THE TRANSFORMATION C OF NETS IN HYPERSPACE*

BY

V. G. GROVE

1. Introduction

It is the purpose of this paper to extend some of the ideas related to the transformation† C of nets in projective space of three dimensions to nets in hyperspace. In three dimensions two nets N_x and N_y are said to be in relation C if the developables of the congruence G of lines joining corresponding points x and y intersect the two sustaining surfaces in those nets, provided however that neither surface is a focal surface of the congruence. If N_x and N_y are in relation C, the tangents at x and y to corresponding curves of the nets intersect.

We shall say that two nets N_x and N_y in space S_n of n dimensions are *in relation* C if the two sustaining surfaces S_x and S_y are such that corresponding tangent planes intersect in a line, and if the developable surfaces of the congruence G of lines g joining corresponding points x and y intersect the two surfaces in the nets. It is to be understood that the two sustaining surfaces are not the focal surfaces of G. Not all nets N_x in S_n for $n \geqq 4$ can have a C transform N_y. A net in S_n which permits of having a C transform will be called *a C net*.

We derive necessary and sufficient conditions that a non-conjugate net be a C net. Another geometrical interpretation is given for two covariant points found by Bompiani.‡

Let S_x and S_y be two surfaces in the same space S_n of n dimensions and with their points in one-to-one point correspondence. Let the parametric equations of these surfaces be

$$x^{(i)} = x^{(i)}(u, v), \quad y^{(i)} = y^{(i)}(u, v) \qquad (i = 1, 2, 3, \cdots, n+1),$$

the parameters being so chosen that corresponding points have the same curvilinear coördinates. Suppose furthermore that the tangent planes to S_x and S_y at corresponding points intersect in a line. There exist therefore scalar functions m, s, f, A, etc., such that

* Presented to the Society, June 13, 1931; received by the editors March 1, 1931.

† V. G. Grove, *Transformations of nets*, these Transactions, vol. 30 (1928), p. 483.

‡ E. Bompiani, *Determinazione delle superficie integrali d'un sistema di equazioni a derivate parziali lineari ed omogenee*, Rendiconti del Reale Istituto Lombardo di Scienze e Lettere, vol. 52 (1919), pp. 610–636. In particular see p. 634. Hereafter referred to as Bompiani, *Surfaces*.

$$(1) \qquad \begin{aligned} y_u &= mx_u + sx_v + fx + Ay, \\ y_v &= tx_u + nx_v + gx + By. \end{aligned}$$

Any point z on the line g joining corresponding points x and y is defined by an expression of the form

$$z = y + \lambda x.$$

Consider the surface S_z generated by the point z, and a curve C on S_z with parametric equations

$$u = u(t), \quad v = v(t).$$

The tangent line to C at z is determined by z and $z' = dz/dt$ where

$$(2) \qquad z' = [(m + \lambda)u' + tv']x_u + [su' + (n + \lambda)v']x_v + (\ )x + (\ )y.$$

Hence the line g generates a congruence in the ordinary sense, that is, the lines of the two-parameter families of lines may be grouped into two one-parameter families of developable surfaces. The curves on S_x corresponding to these developables are defined by

$$(3) \qquad sdu^2 - (m - n)dudv - tdv^2 = 0.$$

We shall assume that these curves are not indeterminate and are distinct. By a change of parameters we may make the curves determined by (3) parametric. Let us suppose that this transformation has been made. *If two surfaces are such that their parametric nets are in relation C, the functions x and y determining the surfaces satisfy equations of the form*

$$(4) \qquad \begin{aligned} y_u &= mx_u + fx + Ay, \\ y_v &= nx_v + gx + By, \quad mn(m - n) \neq 0. \end{aligned}$$

From (2) we find that the focal points of g are defined by

$$(5) \qquad \tau = mx - y, \quad \tau' = nx - y.$$

The nets defined by τ and τ' will be called *the focal nets* of G. The tangent to the curve $v = \text{const.}$ ($u = \text{const.}$) on the focal surface $S_{\tau'}$ (S_τ) will be called *the minus first* (*first*) *derived line of* g, and the congruences generated by them *the minus first* (*first*) *derived congruences of* G.

If we differentiate equations (4) with respect to v and u respectively, we find that x and y satisfy an equation of the form

$$(6) \qquad x_{uv} = ax_u + bx_v + cx + My$$

wherein a, b, c, M are defined by the formulas

$$(7)\qquad \begin{aligned} (m-n)a &= g + Bm - m_v, \quad (m-n)c = Bf - Ag + g_u - f_v, \\ (n-m)b &= f + An - n_u, \quad (m-n)M = B_u - A_v. \end{aligned}$$

Since y is not in the tangent plane to S_x at x, *the vanishing of M is a necessary and sufficient condition that the net N_x be conjugate.*

If the coefficients of the equations corresponding to (4) and (6) with the rôles of x and y interchanged are denoted by $\bar{m}$, $\bar{f}$, etc., we find that

$$\begin{aligned} &\bar{m} = 1/m,\ \bar{n} = 1/n,\ \bar{f} = -A/m,\ \bar{g} = -B/n,\ \bar{A} = -f/m,\ \bar{B} = -g/n, \\ &\bar{a} = a + m_v/m,\ \bar{b} = b + n_u/n, \\ &\bar{c} = -m[aA/m + bB/n + fB/(mn) - M - (A/m)_v], \end{aligned}$$

$$(8)\qquad \bar{M} = -m[af/m + bg/n + fg/(mn) - c - (f/m)_v].$$

2. The relation R in S_n

Denote by $R^{(v)}$ the ruled surface formed by the tangents to the curves $v=$const. at the points where they meet a fixed curve $u=$const. A ruled surface $R^{(u)}$ may be defined similarly.

Let l' be any line lying in the tangent plane to S_x at x, but not passing through x. Let l be a line passing through x but not lying in the tangent plane at x. The line l' intersects the tangent to the curves $v=$const. and $u=$const. in points r and s respectively. If the tangent planes to $R^{(v)}$ and $R^{(u)}$ at r and s respectively intersect in the line l the given lines l and l' will be said to be *in relation* R with respect to N_x.*

The points r and s are defined by expressions of the form

$$(9)\qquad r = x_u - \lambda x, \quad s = x_v - \mu x.$$

A point in the tangent plane to $R^{(v)}$ at r is

$$(10)\qquad r_v + \alpha r + \beta x = x_{uv} + \alpha x_u - \lambda x_v + (\beta - \alpha\lambda - \lambda_v)x.$$

A point in the tangent plane to $R^{(u)}$ at s is

$$(11)\qquad S_u + \alpha' s + \beta' x = x_{uv} + \alpha' x_v - \mu x_u + (\beta' - \alpha'\mu - \mu_u)x.$$

From (10) and (11) we observe that the tangent planes to $R^{(v)}$ and $R^{(u)}$ at r and s intersect in a line joining x to z defined by

$$(12)\qquad z = x_{uv} - \mu x_u - \lambda x_v.$$

* In a footnote on p. 86 of his paper *Memoir on the general theory of surfaces and rectilinear congruences*, these Transactions, vol. 20 (1919), Green defined the relation R between two lines with respect to a net in S_3. The definition we have used for the relation R between two lines with respect to a net in S_n reduces to Green's definition when $n=3$. It is to be noted however that not all lines l in S_n for $n>3$ and protruding from the surface at x have a line l' in relation R to N_x.

The line l in relation R to l' therefore joins x to the point z defined by (12).

If $M \neq 0$, we see readily from (6) that the line g' in relation R to g with respect to N_x joins the points

$$\rho = x_u - bx, \quad \sigma = x_v - ax. \tag{13}$$

The line $\bar{g}'$ in relation R to g with respect to N_y is determined by the points $\bar{\rho}$, $\bar{\sigma}$ defined by

$$\bar{\rho} = y_u - \bar{b}y, \quad \bar{\sigma} = y_v - \bar{a}y. \tag{14}$$

An examination of equations (5) readily shows that *the derived lines of g intersect the tangents to the curves of N_x and N_y in the points determining the lines g' and $\bar{g}'$ in relation R to N_x and N_y.* If N_x and N_y are in relation* F, that is, if they are in relation C and are both conjugate nets, the derived lines intersect the tangents to the curves of the nets in the focal points of these tangents.

The tangent to the curve $u = \text{const.}$ at ρ on the surface generated by that point intersects g in the point

$$(ab + c - b_v)x + My. \tag{15}$$

Similarly the tangent to $v = \text{const.}$ at σ intersects g in the point

$$(ab + c - a_u)x + My. \tag{16}$$

The points defined by (15) and (16) coincide if N_x is conjugate, or if

$$a_u - b_v = 0. \tag{17}$$

3. The third-order differential equations of the problem

Let us assume that the nets N_x and N_y are not conjugate. If we differentiate equation (6) with respect to u and v, and use (4) and (6), we find that the functions x must satisfy two differential equations of the form

$$\begin{aligned} x_{uuv} &= ax_{uu} + Ex_{uv} + Gx_u + Hx_v + Jx, \\ x_{uvv} &= E'x_{uv} + bx_{vv} + G'x_u + H'x_v + J'x, \end{aligned} \tag{18}$$

wherein

$$\begin{aligned} E &= b + A + M_u/M, & E' &= a + B + M_v/M, \\ G &= a_u + c + mM - a(E - b), & H' &= b_v + c + nM - b(E' - a), \\ H &= b_u - b(E - b), & G' &= a_v - a(E' - a), \\ J &= c_u + fM - c(E - b), & J' &= c_v + gM - c(E' - a). \end{aligned} \tag{19}$$

* L. P. Eisenhart, *Transformations of Surfaces*, Princeton University Press, 1923, p. 34.

Similar third-order differential equations are satisfied by the functions y. Hence *if two surfaces S_x and S_y are in one-to-one point correspondence with corresponding tangent planes intersecting in a line, and if the nets on S_x and S_y which are in relation C are parametric, the functions x (and y) satisfy two third-order differential equations of the type* (18). Differential equations of this type have been studied by Bompiani* and Lane.†

Conversely suppose the coördinates x defining a surface S_x satisfy a pair of differential equations of the form (18). In case the two-osculating space $S(2, 0)$ of S_x at x determined by the points $x_{uu}, x_{uv}, x_{vv}, x_u, x_v, x$ is an S_5, the integrability conditions‡ of system (18) are

$$\begin{aligned} aE' + G' - a^2 - a_v &= 0, \quad bE + H - b^2 - b_u = 0, \\ bE' + E_u' + H' &= aE + E_v + G, \\ E'G + bG' + G_u' + J' &= EG' + aG + G_v, \\ EH' + aH + H_v + J &= E'H + bH' + H_u', \\ E'J + bJ' + J_u' &= EJ' + aJ + J_v. \end{aligned} \tag{20}$$

We shall show that *every net N_x whose defining functions x satisfy differential equations of the form* (18) *is a C net*. Since x and y must satisfy equations of the form (4) it follows that the point y is defined by an expression of the form

$$y = x_{uv} - \lambda x_u - \mu x_v + hx. \tag{21}$$

Case I. Suppose that $S(2, 0)$ is an S_5. Differentiating (21) with respect to u and v we find readily that

$$\begin{aligned} y_u = (a - \lambda)x_{uu} &+ [G + h - \lambda_u + \lambda(E - \mu)]x_u + [H + \mu(E - \mu) - \mu_u]x_v \\ &+ [J - h(E - \mu) + h_u]x + (E - \mu)y, \\ y_v = (b - \mu)x_{vv} &+ [G' + \lambda(E' - \lambda) - \lambda_v]x_u + [H' + h - \mu_v + \mu(E' - \lambda)]x_v \\ &+ [J' - h(E' - \lambda) + h_v]x + (E' - \lambda)y. \end{aligned} \tag{22}$$

Hence N_y will be in relation C to N_x if and only if $\lambda = a$, $\mu = b$, h arbitrary. The functions x and y satisfy equations of the form (4) with

$$\begin{aligned} m &= aA + G - a_u + h, & n &= bB + H' - b_v + h, \\ f &= J - Ah + h_u, & g &= J' - Bh + h_v, \\ A &= E - b, & B &= E' - a. \end{aligned} \tag{23}$$

* Bompiani, *Surfaces*, p. 632.

† E. P. Lane, *Integral surfaces of pairs of partial differential equations of the third order*, these Transactions, vol. 32 (1930), pp. 782–793.

‡ Bompiani, *Surfaces*, p. 632.

There exists, therefore, *in this case one and only one line passing through x and not lying in the tangent plane to S_x at x, such that every point y on the line, not a focal point of the line, generates a net N_y in relation C to N_x. The space S_n is such that $n \geqq 5$.*

Case II. Suppose that $S(2, 0)$ is an S_4. It follows that the functions x satisfy an equation of the form

$$P' x_{uu} + Q' x_{uv} + R' x_{vv} + L' x_u + N' x_v + K' x = 0$$

such that not both P' and R' are zero. To fix the notation, suppose that $R' \neq 0$. The functions x therefore satisfy a system of differential equations of the form

$$\begin{aligned} x_{vv} &= P x_{uu} + Q x_{uv} + L x_u + N x_v + K x, \\ x_{uuv} &= a x_{uu} + E x_{uv} + G x_u + H x_v + J x, \\ x_{uvv} &= D' x_{uu} + E' x_{uv} + G' x_u + H' x_v + J' x. \end{aligned} \tag{24}$$

Subcase (a). Suppose that x does not satisfy a differential equation of the form

$$x_{uuu} = \alpha x_{uu} + \beta x_{vv} + \delta x_u + \epsilon x_v + \phi x. \tag{25}$$

Some of the integrability conditions of system (24) are

$$P = D' = 0, \quad aE' + G' - a^2 - a_v = 0, \quad aQ + L = 0.$$

From (21) we find that

$$\begin{aligned} y_u = {} & (a - \lambda) x_{uu} + [G + h - \lambda_u + \lambda(E - \mu)] x_u + [H + \mu(E - \mu) - \mu_u] x_v \\ & + [J - h(E - \mu) + h_u] x + (E - \mu) y, \\ y_v = {} & [G' - \lambda_v - \mu L + \lambda(E' - \lambda - \mu Q)] x_u + [H' - \mu N - \mu_v + h \\ & + \mu(E' - \lambda - \mu Q)] x_v + [J' - \mu K + h_v - h(E' - \lambda - \mu Q)] x \\ & + (E' - \lambda - \mu Q) y. \end{aligned} \tag{26}$$

Hence N_y will be in relation C to N_x if and only if

$$\lambda = a, \quad \mu_u + \mu^2 - E\mu - H = 0, \quad h \text{ arbitrary}.$$

The functions x and y satisfy equations of the form (4) with

$$\begin{aligned} m &= aA + G - a_u + h, & n &= H' - \mu B - \mu N - \mu_v + h, \\ f &= J - Ah + h_u, & g &= J' - \mu K + h_v - Bh, \\ A &= E - \mu, & B &= E' - a - \mu Q. \end{aligned} \tag{27}$$

Hence, in this case, there exist lines g through x such that any point y on any one of the lines generates a net N_y in relation C to N_x. These lines g belong to a pencil *of lines with center at x and in the plane determined by the points* x, x_v, $x_{uv}-ax_u$. *The lines g at a point x are projectively related to the lines g through any other point of the curve* $v=$ *const. through x. The space* S_n *is such that* $n\geqq 5$.

Subcase (b). Suppose that x satisfies a system of differential equations of the form

$$\begin{aligned} x_{vv} &= Px_{uu} + Qx_{uv} + Lx_u + Nx_v + Kx,\\ x_{uuv} &= ax_{uu} + Ex_{uv} + Gx_u + Hx_v + Jx,\\ x_{uvv} &= D'x_{uu} + E'x_{uv} + G'x_u + H'x_v + J'x,\\ x_{uuu} &= \alpha x_{uu} + \beta x_{uv} + \delta x_u + \epsilon x_v + \phi x. \end{aligned} \tag{28}$$

Some of the integrability conditions of system (28) are

$$\begin{aligned} a^2 + a_v + ED' + HP &= \alpha D' + D_u' + aE' + G,\\ D' &= L + aQ + \alpha P + P_u. \end{aligned} \tag{29}$$

We may show from (21) that the net N_y is in relation C to N_x if and only if

$$\begin{aligned} \lambda = a,\ D' - \mu P = 0,\ \mu_u + \mu^2 - E\mu - H = 0,\\ G' - a_v - a^2 + aE' - \mu(L + aQ) = 0. \end{aligned} \tag{30}$$

The value of μ determined by the second of (30), if substituted in the last two of (30), gives two conditions on the coefficients of system (28). These conditions are however a result of the integrability conditions (29). *Hence,* in this case, *there is a unique line joining x to*

$$y = x_{uv} - ax_u - D'x_v/P + hx$$

any point of which generates a net in relation C to N_x. *The space* S_n *is an* S_4.

The functions x and y satisfy equations of the form (4) with

$$\begin{aligned} m &= aA + G_u - a_u + h, & n &= H' - D'N/P - (D'/P)_v + D'B/P + h,\\ f &= J - Ah + h_u, & g &= J' - D'K/P - Bh - h_v,\\ A &= E - D'/P, & B &= E' - a - D'Q/P. \end{aligned}$$

We may state the results of this section as follows: *A non-conjugate net is a C net if and only if the functions x defining the net satisfy a pair of equations of the form* (18).

4. The Bompiani transforms of a C net

Suppose the functions x determining a net N_x satisfy differential equations of the form (18). Bompiani* has shown that there exist two covariant points,

* Bompiani, *Surfaces*, pp. 634–635.

one on each tangent line to the curves of the net, characterized in the following way: The point

$$\rho = x_u - bx$$

is the only point on the tangent to $v=$const. on S_x generating a surface for which the osculating plane to the curve $u=$const. at ρ lies in the S_3 determined by the points x, x_u, x_v, x_{uv} and tangent to the ruled surface $R^{(v)}$ along the generator through x. The point

$$\sigma = x_v - ax$$

has a similar characterization. We shall call the point $\rho(\sigma)$ the *minus first (first) Bompiani transform of* x, and the nets described by $\rho(\sigma)$ the *minus first (first) Bompiani transform of* N_x.

Suppose that the functions x satisfy a system of differential equations of the form (24), but no equation of the form (25). We find that the point ρ, defined by

$$\rho = x_u - \mu x,$$

generates a surface of the type described above for every value of μ. The point σ defined by

$$\sigma = x_v - a\dot{x}$$

is the only point on the tangent to the curve $u=$const. at x generating a surface of the desired type. We shall say in this case that *the minus first Bompiani transform of* x (N_x) *is indeterminate.* The first Bompiani transform of x (N_x) is the point σ (net N_σ). Hence *a necessary and sufficient condition that a net permit of having Bompiani transforms is that the given net be a C net.*

We may state some of the results of this and the preceding section in the following theorem:

Let there be given a net N_x in S_n whose sustaining surface S_x is such that its two-osculating space $S(2, 0)$ is an S_4 or an S_5. There exists a unique line g through x lying in the $S(2, 0)$ of S_x at x such that every point y on g generates a net N_y in relation C to N_x if and only if the given net admits of having uniquely determined Bompiani transforms. If one of the Bompiani transforms is indeterminate there exists a pencil of lines g through x with the above property. The line g is the line in relation R with respect to N_x to the line joining the Bompiani transforms of x. Moreover *the minus first (first) derived line of g intersects the tangent plane of the sustaining surface in the minus first (first) Bompiani transform of x.*

The Bompiani transforms of a C net are C nets. The functions ρ and σ satisfy differential equations of the form (18). As may readily be verified the first Bompiani transform of ρ is the point $\bar{\sigma}$ defined by

$$\bar{\sigma} = \rho_v - a\rho,$$

and the minus first Bompiani transform of σ is the point $\bar{\rho}$ defined by

$$\bar{\rho} = \sigma_u - b\sigma.$$

The points $\bar{\rho}$ *and* $\bar{\sigma}$ *lie on the line* g, *and coincide with the points* (15) *and* (16) *respectively.* From (17) we see that these points coincide* if and only if

$$a_u - b_v = 0. \tag{17bis}$$

* Bompiani, *Surfaces*, p. 635.

MICHIGAN STATE COLLEGE,
EAST LANSING, MICH.