# A SPECIAL INTEGRAL FUNCTION*

BY

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1. Some years ago Collingwood and Valiron† proposed the problem of whether there could exist an integral function whose minimum modulus on every circle $|z|=r$ is bounded, but possessing no asymptotic paths. By an asymptotic path we mean a continuous path tending to infinity along which the value of the function tends to a limit.

In this paper I show how to construct such a function. It is obtained by considering the well known Weierstrassian non-differentiable function

$$\sum_{n=0}^{\infty} c^n z^{a^n}$$

where $c(1<c<2)$ and $a$ (an integer) are suitably chosen. We may observe that, if $a$ is large enough, the Weierstrassian function possesses no asymptotic paths which tend to the boundary $|z|=1$, while, for sufficiently small $c$, its minimum modulus on circles $|z|=r<1$ is bounded, and every point of the unit circle is an essential singularity for the function.

2. Consider the function

$$F_N(z) = \left( \sum_{n=0}^{N} c^n z^{a^n} \right) \exp \left\{ - \left( \frac{z}{1 - a^{-N}} \right)^{2^N a^N} \right\}$$

where $c>1$ (say $c=3/2$), and $a$ is large. We first show that the minimum modulus of $F_N(z)$ on any circle $|z|=r$ is bounded, independently of $N$. Clearly, for $r>1$, $F_N(r)$ does not exceed

$$(1) \qquad Bc^N r^{a^N} \exp(-r - e^{B2^N}),$$

where $B$, here and in the sequel, denotes an absolute positive constant (it may denote a different constant in different contexts). For $r \geqq 1$, $N \geqq 1$, the expression (1) does not exceed a fixed constant, and thus it is sufficient to consider $F_N(z)$ with $|z| \leqq 1$.

Let $r$ be a fixed number less than 1. We consider the value of $F_N(re^{i\theta})$ where $\theta$ is chosen according to the following rules. We first stipulate that

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† E. F. Collingwood and G. Valiron, *Theorems concerning an analytic function which is bounded upon a simple curve passing through an isolated essential singularity*, Proceedings of the London Mathematical Society, (2), vol. 26 (1927), pp. 169–184; p. 182.

$a^N\theta \equiv 0 \pmod{2\pi}$, so that also $2^N a^N \theta \equiv 0 \pmod{2\pi}$, and thus the second factor

$$\exp\left\{-\left(\frac{z}{1-a^{-N}}\right)^{2^N a^N}\right\}$$

of $F_N(z)$ is real and less than 1 in modulus. There are now $a$ possible reduced values of $a^{N-1}\theta \pmod{2\pi}$. We choose that one which makes

$$\left|\sum_{n=N-1}^{N} c^n r^{a^n} \exp(ia^n\theta)\right|$$

a minimum. We now choose that one of the reduced values of $A^{N-2}\theta$ (mod $2\pi$) which makes

$$\left|\sum_{n=N-2}^{N} c^n r^{a^n} \exp(ia^n\theta)\right|$$

a minimum, and so on. We show that if $a$ is sufficiently large the resulting value of

$$\left|\sum_{n=0}^{N} c^n r^{a^n} \exp(ia^n\theta)\right|$$

will not exceed a fixed constant independent of $N$. The argument is almost identical with that given in an earlier paper.* We have at the first stage $a$ possible values of

$$\left|\sum_{n=N-1}^{N} c^n r^{a^n} \exp(ia^n\theta)\right|. \tag{2}$$

There is one of these for which the angle between the lines joining the point $c^N r^{a^N}$ to the origin and to the point

$$\sum_{n=N-1}^{N} c^n r^{a^n} \exp(ia^n\theta)$$

is less than or equal to $\pi/a$. Then the value of (2) can be seen by elementary geometry to lie between

$$c^{N-1} r^{a^{N-1}} \text{ and } c^N r^{a^N} \sec(\pi/a) - c^{N-1} r^{a^{N-1}},$$

which, if $a$ is sufficiently large ($c=3/2$), is certainly not greater than $c^{N-1}r^{a^{N-1}}$. Thus

$$\min_{a^N\theta\equiv 0}\left|\sum_{n=N-1}^{N} c^n r^{a^n} \exp(ia^n\theta)\right| \leq c^{N-1} r^{a^{N-1}}.$$

* R. E. A. C. Paley, *On some problems connected with Weierstrass's non-differentiable function*, Proceedings of the London Mathematical Society, (2), vol. 31 (1930), pp. 301–328; Theorem I, pp. 304–308.

Having now fixed the reduced value of $a^{N-1}\theta$ (mod $2\pi$), we look for the minimum value of

$$(3) \qquad \left| \sum_{n=N-2}^{N} c^n r^{a^n} \exp(ia^n\theta) \right| .$$

There is certainly one value for the expression (3), such that the angle between the lines joining the point

$$\left| \sum_{n=N-1}^{N} c^n r^{a^n} \exp(ia^n\theta) \right|$$

to the origin and to the point

$$\sum_{n=N-2}^{N} c^n r^{a^n} \exp(ia^n\theta)$$

is less than or equal to $\pi/a$. Thus the value of (3) lies between

$$c^{N-2} r^{a^{N-2}} \text{ and } \left| \sum_{n=N-1}^{N} c^n r^{a^n} \exp(ia^n\theta) \right| \sec\left(\frac{\pi}{a}\right) - c^{N-2} r^{a^{N-2}},$$

and, if $a$ is sufficiently large, it does not exceed $c^{N-2}r^{a^{N-2}}$. An inductive process will now show that

$$\min_{a^N\theta\equiv 0} \left| \sum_{n=0}^{N} c^n r^{a^n} \exp(ia^n\theta) \right| \leqq r \leqq 1,$$

and we have shown that, for all values of $r$, the minimum modulus of $F_N(z)$ on $|z| = r$ is less than an absolute constant.

The derivative $F_N'(z)$ of $F_N(z)$ is

$$(4) \quad \exp\left\{ -\left(\frac{z}{1-a^{-N}}\right)^{2^N a^N} \right\} \left\{ \sum_{n=0}^{N} (ac)^n z^{a^n} - \frac{2^N a^N}{z} \left(\frac{z}{1-a^{-N}}\right)^{2^N a^N} \left( \sum_{n=0}^{N} c^n z^{a^n} \right) \right\}.$$

Now suppose that $a$ is sufficiently large, and that $1-3a^{-N} \leqq r \leqq 1-2a^{-N}$. Then the expression (4) is majorized* by the single term

$$(5) \qquad a^N c^N z^{a^N-1}.$$

Indeed the single term (5) exceeds in modulus

$$a^N c^N (1-3a^{-N})^{a^N-1} \geqq \frac{1}{32} a^N c^N, \quad a \geqq 6,$$

* See, e.g., G. H. Hardy, *Weierstrass' non-differentiable function*, these Transactions, vol. 17 (1916), pp. 301–332.

while the difference between the terms (4) and (5) is not greater in modulus than

$$\exp\left(\frac{1-2a^{-N}}{1-a^{-N}}\right)^{2^N a^N}\left\{\sum_{n=0}^{N-1}(ac)^n + 2\cdot 2^N a^N\left(\frac{1-2^{-N}}{1-a^{-N}}\right)^{2^N a^N}\left(\sum_{n=0}^{N} c^n\right)\right\}$$

$$+ a^N c^N\left\{\exp\left(\frac{1-2a^{-N}}{1-a^{-N}}\right)^{2^N a^N} - 1\right\} \leqq 10^{-6} a^N c^N$$

if $a$ and $N$ are sufficiently large.

3. We now write

$$F(z) = \sum_{k=1}^{\infty} f_k(u_k),\quad u_k = \left(\frac{z}{R_k}\right)^{\alpha_k},$$

and set, for abbreviation,

$$f_k(u) \equiv F_{N_k}(u),\ \alpha_k = a^{\lambda_k},\ \beta_k = a^{N_k+\lambda_k},$$

where $\lambda_1 = 0$, $\lambda_{k+1} = 2(N_k + \lambda_k)$, while $N_k$, $R_k$ $(k = 1, 2, \cdots)$ remain to be chosen. We write $N_1 = R_1 = 1$ and give an inductive method for choosing $N_k$, $R_k$ for $k > 1$. Suppose that we have already chosen $N_1, \cdots, N_{k-1}, R_1, \cdots, R_{k-1}$. Since first $F_N(z) = O(|z|)$ for small $z$ uniformly in $N$, we may choose $R_k$ so large that

$$(6)\qquad |f_k(u_k)| \leqq 2^{-n},\ |z| \leqq R_{k-n},\ n = 1, 2, \cdots, k-1,$$

whatever the value of $N_k$ may be. Next since, for $|z| \leqq R/2$ we have, uniformly in $N_k$ and $R$,

$$\frac{d}{dz} f_k\left[\left(\frac{z}{R}\right)^{\alpha_k}\right] = O(|z|^{\alpha_k - 1} R^{-\alpha_k}),$$

we may also assume that $R_k$ is so large that, whatever the value of $N_k$ may be, we have

$$(7)\qquad \left|\frac{d}{dz} f_k(u_k)\right| \leqq 2^{-n} 10^{-6},\ |z| \leqq R_{n-k},\quad n = 1, 2, \cdots, k-1.$$

This finally fixes $R_k$. We now choose $N_k > k$ so great that

$$(8)\qquad \beta_k c^{N_k} R_k^{-1} \geqq 10^6 \max_{|z| \leqq R_k} \left|\frac{d}{dz} \sum_{m=1}^{k-1} f_m(u_m)\right|.$$

We next observe that, for $-(2a)^{-N_k}\pi/4 \leqq \theta \leqq (2a)^{-N_k}\pi/4$, $r \geqq 1$,

$$|f_k(z)| \leqq \left| Bc^{N_k} r^{a^{N_k}} \exp\left(\frac{r}{1-a^{-N_k}}\right)^{2^{N_k} a^{N_k}} e^{\pi i/4} \right\}\right|$$

$$= Bc^{N_k} r^{a^{N_k}} \exp\left\{ -\ 2^{-1/2}\left(\frac{r}{1-a^{-N_k}}\right)^{2^{N_k} a^{N_k}} \right\} \leqq Bc^{N_k} \exp\left\{-\ (Be)^{2^{N_k}}\right\}.$$

For $-a^{-\lambda_{k+1}}\pi \leqq \theta \leqq a^{-\lambda_{k+1}}\pi$, the argument of $u_k$ is $\alpha_k\theta$, and thus in modulus does not exceed

$$a^{\lambda_k - \lambda_{k+1}}\pi = a^{-2N_k - \lambda_k}\pi \leqq (2a)^{-N_k}\pi/4;$$

whence, on the range $|z| \geqq R_k$, $-a^{-\lambda_{k+1}}\pi \leqq \theta \leqq a^{-\lambda\ {}_{+1}}\pi$,

$$\max |f_k(u_k)| \leqq Bc^{N_k} \exp\left\{-\ (Be)^{2^{N_k}}\right\}.$$

We may thus increase $N_k$ if necessary so as to ensure that, on the same range,

$$\max |f_k(u_k)| \leqq 2^{-k}. \tag{9}$$

This finally fixes $N_k$.

4. We observe first that, in virtue of (6), $F(z)$ is in fact an integral function. Next (6) and (9) give us

$$\max\left\{ \sum_{l=1}^{k-1} + \sum_{l=k+1}^{\infty} |f_l(u_l)| \right\} \leqq B, \tag{10}$$

$$R_k \leqq |z| \leqq R_{k+1},\ -\ a^{-\lambda_{k+1}}\pi \leqq \theta \leqq a^{-\lambda_{k+1}}\pi.$$

Also $F_{N_{k+1}}$ is so constructed that for fixed $r$, satisfying $R_k \leqq r \leqq R_{k+1}$,

$$\min |f_{k+1}(u_{k+1})| \leqq B, \tag{11}$$

$$-\ a^{-\lambda_{k+1}}\pi \leqq \theta \leqq a^{-\lambda_{k+1}}\pi.$$

The equations (10) and (11) show that if $r$ is fixed with $R_k \leqq r \leqq R_{k+1}$, then

$$\min_{|z|=r} |F(z)| \leqq B,$$

where $B$ is independent of $r$, $k$. Thus, the minimum modulus of $F(z)$ on circles $|z| = r$ is bounded.

5. We have now to show that $F(z)$ has no asymptotic path. To do this we show that in certain regions the differential coefficient of $F(z)$ is not only large but so large that there can be no continuous path passing through all these regions on which $F(z)$ is bounded.

Consider $F'(z)$ in the annulus

$$1 - 3a^{-N_k} \leqq u_k \leqq 1 - 2a^{-N_k}. \tag{12}$$

We have

$$\frac{d}{dz}f_k(u_k) = \frac{\alpha_k}{z}u_k\frac{d}{du_k}f_k(u_k) = \frac{\alpha_k}{z}u_k(ac)^{N_k}\left(\frac{z}{R_k}\right)^{\beta_k-\alpha_k}(1+\epsilon),$$

where $|\epsilon| \leqq |10^{-k}$, in virtue of the remarks at the end of §2. Now, in the annulus considered, when $a>6$,

$$\left|\frac{\alpha_k}{z}u_k(ac)^{N_k}\left(\frac{z}{R_k}\right)^{\beta_k-\alpha_k}\right| = \beta_k c^{N_k}R_k^{-1}\left|\frac{z}{R_k}\right|^{\beta_k-1}$$
$$\geqq \beta_k c^{N_k}R_k^{-1}(1-3a^{-N_k})^{a^{N_k}} \geqq 10^{-2}\beta_k c^{N_k}R_k^{-1}.$$

Also, by (7) and (8), in the annulus considered,

$$\left|\frac{d}{dz}\left\{\sum_{m=1}^{k-1}+\sum_{m=k+1}^{\infty}f_m(u_m)\right\}\right| \leqq 10^{-6}\beta_k c^{N_k}R_k^{-1}+10^{-6},$$

and thus

$$F'(z) = \beta_k c^{N_k}R_k^{-\beta_k}z^{\beta_k-1}(1+\epsilon'), \tag{13}$$

where $|\epsilon'| \leqq 3\cdot 10^{-4} \leqq 10^{-3}\pi^{-1}$.

Now let $\zeta$ and $\zeta'$ be two points of the annulus (12) and let

$$\left|\frac{\zeta'}{\zeta}-1\right| = 10^{-1}\beta_k^{-1}.$$

Then (13) shows that

$$\begin{aligned} F(\zeta')-F(\zeta) &= \int_{\zeta'}^{\zeta}F'(z)dz \\ &= \int_{\zeta'}^{\zeta}\beta_k c^{N_k}R_k^{-\beta_k}z^{\beta_k-1}(1+\epsilon')dz \\ &= c^{N_k}\zeta^{\beta_k}R_k^{-\beta_k}\left\{\left(\frac{\zeta'}{\zeta}\right)^{\beta_k}-1\right\}+R, \end{aligned} \tag{14}$$

where

$$\begin{aligned} |R| &\leqq 10^{-3}|\zeta'-\zeta| \max_{|z|\leqq R_k}|\beta_k c^{N_k}R_k^{-\beta_k}z^{\beta_k-1}| \\ &\leqq 10^{-3}|\zeta'-\zeta|\beta_k c^{N_k}R_k^{-1} \leqq 10^{-4}c^{N_k}, \end{aligned} \tag{15}$$

for we may certainly find a path joining $\zeta$ and $\zeta'$ of length not exceeding $|\zeta'-\zeta|\pi$, entirely interior to the annulus (12). The first term of (14) is

$$c^{N_k}\zeta^{\beta_k}R_k^{-\beta_k}\{\beta_k(\zeta'-\zeta)/\zeta\}(1+\epsilon''), \tag{16}$$

where

$$|\epsilon''| \leqq 10\{(1 + 10^{-1}\beta_k^{-1})^{\beta_k} - 1 - 10^{-1}\} < 1/10.$$

Since, finally, in virtue of (12), if $a$ is large enough,

$$|c^{N_k}\zeta^{\beta_k}R_k^{-\beta_k}\cdot\beta_k(\zeta' - \zeta)/\zeta| \geqq 10^{-3}c^{N_k},$$

it follows from (14), (15), (16) that

$$|F(\zeta') - F(\zeta)| \geqq 2\cdot 10^{-4}c^{N_k}. \tag{17}$$

Since, for sufficiently large $a$, the breadth of the strip (12) exceeds

$$10^{-1}R_k\beta_k^{-1},$$

(17) shows that there can be no continuous path, crossing the strip, for which the minimum modulus of $F(z)$ is less than $10^{-4}c^{N_k}$ which is arbitrarily large with $k$. Thus there can be no asymptotic path tending to infinity.

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