, and this fact is indicated by the notation $X<Y$, provided that for at least one set of decompositions satisfying (i), (ii) and (iii) it is true that $X \subset M_1(Y)$. We shall now show that this definition gives a natural order to the elements of $E(a, b)$.

First, for any two elements $X$ and $Y$ of $E(a, b)$, at least one of the relations $X<Y$ and $Y<X$ must be valid. For if $X$ does not precede $Y$, then by (2.2), (a), $r=2$ and hence $s=1$. By (b) and (iii) it follows that $j=2$ and hence $i=1$. Therefore by (a), $Y \subset M_1(X)$, which means $Y<X$.

Second, only one of the relations $X<Y$ and $Y<X$ can be valid. For if $X<Y$, then [for any set of decompositions whatever satisfying (i), (ii), (iii)], in (2.2), $r=1$ and hence $s=2$. By (b) and (iii) it follows that $j=1$ and hence $i=2$. Therefore by (a), $Y\subset M_2(X)$, which is incompatible with $Y<X$.

Finally, for any three elements $Z$, $X$ and $Y$ of $E(a, b)$, the relations $Z<X$, $X<Y$ imply that $Z<Y$. For then $Z\subset M_1(X)$ and $X\subset M_1(Y)$. Hence in (2.2), $r=1$ and $s=2$. By (b) and (iii) it follows that $j=1$. Therefore by the second relation in (b), $Z\subset M_1(X)+X\subset M_1(Y)$, which gives $Z<Y$.

Thus we have proved the following result:

(2.3) *If each element of the non-separated collection $E(a, b)$ of subsets of $M$ separates the two points $a$ and $b$ in $M$, then the collection $E(a, b)$ possesses a natural order.*

For convenience we give here a lemma concerning ordered sets due to Zarankiewicz* which will be used below.

LEMMA (Zarankiewicz). *If $K$ is any ordered subset of $M$, then the set $H$ of all points $p$ of $K$ such that $p$ is not at the same time a limit point of the set $P_p$ of all points of $K$ preceding $p$ and also of the set $F_p$ of all points of $K$ following $p$ is countable.*

The space $M$ being separable and metric, it therefore contains a countable sequence $R_1, R_2, R_3, \cdots$ of open sets which is equivalent to the set of all open subsets of $M$. Now let $H_1$ be the set of all points of $K$ which are not limit points of their predecessors, and let $H_2=H-H_1$. For each point $p$ of $H_1$ let $n(p)$ be the least positive integer such that $R_{n(p)}$ contains $p$ but contains no point of $K$ which precedes $p$. Then if $p$ and $q$ are distinct points of $H_1$ and

* See Fundamenta Mathematicae, vol. 12 (1928), p. 119.

$p<q$, then since $R_{n(q)}$ does not contain $p$, it follows that $n(p)\neq n(q)$, and therefore $H_1$ is countable. A similar argument proves $H_2$ countable; and hence $H$ is countable.

3. THEOREM. *If $G$ is any uncountable non-separated collection of cuttings of a connected, metric, and separable space $M$, then all save possibly a countable number of the elements of $G$ have the potential order 2 in $M$ relative to $G$.*

Suppose, on the contrary, that $G$ contains an uncountable subcollection $G_1$ no element of which has the potential order 2 in $M$ relative to $G$. Now there exist two points $a$ and $b$ of $M$ such that the collection $E(a, b)$ of all those elements of $G_1$ which separate $a$ and $b$ in $M$ is uncountable; for $M$ being separable, there exists a countable subset $D$ of $M$ such that $\overline{D}=M$; and since every element of $G_1$ which contains no point of $D$ must separate some pair of points of $D$ in $M$, and since the set of all pairs of points of $D$ is countable, it follows that for at least one pair of points $a$, $b$ of $D$, the set $E(a, b)$ is uncountable.

By §2 the elements of the collection $E(a, b)$ possess a natural order; and if $K$ is a point set which contains exactly one point $x$ of each element $X$ of $E(a, b)$ and contains no other points, then $K$ is an ordered point set. Indeed for each pair $x$, $y$ of points of $K$, set $x<y$ provided that $X<Y$. By the Zarankiewicz lemma, the set $H$ of all points $p$ of $K$ which are not at the same time a limit point both of their predecessors and of their successors is countable. Let $H(a, b)$ be the collection of all those sets $X$ of $E(a, b)$ such that the corresponding point $x$ in $K$ belongs to $K-H$. Then $H(a, b)$ is uncountable and each element $X$ of $H(a, b)$ contains a point $x$ which is a limit point of the sum of the predecessors of $X$ and also of the sum of the successors of $X$.

Now for each element $X$ of $H(a, b)$, there exist mutually separated sets $M_1(X)$ and $M_2(X)$ such that

$$M - X = M_1(X) + M_2(X),\ M_1(X) \supset a \text{ and } M_2(X) \supset b.$$

And with the aid of what has just been shown it follows immediately that there exist two infinite sequences of elements $X_1, X_2, X_3, \cdots$ and $Y_1, Y_2, Y_3, \cdots$ of $H(a, b)$ such that, for each $n$,

$$X_n < X_{n+1} < X < Y_{n+1} < Y_n, \tag{1}$$

and such that $X$ contains a point which is a limit point both of $\sum X_n$ and $\sum Y_n$.

Since by supposition no element of $H(a, b)$ can have the potential order 2 in $M$ relative to $G$, it follows that if for each element $X$ of $H(a,b)$, $V_n(X)$ denotes the set of points $M-[M_1(X_n)+M_2(Y_n)]$, then there exists at least one point $p_x$ belonging to the point set

$$\prod_1^\infty V_n(X) - X,$$

for if this were not the case, then by virtue of (1) and equation (2.1) in which substitute $X_n$ for $X$, $Y_n$ for $Y$, 1 for $i$ and 2 for $r$, it follows that $V_n(X) \supset M_2(X_n)\cdot M_1(Y_n) \supset X$; and if for each point $p$ of $M_2(X_n)\cdot M_1(Y_n)$ we take a neighborhood $N_p$ of $p$ of diameter less than 1/4 the distance from $p$ to the set of points $\overline{M_1(X_n)}+\overline{M_2Y(_n)}$, and call $U_n(X)$ the sum of all the neighborhoods $N_p$, then it follows readily that

$$X \subset M_2(X_n)\cdot M_1(Y_n) \subset U_n(X) \subset \overline{U_n(X)} \subset V_n(X);$$

and hence $F[U_n(X)] \subset X_n + Y_n$, $U_n(X) \subset U_{n-1}(X)$ and $X = \prod_1^\infty \overline{U_n(X)}$; but then $X$ has the potential order 2 in $M$ relative to $G$, contrary to supposition.

Now if $X$ and $Y$ are any two elements of $H(a, b)$, $X \neq Y$, it follows that $p_x \neq p_y$. For suppose $X < Y$. Then since $X$ contains a limit point of the sum of its successors in $E(a, b)$ but contains no limit point of $M_2(Y)$, it follows that there exist two elements $Y_k$ and $Y_m$ in the "$Y$-sequence" in (1) for the element $X$ such that

$$X < Y_k < Y_m < Y;$$

and since $Y$ contains a limit point of the sum of its predecessors in $E(a, b)$ but contains no limit point of $M_1(Y_m)$, it follows that there exists an element $X_n$ of the "$X$-sequence" for $Y$ in (1) such that

$$X < Y_k < Y_m < X_n < Y.$$

Consequently it follows with the aid of (2.2) that

$$p_x \subset M_1(Y_k) + Y_k \subset M_1(Y_m)$$

and

$$p_y \subset M_2(X_n) + X_n \subset M_2(Y_m),$$

and hence $p_x \neq p_y$.

Now let $L$ denote the set of all points $[p_x]$ for all elements $X$ of $H(a, b)$. Then $L$ is uncountable and is an ordered set; indeed, it is only necessary to set $p_x < p_y$ when $X < Y$. Therefore by the Zarankiewicz lemma, there exists a point $p_x$ of $L$ which is a limit point both of its predecessors and of its followers, and hence both of $\sum X_n$ and of $\sum Y_n$, where the sequences $[X_n]$ and $[Y_n]$ satisfy (1). But $\sum X_n \subset M_1(X)$ and $\sum Y_n \subset M_2(X)$; and $p_x$ must then belong either to $M_1(X)$ or to $M_2(X)$ and be a limit point of the other, contrary to the fact that these two sets are mutually separated. Thus the supposition that our theorem is false leads to a contradiction.

4. **Consequences of §3.** Let $G$ be any uncountable non-separated collection of cuttings of $M$. Then since the product of any family $[\overline{U}_n]$ of closed sets is closed, §3 yields at once the result

($\alpha$) *All save a countable number of the elements of G are closed point sets.*

Now if $X$ is any element of $G$ such that $M-X$ is not the sum of two connected point sets, $X$ cannot have a potential order 2 in $M$ relative to $G$. For $M-X=M_1(X)+M_2(X)+M_3(X)$, where the sets $M_1(X)$, $M_2(X)$, and $M_3(X)$ are mutually separated and contain points $a_1$, $a_2$ and $a_3$ respectively; and if $X$ had the potential order 2 relative to $G$, there would exist two elements $X_1$ and $X_2$ of $G$ and a neighborhood $U$ of $X$ such that $F(U)\subset X_1+X_2$, $X_1\subset M_1(X)$, $X_2\subset M_2(X)$ and $\overline{U}\cdot(a_1+a_2+a_3)=0$; but then it would readily follow that the point set $M_3(X)\cdot(M-\overline{U})$ is non-vacuous and is both open and closed, contrary to the fact that $M$ is connected. Thus in consequence of the theorem in §3 we have

($\beta$) *The complement of each element of G, with the exception of a countable number of such elements, consists of exactly two components.*

Let us denote by $\rho$ the property of any subset $N$ of $M$ not to be separated in $M$ by any single element of $G$. Clearly each element $X$ of $G$ has the property $\rho$. We shall now show that

($\gamma$) *All save a countable number of the elements of G are saturated in M relative to the property $\rho$.*

If, on the contrary, $G$ contains an uncountable subcollection $G_1$ no element of which is saturated relative to the property $\rho$, then for each element $Z$ of $G_1$ there exists at least one point $p_z$ which is not separated from $Z$ in $M$ by any single element of $G$. Under these conditions it follows by the theorem and proof in §3 that there exist two points $a$ and $b$ of $M$ and three elements $Z$, $X$ and $Y$ of $E(a, b)$ ( the collection of all those elements of $G_1$ which separate $a$ and $b$) such that $X<Z<Y$, and $M_2(X)\cdot M_1(Y)$ contains $Z$ but does not contain the point $p_z$ and also such that $X+Y$ does not contain $p_z$. But then by equation (2.1) we have either $p_z\subset M_1(X)$ or $p_z\subset M_2(Y)$. This is impossible because in the first case $X$ separates $p_z$ and $Z$ in $M$ and in the second case $Y$ separates $p_z$ and $Z$ in $M$.

A cutting $X$ of $M$ is said to be an irreducible cutting of $M$ provided that no proper subset of $X$ is a cutting of $M$.

($\delta$) *All save a countable number of the elements of G are irreducible cuttings of M.*

If this is not so, there exists an uncountable collection $G^0$ of cuttings of $M$ such that for each element $X^0$ of $G^0$ there exists an element $X$ of $G$ and a **point $p_x$ of** $X$ such that $X^0\subset X-p_x$. Since $G$ is non-separated, it follows at once that $G^0$ is non-separated. Therefore by ($\gamma$) there exists an element $X^0$

of $G^0$ which is saturated relative to the property $\rho$ defined by the collection $G^0$. Consequently there exists an element $Y^0$ of $G^0$ which separates $X^0$ and $p_x$ in $M$, and one has $M-Y^0=M_1(Y^0)+M_2(Y^0)$, where $M_1(Y^0)\supset X^0$ and $M_2(Y^0)\supset p_x$. But then $M-Y=M_1(Y^0)\cdot(M-Y)+M_2(Y^0)\cdot(M-Y)$, and thus $Y$ separates $X$ in $M$ (for $Y\cdot(X^0+p_x)\subset Y\cdot X=0$), which contradicts the non-separatedness of $G$.

We prove now the following general theorem:

THEOREM. *Every uncountable non-separated collection $G$ of cuttings of a connected, metric, and separable space $M$ contains a subcollection $Q$ which contains all save possibly a countable number of the elements of $G$ and such that each element $X$ of $Q$ has the following properties*: (a) *$X$ is closed*; (b) *$M-X$ is the sum of two mutually separated connected point sets*; (c) *$X$ is saturated in $M$ relative to the property $\rho$ defined by the collection $Q$, i.e., for every point $p$ of $M-X$, there exists an element $Y$ of $Q$ which separates $X$ and $p$ in $M$*; (d) *$X$ is an irreducible cutting of $M$; and* (e) *$X$ has the potential order 2 in $M$ relative to $Q$.*

To obtain the collection $Q$, let $D$ be a countable subset of $M$ which is dense in $M$ and let us omit from $G$: (1) every element which does not possess each of the properties (a), (b), and (d); (2) every element which separates in $M$ some pair of points $a$, $b$ of $D$ which are separated by only a countable number of elements of $G$; (3) every element which separates some pair $a$, $b$ of points of $D$ and contains no point $p$ having the property that every neighborhood of $p$ contains points of uncountably many distinct elements of $G$ which separate $a$ and $b$. Let $G_1$ denote the collection of the elements of $G$ remaining after these omissions. Then by virtue of ($\alpha$), ($\beta$) and ($\delta$), together with the facts that there are only a countable number of pairs of points of $D$ and that in the space $M$ every uncountable set of points contains a point of condensation of itself, it follows that $G_1$ contains all save possibly a countable number of the elements of $G$.

Now let us omit from $G_1$ every element which is not saturated in $M$ relative to the property $\rho$ defined by the collection $G_1$ and also every element which does not have the potential order 2 in $M$ relative to $G_1$. Let $Q$ be the collection of elements of $G_1$ remaining after these omissions. Then $Q$ contains all save a countable number of the elements of $G_1$ and hence also of $G$, and every element $X$ of $Q$ has the desired properties (a)-(e). Clearly $X$ has properties (a), (b) and (d), for every element of $G_1$ has these properties. It remains to show that $X$ has properties (c) and (e).

To show that $X$ has property (c), let $p$ be any point of $M-X$. There exists an element $Y$ of $G_1$ which separates $X$ and $p$, because every element of $Q$ is saturated in $M$ relative to the property $\rho$ defined by $G_1$. Hence $M-Y$

$=M_1(Y)+M_2(Y)$, where $M_1(Y)\supset X$ and $M_2(Y)\supset p$. Also $M-X=M_1(X)+M_2(X)$, where $M_2(X)\supset Y$. Thus if $a$ and $b$ are points of $M_1(X)$ and $M_2(Y)$ respectively belonging to $D$, both $X$ and $Y$ separate $a$ and $b$ in $M$, and we have $X<Y$ in the order from $a$ to $b$. Now there exists also an element $Z$ of $G_1$ which separates $X$ and $Y$ in $M$, and it follows from §2 that $Z$ also separates $a$ and $b$ in $M$, and we have the order $X<Z<Y$. Thus $Z\subset M_2(X)\cdot M_1(Y)$. Since $X$ and $Y$ are closed, $M_2(X)\cdot M_1(Y)$ is a neighborhood of $Z$, and hence it contains points of (and therefore contains all of) uncountably many elements of $G$ which separate $a$ and $b$ in $M$. Therefore there exists at least one of these elements, say $W$, which belongs to $Q$, for all but a countable number of the elements of $G$ belong to $Q$. Thus we have the order $X<W<Y$; and since $p\subset M_2(Y)$, it follows that $W$ separates $X$ and $p$ in $M$. Consequently $X$ has property (c).

Since $X$ has the potential order 2 in $M$ relative to $G_1$, there exist, as shown in §3, two points $a$ and $b$ of $M$ such that $X$ belongs to the collection $E(a, b)$ of all those elements of $G_1$ which separate $a$ and $b$ in $M$ and such that there exist two sequences $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ of elements of $E(a, b)$ so that

$$X_n < X_{n+1} < X < Y_{n+1} < Y_n$$

and such that if $U_n=M_2(X_n)\cdot M_1(Y_n)$, then $X=\prod_1^\infty \overline{U}_n$. Now for each $n$ there exist, by virtue of property (c), two elements $X_n'$ and $Y_n'$ of $Q$ belonging to $E(a, b)$ such that $X_n<X_n'<X<Y_n'<Y_n$. Hence if $U_n'$ denotes the point set $M_2(X_n')\cdot M_1(Y_n')$, one has $U_n'\subset U_n$. Hence $X=\prod_1^\infty \overline{U}_n'$, and since $F(U_n')\subset X_n'+Y_n'$ and since clearly the sequence $[U_n]$ contains an infinite subsequence $[U_{n_i}]$ such that $U_{n_{i+1}}\subset U_{n_i}$, it follows that $X$ has the potential order 2 in $M$ relative to $Q$. This completes the proof.

5. **Decomposition of $M$ by means of a non-separated collection $G$ every element of which is saturated relative to property $\rho$.** Let $G$ be any non-separated collection of subsets of $M$ each of which is saturated in $M$ relative to the property $\rho$ defined by $G$. For each point $e$ of $M$ which belongs to no element of $G$, let $E$ denote the point set consisting of $e$ together with all points $p$ of $M$ which are not separated in $M$ from $e$ by any single element of $G$. Let $S$ denote the collection whose elements are the elements of $G$ together with all such point sets $E$ thus defined. Clearly each element of $S$ is closed and every point of $M$ belongs to exactly one element of $S$. We shall show next that the collection $S$ is non-separated.

Suppose, on the contrary, that some element $X$ of $S$ separates some pair of points $p$ and $q$ belonging to an element $Y$ of $S$. Then $M-X=M_1(X)+M_2(X)$, where $M_1(X)\supset p$ and $M_2(X)\supset q$. Now by virtue of the definition of the collections $G$ and $S$, it follows that there exists an element $Z$ of $G$ which separates $X$ and $p$ in $M$. Hence $M-Z=M_1(Z)+M_2(Z)$, where $M_1(Z)\supset X$ and $M_2(Z)\supset p$. Since $Z$ belongs to $G$, it cannot separate $Y$ in $M$; and therefore $p+q\subset Y\subset M_2(Z)$. But then

$$M - Z = [M_1(Z) + M_1(X)\cdot M_2(Z)] + M_2(X)\cdot M_2(Z),$$

and we have a separation of $M-Z$ into two mutually separated sets containing the points $p$ and $q$ respectively of $Y$, contrary to the fact that since $Z$ belongs to $G$ it cannot separate $Y$ in $M$. Therefore $S$ is non-separated.

Now clearly every element of $S$ is saturated in $M$ relative to the property $\rho$ defined by the collection $S$. Consequently every two elements $X$ and $Y$ of $S$ are separated in $M$ by some third element of $S$. With the aid of this property it follows immediately that the collection $S$ is upper semi-continuous,* i.e., there does not exist a sequence $X_1, X_2, X_3, \cdots$ of elements of $S$ and two sequences $[p_i]$ and $[q_i]$ of points such that $p_i+q_i\subset X_i$ and which have sequential limit points $p$ and $q$ respectively belonging to two different elements $P$ and $Q$ respectively of $S$. For there exists an element $X$ of $S$ such that $M-X=M_1(X)+M_2(X)$ where $M_1(X)\supset P$ and $M_2(X)\supset Q$; and since for each $i$, $X_i$ is a subset either of $M_1(X)$ or of $M_2(X)$, either $M_1(X)$ or $M_2(X)$ contains $X_i$ for infinitely many $i$'s; but this is impossible, for both $p$ and $q$ are limit points of every infinite subsequence of $X_1, X_2, X_3, \cdots$.

Now in case the space $M$ is compact, the elements of $S$ are closed and compact, and if for each pair of elements $X$ and $Y$ of $S$ we define the distance $\rho(X, Y)$ between $X$ and $Y$ as the upper limit of the distances $\rho(x, y)$, where $x$ and $y$ are points of $X$ and $Y$ respectively, it readily follows that the space $S'$ so obtained is compact, separable, metric and connected; and since it readily follows that every two "points" of $S'$ are separated in $S'$ by some third "point" of $S'$, therefore† $S'$ is an acyclic continuous curve.

6. EXISTENCE THEOREM. *If the space $M$ is connected im kleinen, there exists an uncountable non-separated collection of cuttings of $M$.*

Let $a$ and $b$ be any two points of $M$, and for each positive number $r$ which is less than the distance from $a$ to $b$, let $S(a, r)$ denote the set of all points of $M$ whose distance from $a$ is equal to $r$ and let $I(a, r)$ denote the set of all points at a distance $<r$ from $a$. Then for each $r$, $S(a, r)$ separates $a$ and $b$ in $M$. Let $R(a, r)$ denote the component of $M-S(a, r)$ containing $a$, let $R(b, r)$ denote the component of $M-\overline{R(a, r)}$ containing $b$, and let $X_r$ denote the point set $\overline{R(a, r)}\cdot\overline{R(b, r)}$. Then clearly $X_r$ separates $a$ and $b$ in $M$ and we have

(i) $$X_r \subset F[R(a, r)] \subset S(a, r), \text{ and } X_r = F[R(b, r)];$$

(ii) $$R(a, r) \subset I(a, r).$$

* See R. L. Moore, these Transactions, vol. 27 (1925), pp. 416–428.

† See R. L. Moore, Fundamenta Mathematicae, vol. 7 (1925), pp. 302–307.

Obviously the collection of cuttings $[X_r]$ is uncountable. It remains to show that it is non-separated. Let $X_{r_1}$ and $X_{r_2}$ be any two elements of this collection and suppose $r_1 < r_2$. By (ii) it follows that $\overline{R(a, r_1)} \subset R(a, r_2)$. Thus $X_{r_1} \subset R(a, r_2)$, and therefore $X_{r_1}$ does not separate $X_r$ in $M$. From the inclusion $\overline{R(a, r_1)} \subset R(a, r_2)$ and (i) it follows that $\overline{R(b, r_2)} = R(b, r_2) + X_{r_2} \subset R(b, r_1)$, and consequently $X_r$ does not separate $X_{r_2}$ in $M$. Thus the collection $[X_r]$ is non-separated, and the theorem is proved.

Now since $a$ may be any point whatever of $M$ and since every neighborhood of $a$ contains uncountably many of the sets $[X_r]$, it follows by §4, ($\delta$), that every such neighborhood contains at least one set $X_r$ which is an irreducible cutting of $M$. Thus we have the following

COROLLARY. *Every open subset of a connected and connected im kleinen point set $M$ lying in a separable metric space contains an irreducible cutting $I$ of $M$.*

This corollary answers a question raised by the author.*

As a result of this existence theorem it follows that the decomposition treated in §5 is always realisable in case $M$ is locally connected; and in case $M$ is a continuous curve, $M$ may be decomposed upper semi-continuously into a collection $S$ of the type attained in §5, and the decomposition space $S'$ is a non-degenerate acyclic continuous curve.

7. **Concluding remarks.** Although it is easily seen with the aid of a very simple example that two cuttings $X$ and $Y$ of $M$ may have the property that neither of them separates the other in $M$ and yet the set $M_2(X)+X$ not be connected, where $M_2(X) \supset Y$, nevertheless the following lemma is true.

LEMMA. *If $a$ and $b$ are two points of $M$ and $X_1, X_2, X_3, \cdots$ is any infinite sequence of distinct mutually exclusive sets each of which separates $a$ and $b$ in $M$ and no one of which separates any other one, and we have*

$$X_1 < X_2 < X_3 < \cdots,$$

*then the set of points $\sum_1^\infty M_1(X_i)$ is connected.*

For if on the contrary this set of points is the sum of two mutually separated sets $N_1$ and $N_2$, then since $a$ belongs to all of the sets $M_1(X_i)$, there exists an integer $n$ such that $N_1 \cdot M_1(X_n) \neq 0 \neq N_2 \cdot M_1(X_n)$. Since by (2.2), (b), it follows that $M_1(X_n) \subset M_1(X_{n+1})$, therefore $N_1 \cdot M_1(X_{n+1}) \neq 0 \neq N_2 \cdot M_1(X_{n+1})$. Since these two sets are mutually separated, one of them, say $N_1 \cdot M_1(X_{n+1})$,

* See Fundamenta Mathematicae, vol. 13 (1929), p. 50, where the question is raised for *continuous curves $M$*. A solution of this problem for the case where $M$ is a plane continuous curve has been given by J. H. Roberts; see these Transactions, vol. 32 (1930), p. 19.

contains $X_n$. But then it is easily seen that the sets $N_2 \cdot M_1(X_n)$ and $M - N_2 \cdot M_1(X_n)$ are mutually separated, contrary to the fact that $M$ is connected.

With the aid of this lemma it can be shown without great difficulty that if $X$ is any element of a non-separated collection $G$ of subsets of $M$ which is saturated in $M$ relative to the property $\rho$ defined by the collection $G$, then

(1) each component of $M - X$ is open in $M$;

(2) the components of $M - X$ are countable;

(3) $X$ is a potentially regular element of $G$ in $M$ relative to $G$, i.e., a monotone decreasing sequence of neighborhoods $[U_i]$ of $X$ exists such that $F(U_i)$ is a subset of a finite number of the elements of $G$ and $X = \prod_1^\infty \overline{U}_i$;

(4) the potential order of $X$ in $M$ relative to $G$ is equal to the number of components of $M - X$ when this number is finite, and is equal to $\omega$ (i.e., $X$ is of increasing order) when and only when this number is infinite.

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