CONTRIBUTIONS TO THE THEORY OF TRANSFORMATIONS OF NETS IN A SPACE S_n*

BY

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1. Introduction

Let there be given a surface S in euclidean space of $n \geqq 3$ dimensions. Suppose that through each point x of S there passes a line g of a congruence G. The developables of G intersect S in a net of curves N. We have called such a net a *C net.*†

Let $\bar{S}$ be another surface in the same space S_n, in one-to-one point correspondence with S, corresponding points lying on the lines g of G. The developables of G intersect $\bar{S}$ in a C net of curves $\bar{N}$. The two nets N and $\bar{N}$ are said to be *in relation*‡ C.

The tangent planes to S and $\bar{S}$ intersect in a line h. If the points of h are each equidistant from the corresponding points of S and $\bar{S}$, we shall say that the nets N and $\bar{N}$ are *in relation E.*

We propose in this paper to develop a theory of the relations defined above which is independent of the dimension of the space S_n for $n \geqq 3$.

Let the coordinates of the point x on S be $x_1, x_2, \cdots, x_n$, the coordinates of the corresponding point $\bar{x}$ on $\bar{S}$ be $\bar{x}_1; \bar{x}_2, \cdots, \bar{x}_n$, and the direction cosines of the line g joining them be $\lambda_1, \lambda_2, \cdots, \lambda_n$, where

$$\sum_{i=1}^{n} \lambda_i^2 = 1.$$

Let the parametric curves on S and $\bar{S}$ be chosen as the curves of the given nets N, $\bar{N}$ on these surfaces. The pairs of functions $(x, \bar{x})$ and the number pair $(1, 1)$ are solutions of a system of differential equations of the form§

$$\begin{aligned} \bar{x}_u &= m x_u - A x + A\bar{x}, \\ \bar{x}_v &= n x_v - B x + B\bar{x}. \end{aligned} \tag{1}$$

* Presented to the Society, April 14, 1933; received by the editors January 15, 1933.

† V. G. Grove, *The transformation C of nets in hyperspace*, these Transactions, vol. 33 (1931), pp. 733–741. Hereafter referred to as C.

‡ C, p. 733.

§ C, p. 734.

The coordinates of the point $\bar{x}$ are of the form

$$\bar{x} = x + \lambda\delta. \tag{2}$$

The pairs of functions (x, λ) are solutions of the following system of differential equations:

$$\begin{aligned} \lambda_u &= \mu x_u + \alpha\lambda, \\ \lambda_v &= \nu x_v + \beta\lambda, \end{aligned} \tag{3}$$

wherein

$$\mu = (m-1)/\delta, \quad \alpha = -\mu E^{1/2}\cos\theta^{(u)}, \quad E = \sum_{i=1}^{n} x_{iu}^2,$$

$$\nu = (n-1)/\delta, \quad \beta = -\nu G^{1/2}\cos\theta^{(v)}, \quad G = \sum_{i=1}^{n} x_{iv}^2,$$

and $\theta^{(u)}$ and $\theta^{(v)}$ are the angles between g and the tangents to $v=$const. and $u=$const. respectively.

From (2) we find that

$$\begin{aligned} \bar{E}^{1/2}\cos\bar{\theta}^{(u)} &= E^{1/2}\cos\theta^{(u)} + \delta_u, \\ \bar{G}^{1/2}\cos\bar{\theta}^{(v)} &= G^{1/2}\cos\theta^{(v)} + \delta_v, \end{aligned} \tag{4}$$

wherein $\bar{E}$, $\bar{G}$ etc. bear the same relation to $\bar{N}$ as the corresponding quantities bear to N.

The focal points ξ and η of the line g have the coordinates

$$\xi = x - \lambda/\mu, \ \eta = x - \lambda/\nu, \ \mu\nu \neq 0. \tag{5}$$

If $\mu\nu=0$ one or both of the families of developables of G are cylinders.

The tangent planes to S and $\bar{S}$ at x and $\bar{x}$ intersect in the line h determined by the two points

$$r = x - mx_u/A, \ s = x - nx_v/B, \ AB \neq 0. \tag{6}$$

If $AB=0$, one or both of the curves of the net N are parallel to the corresponding curves of $\bar{N}$. The points r and s are equidistant from x and $\bar{x}$ if

$$A\delta + 2mE^{1/2}\cos\theta^{(u)} = 0, \quad B\delta + 2nG^{1/2}\cos\theta^{(v)} = 0. \tag{7}$$

We may readily verify that equations (7) are necessary and sufficient conditions that the nets N and $\bar{N}$ be in relation E if not both A and B are zero. Equations (7) are of the form

$$\delta_u + P\delta = Q, \quad \delta_v + P'\delta = Q',$$

wherein P, Q, P', Q' are independent of δ.

If we differentiate the first of equations (1) with respect to v and the second with respect to u, we find that if $m-n\neq 0$,

$$x_{uv} = ax_u + bx_v - Mx + M\bar{x}, \tag{8}$$

wherein a, b, M are defined by

$$\begin{aligned} (m-n)a &= B(m-1) - m_v, \quad (m-n)M = B_u - A_v,\\ (n-m)b &= A(n-1) - n_u. \end{aligned} \tag{9}$$

If $m-n=0$, we find that

$$B(m-1) - m_v = 0, \quad A(n-1) - n_u = 0. \tag{10}$$

In case that N is not conjugate, and C is not radial, we find from (8) that

$$\begin{aligned} \bar{x} &= x + (x_{uv} - ax_u - bx_v)/M,\\ \lambda &= (x_{uv} - ax_u - bx_v)/(\delta M). \end{aligned} \tag{11}$$

If the net N is conjugate, or if C is radial, the congruence G is not determined by the net N alone.

2. Congruences semi-normal to a net

A congruence G will be said to be *semi-normal to the net corresponding to the developables of the congruence* if the lines g of G are perpendicular to the tangents of one (only) of the families of curves of the net. In particular suppose that the line g is perpendicular to the tangent at x to the curve $v=$ const. Suppose that the transformation C is a transformation E. It follows from (2) and (3) that

$$A = \delta_u = 0.$$

Hence *if a congruence is semi-normal to the net N in which the developables of the congruence intersect the sustaining surface S of N, the congruence is semi-normal to any E transform of N.* Moreover *the distance between corresponding points x and $\bar{x}$ on the curves of N and $\overline{N}$ to whose tangents the lines g are normal, is a constant, and the tangent lines to these curves are parallel.*

3. Two-parameter families of lines normal to a surface

Let Γ be a two-parameter family of lines, such that through each point x of S there passes one and only one line l of Γ. Suppose furthermore that this line l is perpendicular to the tangent plane to S at x for all points x on S. We shall say that Γ is *normal to S*. Let the direction cosines of l be $l_1, l_2, \cdots, l_n$. It follows therefore that

$$\sum lx_u = 0, \quad \sum lx_v = 0.$$

Consider a curve C on S with parametric equations

$$u = u(t), \quad v = v(t).$$

Any point y on the tangent t to C at x has coordinates defined by an expression of the form

$$y = x + p(x_u u' + x_v v'), \quad u' = \frac{du}{dt}\cdot$$

As x moves along C the point y describes a curve, the direction cosines of whose tangent are proportional to expressions of the form

$$p(x_{uu}u'^2 + 2x_{uv}u'v' + x_{vv}v'^2) + L(x_u, x_v),$$

wherein $L(x_u, x_v)$ is a homogeneous linear function of the indicated arguments. The line l is perpendicular to the tangent to the locus of the point y if and only if C is an integral curve of the differential equation

$$Ddu^2 + 2D'dudv + D''dv^2 = 0, \tag{12}$$

wherein

$$D = \sum l x_{uu}, \; D' = \sum l x_{uv}, \; D'' = \sum l x_{vv}. \tag{13}$$

We shall call the net defined by (12), in case a net is so defined, *the A net of* Γ. We readily verify that *the line l is normal to the osculating plane at x of any curve of the A net of* Γ. *The A net of* Γ *is indeterminate in case* Γ *is normal to every plane of the two-osculating space $S_{(2,0)}$ of S at x.* If the parametric net is a conjugate net it follows that $D'=0$ for the A net of every two-parameter family of lines Γ normal to S.

Suppose now that Γ is a congruence G. Let the parametric curves be the curves in which the developables of G intersect S. Equations (3) may be written

$$\lambda_u = \mu x_u, \quad \lambda_v = \nu x_v. \tag{14}$$

It follows therefore, that, if C is not radial, the functions x and λ each satisfy differential equations of the Laplace type. Moreover

$$F = \sum x_u x_v = 0,$$

$$\mathcal{J} = \sum \lambda_u \lambda_v = 0.$$

Hence *if a congruence is normal to a surface its developables intersect the surface in an orthogonal conjugate net. Moreover the net of curves of the spherical indicatrix of G corresponding to the developables of G is an orthogonal conjugate net.*

If the parametric curves are not the curves in which the developables of G intersect S, the curves in which these developables do intersect S are defined by the differential equation

$$(ED' - FD)du^2 + (ED'' - GD)du dv + (FD'' - GD')dv^2 = 0. \tag{15}$$

We remark at this point that *a given surface S cannot possess a normal congruence unless it sustains an orthogonal conjugate net. Moreover it cannot possess more than one such normal congruence unless the developables of such other congruence intersect the surface in the same net* (15). The tangents to the curves of the A nets of such other congruences belong to the same involution, namely that determined by the tangents to the minimal curves and the tangents to the curves of the A net of the given normal congruence.

4. The radial transformation E

Suppose that the transformation C is radial. It follows that

$$m - n = 0.$$

Suppose also that C is an E transformation. From (7) and (10), we find that

$$\delta^2 = k^2(m - 1)^2/m, \tag{16}$$

wherein k is an arbitrary constant different from zero. Conversely if $m-n=0$, and equation (16) is satisfied, so also are equations (7). If r and $\bar{r}$ denote the distances from the points x and $\bar{x}$ respectively to the focal point of g, we find readily that

$$r\bar{r} = \delta^2 m/(m - 1)^2 = k^2.$$

Hence *if two nets are radial transforms in relation E, they are transforms of one another by a transformation by reciprocal radii, and conversely.*

Equations (1) for a transformation by reciprocal radii assume the following simple form:

$$\begin{aligned} \bar{x}_u &= k^2\mu^2 x_u - (x - \bar{x})\frac{\partial}{\partial u}\log(k^2\mu^2 - 1), \\ \bar{x}_v &= k^2\mu^2 x_u - (x - \bar{x})\frac{\partial}{\partial v}\log(k^2\mu^2 - 1). \end{aligned} \tag{17}$$

Equations (17) may readily be integrated. The solution for a proper choice of the constants of integration may be written in the form

$$\bar{x} = k^2\mu^2 x. \tag{18}$$

The lines joining x to $\bar{x}$ evidently pass through the origin. Moreover from (5) the quantity $-1/\mu$ is the distance from the point x to the fixed focal point of g. Hence

$$\bar{x} = \frac{k^2 x}{\sum x^2}.$$

These of course are the familiar formulas of a transformation by reciprocal radii.

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