SUFFICIENT CONDITIONS IN THE PROBLEM OF THE CALCULUS OF VARIATIONS IN n-SPACE IN PARAMETRIC FORM AND UNDER GENERAL END CONDITIONS*

BY

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1. **Introduction.** Sufficient conditions in the general problem of the calculus of variations in parametric form are given here. The results are in terms of the characteristic roots of a linear boundary value problem, and are in close relation to the conditions recently given by Morse† in the corresponding problem in non-parametric form.

An important feature of the results is that the usual "non-tangency" hypothesis is not made. For example, if these results were applied to the problem of minimizing an integral along curves joining a point to a manifold, we would obtain sufficient conditions for a minimum even in the case that the minimizing curve is tangent to the manifold.

The essential idea in the methods used in the paper is the treatment of the parametric problem as the limiting case of a series of non-singular non-parametric problems by means of a suitable modification of the integrand. Although they lack the geometric invariance of methods now being developed by Morse, in which the parametric problem is approximated by means of a series of parametric problems of the same nature as the original problem, the methods and results of this paper derive advantage from the non-singularity of the approximating non-parametric problems and from the fact that the cases of "non-tangency" and "tangency" are treated together. The work of the author and that of Morse are thus complementary, and constitute the first complete treatment of sufficient conditions in the general parametric problem.

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† Certain results in the following papers will be used.

Morse, *Sufficient conditions in the problem of Lagrange with variable end conditions*, American Journal of Mathematics, vol. 53 (1931), pp. 517–546.

Morse and Myers, *The problems of Lagrange and Mayer with variable end points*, Proceedings of the American Academy of Arts and Sciences, vol. 66 (1931), pp. 235–253.

Bliss, *Jacobi's condition for problems of the calculus of variations in parametric form*, these Transactions, vol. 17 (1916), pp. 195–206.

Further references can be found in the three papers just cited.

2. **The Euler equations and the transversality conditions.** In the space of the variables

$$(x) = (x_1, \cdots, x_n)$$

let there be given an ordinary arc g

$$x_i = \bar{x}_i(t), \quad t^{(1)} \leqq t \leqq t^{(2)} \qquad (i = 1, \cdots, n), \tag{2.1}$$

of class C'.

We consider ordinary arcs of class D' neighboring g. The initial and final end points of such arcs will be denoted respectively by

$$(x^s) = (x_1^s, \cdots, x_n^s) \qquad (s = 1, 2)$$

and the end values of the parameter t will be denoted respectively by t^s ($s=1, 2$), where $s=1$ at the initial end point and $s=2$ at the final end point. An ordinary arc of class D' neighboring g will be said to be *admissible* if its end points are given for some value of

$$(\alpha) = (\alpha_1, \cdots, \alpha_r)$$

by the functions

$$x_i^s = x_i^s(\alpha_1, \cdots, \alpha_r), \quad 0^* \leqq r \leqq 2n \quad (i = 1, \cdots, n; s = 1, 2). \tag{2.2}$$

These functions of (α) are of class C'' for (α) near (0) and reduce to the end points of g for $(\alpha) = (0)$. We assume that the functional matrix of the functions in (2.2)

$$\| x_{ih}^s \| \quad (h = 1, \cdots, r; i = 1, \cdots, n; s = 1, 2)$$

is of rank r for $(\alpha) = (0)$. Here and henceforth the subscript h attached to x_i^s shall denote differentiation with respect to α_h.

We seek conditions under which the arc g and the set $(\alpha) = (0)$ afford a minimum to the expression

$$J = \int_{t^1}^{t^2} F(x, \dot{x})dt + \theta(\alpha) \tag{2.3}$$

among sets (α) near (0) and admissible arcs neighboring g with end points determined by these sets (α). The function $F(x, \dot{x})$ is defined for (x) in an open region containing g and for $(\dot{x})$ any set not (0), and is to be of class C'''. The function θ is to be of class C'' for (α) near (0).

Furthermore, the function F is to satisfy the usual homogeneity relation

$$F(x, k\dot{x}) \equiv kF(x, \dot{x}), \qquad k > 0. \tag{2.4}$$

* The case $r=0$ yields the fixed end point problem. This case will be treated separately at the end of the paper, so that until then we shall assume that $r>0$.

Certain necessary conditions are obtained immediately by treating the problem as a non-parametric problem of minimizing J among curves of class D' in the $(n+1)$-space of the variables (t, x) whose end points satisfy (2.2) and the conditions $t=t^{(s)}$.*

THEOREM 1. *If g affords a minimum to J in the problem, then along g the following equations must be satisfied*:

$$\frac{d}{dt}\left[\frac{\partial F}{\partial \dot{x}_i}\right] - \frac{\partial F}{\partial x_i} = 0 \qquad (i = 1, \cdots, n), \tag{2.5}$$

while the following transversality relations must hold:

$$\left[\frac{\partial F}{\partial \dot{x}_i} \overset{s}{x}_{ih}\right]_1^2 + \theta_h = 0\dagger \quad (h = 1, \cdots, r; i = 1, \cdots, n). \tag{2.6}$$

We shall now state and prove a theorem which will be useful later.

THEOREM 2. *For an arbitrary set of functions $\eta_i(t)$ of class D' such that $\eta_i(t^{(s)}) = x^s_{ih}u_h$ $(i=1, \cdots, n;\ h=1, \cdots, r;\ s=1, 2)$ for some set of numbers $(u_1, \cdots, u_r)$, there exists a one-parameter family of admissible arcs*

$$x_i = x_i(t, e), \quad \alpha_h = \alpha_h(e) \quad (i = 1, \cdots, n;\ h = 1, \cdots, r), \tag{2.7}$$

containing g for $e=0$, with $\eta_i(t)$ and u_h as its respective variations; that is, the functions in (2.7) *will have the following properties*:

$$\begin{aligned} x_i(t, 0) &= \bar{x}_i(t), \\ x_i(\bar{t}^s, e) &= \overset{s}{x}_i[\alpha(e)], \\ \alpha_h(0) &= 0, \quad x_{ie}(t, 0) = \eta_i(t), \quad \alpha_h'(0) = u_h \\ & \qquad (i = 1, \cdots, n;\ h = 1, \cdots, r;\ s = 1, 2). \end{aligned} \tag{2.8}$$

Furthermore, the functions $x_i(t, e)$ and $x_{ie}(t, e)$ are continuous and have continuous derivatives with respect to e for e near 0 *and t in the interval $t^{(1)} \leqq t \leqq t^{(2)}$, while the functions $x_{it}(t, e)$ and $x_{iet}(t, e)$ have the same properties except possibly at the values of t defining the corners of (η). The functions $\alpha_h(e)$ are of class C''.*

For the following is such a family:

$$\begin{aligned} x_i &= \bar{x}_i(t) + e[\eta_i(t) - \eta_i^1 h^2(t) - \eta_i^2 h^1(t)] + [x_i^1(eu) - \bar{x}_i^1]h^2(t) \\ &\quad + [x_i^2(eu) - \bar{x}_i^2]h^1(t), \\ \alpha_h &= eu_h \qquad (h = 1, \cdots, r;\ i = 1, \cdots, n), \end{aligned} \tag{2.9}$$

* See Morse and Myers, p. 245, loc. cit.

† Here and henceforth $[\ \ ]_1^2$ shall mean the difference between the value of the bracket evaluated for $s=2$ and $(x, \dot{x})$ at the final end point of g, and the corresponding evaluation at the initial end point of g. Also, an index repeated in the same term shall always mean summation with respect to that index. The notation θ_h stands for $(\partial\theta/\partial\alpha_h)(0)$.

where $h^1(t)$, $h^2(t)$ are any functions of class C' such that

$$h^1(t^{(1)}) = 0, \quad h^1(t^{(2)}) = 0, \quad h^2(t^{(1)}) = 0, \quad h^2(t^{(2)}) = 0,$$

while η_i^s is an abbreviation for $\eta_i(t^{(s)})$ and $\bar{x}_i^s$ is an abbreviation for $\bar{x}_i(t^{(s)})$.

3. **The accessory boundary problem and a further necessary condition.** We assume now that g is an extremal satisfying the transversality conditions (2.6). We shall use permanently the notations

$$\begin{aligned} \eta_i(t) &= x_{ie}(t, 0), \\ \overset{s}{\eta}_i &= \eta_i(t^{(s)}), \\ u_h &= \alpha_h'(0) \qquad (i = 1, \cdots, n;\ s = 1, 2;\ h = 1, \cdots, r). \end{aligned}$$

Consider now a family of admissible arcs of form (2.7) satisfying the first three conditions of (2.8) and possessing the differentiability properties of Theorem 2. If we consider this family momentarily as a family of arcs in (t, x)-space satisfying the end conditions

$$\begin{aligned} \overset{s}{x}_i &= \overset{s}{x}_i(\alpha), \\ t^s &= t^{(s)} \qquad (i = 1, \cdots, n;\ s = 1, 2), \end{aligned} \tag{3.1}$$

then we can apply known results* to obtain the second variation of J along this family. We find that

$$J''(0) = b_{hk}u_h u_k + 2\int_{t^{(1)}}^{t^{(2)}} \omega(\eta, \dot{\eta})dt \qquad (h, k = 1, \cdots, r), \tag{3.2}$$

where

$$2\omega(\eta, \dot{\eta}) = \frac{\partial^2 F}{\partial \dot{x}_i \partial \dot{x}_j}\dot{\eta}_i\dot{\eta}_j + 2\frac{\partial^2 F}{\partial \dot{x}_i \partial x_j}\dot{\eta}_i\eta_j + \frac{\partial^2 F}{\partial x_i \partial x_j}\eta_i\eta_j \ (i, j = 1, \cdots, n) \tag{3.3}$$

and

$$b_{hk} = \left[\frac{\partial F}{\partial \dot{x}_i}\overset{s}{x}_{ihk}\right]_1^2 + \theta_{hk} \ \ (h, k = 1, \cdots, r;\ i = 1, \cdots, n). \tag{3.4}$$

With the idea of dominating the sign of the second variation by adding new terms, we are led to consider the accessory problem of minimizing

$$I(\eta, u, \sigma) = b_{hk}u_h u_k + \int_{t^{(1)}}^{t^{(2)}} [2\omega - \sigma(\dot{\eta}_i\dot{\eta}_i + \eta_i\eta_i)]dt \qquad (i = 1, \cdots, n;\ h, k = 1, \cdots, r) \tag{3.5}$$

for a given number σ, relative to constants (u) and functions (η) of class D' satisfying

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* See Morse, p. 521, loc. cit.

$$\overset{s}{\eta}_i = \overset{s}{x}_{ih} u_h \qquad (i = 1, \cdots, n; s = 1, 2; h = 1, \cdots, r). \tag{3.6}$$

A *solution* (η), (u) of this new minimum problem in which the functions (η) are of class C'' must satisfy the conditions of the following boundary value problem:

$$\frac{d}{dt}\left[\frac{\partial \Omega}{\partial \dot{\eta}_i}\right] - \frac{\partial \Omega}{\partial \eta_i} = 0 \qquad (i = 1, \cdots, n), \tag{3.7}$$

$$b_{hk} u_k + \left[\frac{\partial \Omega}{\partial \dot{\eta}_i} \overset{s}{x}_{ih}\right]_1^2 = 0 \qquad (h, k = 1, \cdots, r), \tag{3.8}$$

$$\overset{s}{\eta}_i = \overset{s}{x}_{ih} u_h \qquad (s = 1, 2), \tag{3.9}$$

where

$$2\Omega(\eta, \dot{\eta}, \sigma) = 2\omega(\eta, \dot{\eta}) - \sigma(\dot{\eta}_i \dot{\eta}_i + \eta_i \eta_i) \qquad (i = 1, \cdots, n). \tag{3.10}$$

This boundary problem we shall call the *accessory boundary problem*. By a *solution* of the accessory boundary problem is meant a set of functions $\eta_i(t)$ of class C'' which with constants (u) and σ satisfy the conditions of the problem. A *characteristic* solution is one for which $(\eta) \not\equiv (0)$.

The corresponding value of σ will be called a *characteristic root*.

The following lemma and theorem can be proved in a manner similar to that used by Morse in his proof of the corresponding results for the non-parametric problem.† In the proof of Theorem 3, Theorem 2 must be used.

LEMMA 1. *If (η) is a characteristic solution with constants (u) and σ, $I(\eta, u, \sigma) = 0$.*

THEOREM 3. *If g furnishes a minimum for the given problem, there can exist no characteristic root $\sigma < 0$.*

4. **The function $J(\alpha)$ and the quadratic form $H(u, \sigma)$.** By the *Legendre sufficient condition* we mean the condition

$$\frac{\partial^2 F}{\partial \dot{x}_i \partial \dot{x}_j} \pi_i \pi_j > 0 \qquad (i, j = 1, \cdots, n) \tag{4.1}$$

along g, for all sets $(\pi) \neq (0)$ and not proportional to $(\partial \dot{x}/\partial t)$.

By the *Weierstrass sufficient condition* we mean the condition

$$E(x, \dot{x}, \dot{y}) \equiv F(x, \dot{y}) - \dot{y}_i \frac{\partial F}{\partial \dot{x}_i}(x, \dot{x}) > 0 \qquad (i = 1, \cdots, n) \tag{4.2}$$

for all (x), $(\dot{x})$ on g, and for all $(\dot{y}) \neq (0)$ and not proportional to $(\bar{x}')$.

† See Morse, p. 254, loc. cit.

We shall assume henceforth that g is an extremal along which the Legendre sufficient condition holds. Among the well known consequences of this assumption are the following:

(1) The determinant

$$\begin{vmatrix} \dfrac{\partial^2 F}{\partial \dot{x}_i \partial \dot{x}_j} & \bar{x}_i' \\ \bar{x}_j' & 0 \end{vmatrix} \neq 0$$

along g.

(2) The functions $\bar{x}_i(t)$ are of class C'''.

(3) The characteristic determinant

$$\left| \frac{\partial^2 F}{\partial \dot{x}_i \partial \dot{x}_j} - \sigma \delta_i^j \right|$$

does not vanish for $\sigma < 0$, where δ_i^j is the Kronecker delta.

A set (α) neighboring $(\alpha) = (0)$ determines through (2.2) two end points P and Q near the respective end points of g. If we assume for the moment that the end points of g are not conjugate, then P and Q can be joined by a unique extremal E, which is thus determined by the set (α). We can thus obtain a family of extremals determined by values of (α) near (0), and this family can be represented in the following form:

$$x_i = x_i^*(t, \alpha) \qquad (i = 1, \cdots, n), \tag{4.3}$$

where x_i^* and x_{it}^* are of class C'' in (α) and satisfy the following conditions:

$$x_i^*(t, 0) = \bar{x}_i(t) \qquad (i = 1, \cdots, n), \tag{4.4}$$

$$x_i^*(t^{(s)}, \alpha) = \overset{s}{x}_i(\alpha) \qquad (s = 1, 2). \tag{4.5}$$

The expression J taken along the extremals of the family (4.4) becomes a function $J(\alpha)$ of class C''.

The Euler equations (2.5) and the transversality conditions (2.6) enable us to prove that $J(\alpha)$ has a critical point for $(\alpha) = (0)$.

The terms of the second order of $J(\alpha)$ are obtained by means of the following identity in the variables $(u_1, \cdots, u_r)$:

$$J_{\alpha_h \alpha_k}(0) u_h u_k \equiv \frac{d^2 J}{de^2}(eu), \quad (e = 0) \qquad (h, k = 1, \cdots, r). \tag{4.6}$$

The right hand side of (4.6) is nothing but the second variation of the one-parameter family of extremals obtained from the family (4.3) by setting $\alpha_h = eu_h$, where u_h is fixed and e is variable. This one-parameter family has the form

$$x_i = x_i(t, e), \quad \alpha_h = eu_h \quad (i = 1, \cdots, n;\ h = 1, \cdots, r), \tag{4.7}$$

where

$$x_i(t^{(s)}, e) = x_i^s(eu) \qquad (i = 1, \cdots, n;\ s = 1, 2). \tag{4.8}$$

The second variation of the family (4.7) has the form (3.2), so that

$$J_{\alpha_h\alpha_k}(0)u_hu_k = b_{hk}u_hu_k + 2\int_{t^{(1)}}^{t^{(2)}} \omega(\eta, \dot\eta)dt \quad (h, k = 1, \cdots, r). \tag{4.9}$$

A curve $\eta_i = \eta_i(t)$ of class C'' in the space of the variables (t, η) will be called a *secondary extremal* if the functions (η) satisfy (3.7) for some σ. At present we are concerned only with secondary extremals for $\sigma = 0$.

To show the complete relation between (u) and (η) in (4.9), we need the following lemma.

Lemma 2. *The integral $\int_{t_1}^{t_2}\omega dt$ has the same value if evaluated along any two secondary extremals joining the same end points $A:(t_1, a)$ and $B:(t_2, b)$.*

Suppose that $(\bar\eta)$ and $(\bar{\bar\eta})$ are the two secondary extremals. Then

$$\eta_i = \bar\eta_i + e(\bar\eta_i - \bar{\bar\eta}_i) \qquad (i = 1, \cdots, n)$$

is a one-parameter family of secondary extremals joining A and B and containing $(\bar\eta)$ and $(\bar{\bar\eta})$. But the value of an integral taken along the members of a one-parameter family of extremals joining the same end points is the same for each extremal.

Returning now to (4.9), we note that the functions $\eta_i(t)$ in the argument of ω define a secondary extremal E', since they are the variations of a family of extremals. The set (u) in (4.9) determines the end points of E'; for upon differentiating (4.8) with respect to e and setting $e = 0$, we obtain

$$\eta_i^s = x_{ih}^s u_h \quad (i = 1, \cdots, n;\ h = 1, \cdots, r;\ s = 1, 2), \tag{4.10}$$

and it is in this sense that the set (u) *determines* the end points of E'.

From (4.9) and Lemma 2 we obtain the following theorem:

Theorem 4. *Under the assumption that the end points of g are not conjugate, let $J(\alpha)$ represent the value of J taken along the extremal determined by (α). Then the terms of second order of $J(\alpha)$ have the form*

$$J_{\alpha_h\alpha_k}(0)u_hu_k = b_{hk}u_hu_k + 2\int_{t^{(1)}}^{t^{(2)}} \omega(\eta, \dot\eta)dt \quad (h, k = 1, \cdots, r) \tag{4.11}$$

where (η) may be taken along any secondary extremal with end points determined by (u).

In order to bring the parameter σ into the second variation as in (3.5), we replace the integrand F by a one-parameter family of integrands

$$\overline{F} \equiv F - \frac{\sigma}{2}\{[x_i - \bar{x}_i(t)][x_i - \bar{x}_i(t)] + [\dot{x}_i - \bar{x}_i'(t)][\dot{x}_i - \bar{x}_i'(t)]\} \quad (i = 1, \cdots, n) \tag{4.12}$$

which we consider only for $\sigma \leqq 0$. For $\sigma = 0$ we have our original problem in (x)-space, but for each $\sigma<0$ we consider a non-parametric problem in (t, x)-space, the problem with the integral

$$\int_{t^{(1)}}^{t^{(2)}} \overline{F} dt$$

and the end conditions

$$x_i^s = x_i^s(\alpha), \ t^s = t^{(s)} \qquad (i = 1, \cdots, n; s = 1, 2). \tag{4.13}$$

When we talk about extremals, conjugate points, etc., for $\sigma<0$, these terms will always be understood to refer to the non-parametric problem in (t, x)-space.

For each $\sigma<0$, $g: x_i = \bar{x}_i(t)$ is still an extremal. We note that the problem for each $\sigma<0$ is *non-singular*; that is, along g the determinant

$$\left|\frac{\partial^2 \overline{F}}{\partial \dot{x}_i \partial \dot{x}_j}\right| \equiv \left|\frac{\partial^2 F}{\partial \dot{x}_i \partial \dot{x}_j} - \sigma \delta_i^j\right| \neq 0, \qquad \sigma < 0. \tag{4.14}$$

This is a consequence of the Legendre sufficient condition. If, then, we assume momentarily that for each $\sigma<0$ the end points of g are not conjugate, a set (α) neighboring (0) will determine for each $\sigma<0$ a unique extremal, and the expression

$$\overline{J} = \int_{t^{(1)}}^{t^{(2)}} \overline{F} dt + \theta(\alpha)$$

becomes a function $\overline{J}(\alpha, \sigma)$. The following theorem is proved as was Theorem 4.

THEOREM 4a. *Under the assumption that the end points of g are not conjugate for any $\sigma \leqq 0$, let $\overline{J}(\alpha, \sigma)$ represent the value of $\overline{J}$ taken along the extremal determined by (α) for any $\sigma \leqq 0$. Then the terms of second order of $\overline{J}(\alpha, \sigma)$ have the form*

$$H(u, \sigma) \equiv \overline{J}_{\alpha_h \alpha_k}(0, \sigma) u_h u_k = b_{hk} u_h u_k + 2 \int_{t^{(1)}}^{t^{(2)}} \Omega(\eta, \dot{\eta}, \sigma) dt \quad (h, k = 1, \cdots, r). \tag{4.15}$$

For $\sigma<0$, (η) is taken along the secondary extremal determined by (u) through (4.10), *while for $\sigma = 0$, (η) may be taken along any secondary extremal with end points determined by (u) through* (4.10).

5. Sufficient conditions for a minimum. Consider the expression

$$I(\eta, u, \sigma) = b_{hk}u_hu_k + 2\int_{t^{(1)}}^{t^{(2)}} \Omega(\eta, \dot\eta, \sigma)dt \quad (h, k = 1, \cdots, r).$$

By an admissible set (u, η) will be meant a set of constants (u) and a set of functions (η) of class D' which together satisfy (3.9).

THEOREM 5. *For sufficiently large negative values of σ, the expression $I(\eta, u, \sigma)$ is positive for all admissible sets $(u, \eta)\neq(0, 0)$.*

First we note that since $\|x_{ih}\|$ is of rank r, equations (3.9) can be solved for u_h in terms of a subset of the variables η_i^s. Hence for all admissible sets (u, η)

$$I(\eta, u, \sigma) = q(\eta) + 2\int_{t^{(1)}}^{t^{(2)}} \Omega(\eta, \dot\eta, \sigma)dt \tag{5.1}$$

where $q(\eta)$ is a form quadratic in the variables η_i^s. From this it follows† that for all admissible sets (u, η)

$$I(\eta, u, \sigma) \geqq \int_{t^{(1)}}^{t^{(2)}} [2\omega(\eta, \dot\eta) + M(\eta, \dot\eta) - \sigma\eta_i\eta_i - \sigma\dot\eta_i\dot\eta_i]dt \quad (i = 1, \cdots, n), \tag{5.2}$$

where $M(\eta, \dot\eta)$ is a suitably chosen form quadratic in the variables $(\eta, \dot\eta)$ with coefficient continuous in t.

But any such form as the integrand in (5.2) can be made positive definite by making σ negative and sufficiently large, independently of t.

Thus for such a σ,

$$I(\eta, u, \sigma) > 0 \tag{5.3}$$

for all admissible sets $(u, \eta)\neq(0, 0)$.

LEMMA 3. *Let (u, η) be any admissible set. If there is no point on g conjugate to its initial point for $\sigma=\sigma_0<0$, then*

$$H(u, \sigma_0) \leqq I(\eta, u, \sigma_0), \tag{5.4}$$

the equality holding if and only if (η) is a secondary extremal for $\sigma=\sigma_0$.

By Theorem 4a the equality holds if (η) is a secondary extremal for $\sigma=\sigma_0$. If (η) is not a secondary extremal, let $(\bar\eta)$ be the secondary extremal determined by (u) for $\sigma=\sigma_0$.

We note that along any arc (η) $(t^{(1)}\leqq t\leqq t^{(2)})$,

$$\frac{\partial^2\Omega}{\partial\dot\eta_i\partial\dot\eta_j}(\eta, \dot\eta, \sigma_0)\pi_i\pi_j = \frac{\partial^2\overline{F}}{\partial\dot x_i\partial\dot x_j}\pi_i\pi_j = \frac{\partial^2 F}{\partial\dot x_i\partial\dot x_j}\pi_i\pi_j - \sigma_0\pi_i\pi_i \qquad (i, j = 1, \cdots, n),$$

† See Morse, p. 534, loc. cit.

which, by the Legendre condition, is positive for all $(\pi)\neq(0)$. Also, if we use Taylor's formula we see that the Weierstrass E-function

$$\Omega(\eta, \bar{\eta}', \sigma_0) - \Omega(\eta, \dot{\eta}, \sigma_0) - \frac{\partial\Omega}{\partial\dot{\eta}_i}(\eta, \dot{\eta}, \sigma_0)(\bar{\eta}_i' - \eta_i) \qquad (i = 1, \cdots, n)$$

is equal to

$$\frac{\partial^2\Omega}{\partial\dot{\eta}_i\partial\dot{\eta}_j}(\eta, \dot{\eta}, \sigma_0)(\bar{\eta}_i' - \dot{\eta}_i)(\bar{\eta}_j' - \dot{\eta}_j) \qquad (i, j = 1, \cdots, n)$$

and so is positive along any arc (η) and for all $(\bar{\eta}')\neq(\dot{\eta})$.

These facts, together with the hypothesis that there is no point on g conjugate to its initial point for $\sigma=\sigma_0$, enable us to infer that the secondary extremal $(\bar{\eta})$ minimizes $I(\eta, u, \sigma_0)$ in the fixed end point problem; that is, $I(\bar{\eta}, u, \sigma_0)<I(\eta, u, \sigma_0)$. The lemma follows from Theorem 4a.

LEMMA 4. *If $I(\eta, u, \sigma_0)$ $(\sigma_0<0)$ is positive for all admissible sets $(u, \eta)\neq(0,0)$, then there is no pair of conjugate points on g for $\sigma=\sigma_0$.*

For if t_2 were conjugate to t_1 on g for $\sigma=\sigma_0$, there would exist a secondary extremal $(\bar{\eta})\not\equiv(0)$ vanishing at t_1 and t_2. Then $I(\eta, u, \sigma_0)$ would be zero if evaluated for $(u)=(0)$ and for (η) taken along the broken secondary extremal consisting of $(\bar{\eta})$ in the interval t_1t_2 and the t-axis in the remainder (if any) of the interval $t^{(1)}t^{(2)}$. This is contrary to hypothesis.

LEMMA 5. *If there is no point on g conjugate to its initial point for $\sigma=\sigma_0<0$, then there is no point on g conjugate to its initial point for σ in the neighborhood of σ_0.*

For each $\sigma<0$, the points conjugate to $t=t^{(1)}$ are defined by the zeros $t\neq t^{(1)}$ of the determinant $D(t, \sigma)=|\eta_{ij}(t, \sigma)|$, where $\|\eta_{ij}(t, \sigma)\|$ is a matrix each column of which represents a secondary extremal for $\sigma=\sigma$, and which satisfies the conditions

$$\|\eta_{ij}(t^{(1)}, \sigma)\| = \|0\|, \|\dot{\eta}_{ij}(t^{(1)}, \sigma)\| = \delta_i^j \ (i, j = 1, \cdots, n; \delta_i^j = \text{Kronecker delta}).$$

Now by means of the integral Law of the Mean, the function $\eta_{ij}(t, \sigma)$ can be expressed in the form

$$\begin{aligned}\eta_{ij}(t, \sigma) &= (t - t^{(1)})\int_0^1 \dot{\eta}_{ij}[t + \theta(t - t^{(1)}), \sigma]d\theta \ (i, j = 1, \cdots, n)\\ &= (t - t^{(1)})a_{ij}(t, \sigma),\end{aligned}$$

where $a_{ij}(t, \sigma)$ is continuous for $t^{(1)}\leqq t\leqq t^{(2)}$ and $\sigma<0$, and where

$$\|a_{ij}(t^{(1)}, \sigma)\| = \|\dot{\eta}_{ij}(t^{(1)}, \sigma)\| = \delta_i^j \qquad (i, j = 1, \cdots, n).$$

Thus

$$D(t, \sigma) = (t - t^{(1)})^n \left| a_{ij}(t, \sigma) \right| .$$

Since $D(t, \sigma_0) \neq 0$ for $t^{(1)} < t \leqq t^{(2)}$ by hypothesis, we see that $|a_{ij}(t, \sigma_0)| \neq 0$ for $t^{(1)} \leqq t \leqq t^{(2)}$. It follows from the continuity of $a_{ij}(t, \sigma)$ that $|a_{ij}(t, \sigma)|$ is $\neq 0$ in the interval $t^{(1)} \leqq t \leqq t^{(2)}$ for σ near σ_0. Hence $D(t, \sigma) \neq 0$ for σ near σ_0 in the interval $t^{(1)} < t \leqq t^{(2)}$, and the theorem is proved.

THEOREM 6. *If there exist no negative characteristic roots, then $I(\eta, u, 0) \geqq 0$ for all admissible sets (u, η).*

For σ negative and sufficiently large, $I(\eta, u, \sigma)$ is, by Theorem 5, positive for all admissible sets $(u, \eta) \neq (0, 0)$. Suppose we now increase σ towards zero. Then $I(\eta, u, \sigma)$ either remains positive for $\sigma < 0$ and for all admissible sets $(u, \eta) \neq (0, 0)$, or else there is a least upper bound $\sigma_0 < 0$ of the values of σ for which $I(\eta, u, \sigma)$ is positive for all admissible sets $(u, \eta) \neq (0, 0)$. We shall show that the latter case is impossible.

Suppose there does exist such a least upper bound σ_0. Then either $I(\eta, u, \sigma_0)$ is positive for all admissible sets $(u, \eta) \neq (0, 0)$, or else $I(\eta, u, \sigma_0)$ is zero for some such sets. If $I(\eta, u, \sigma_0)$ is zero for an admissible set $(\bar{u}, \bar{\eta}) \neq (0, 0)$ then $(\bar{u}, \bar{\eta})$ must minimize $I(\eta, u, \sigma_0)$ among admissible sets (u, η). Hence $(\bar{\eta})$ must be a secondary extremal for $\sigma = \sigma_0$ satisfying (3.8) and (3.9), contrary to the hypothesis that there exist no negative characteristic roots. Thus $I(\eta, u, \sigma_0)$ must be positive for all admissible sets $(u, \eta) \neq (0, 0)$.

Lemma 4 then enables us to set up the quadratic form $H(u, \sigma_0)$, which must be positive definite. By Lemma 5, we can set up $H(u, \sigma)$ for σ slightly greater than σ_0, and it must be positive definite for σ slightly greater than σ_0. By Lemma 3, $I(\eta, u, \sigma)$ must then be positive for all admissible sets $(u, \eta) \neq (0, 0)$ for σ slightly greater than σ_0. This contradicts the hypothesis that σ_0 is the least upper bound of the values of σ for which $I(\eta, u, \sigma)$ is positive for all admissible sets $(u, \eta) \neq (0, 0)$.

We conclude, then, that $I(\eta, u, \sigma)$ is positive for all $\sigma < 0$ and for all admissible sets $(u, \eta) \neq (0, 0)$.

It follows, then, that $I(\eta, u, 0) \geqq 0$ for all admissible sets (u, η).

A set of functions (η) will be called *tangential* if they are of the form

$$\eta_i = \rho(t) \bar{x}_i'(t) \tag{5.5}$$

where $\rho(t)$ is any function of class D'.

LEMMA 6. *A set of tangential functions of class C'' represents a secondary extremal for $\sigma = 0$.*

The one-parameter family

$$x_i = \bar{x}_i[t + e\rho(t)] \qquad (i = 1, \cdots, n),$$

where $\rho(t)$ is any function of class C'', is certainly a family of extremals, for its members are simply different representations of the same extremal g. Hence the variations $\eta_i(t)$ of this family represent a secondary extremal. But for this family

$$\eta_i = \bar{x}_i'(t)\rho(t) \qquad (i = 1, \cdots, n)$$

and so the lemma is proved.

Such a secondary extremal we shall call *tangential*.

LEMMA 7. *A tangential secondary extremal (not the t-axis) vanishing at $t^{(1)}$ and $t^{(2)}$ is a characteristic solution for $\sigma = 0$.*

That such a secondary extremal satisfies (3.8) for $\sigma = 0$, $(u) = (0)$, follows from the relation

$$\frac{\partial^2 F}{\partial \dot{x}_i \partial \dot{x}_j} \bar{x}_j' = 0 \qquad (i, j = 1, \cdots, n).$$

THEOREM 7. *If there exist no negative characteristic roots, and no non-tangential characteristic solutions for $\sigma = 0$, there is no point on g conjugate to its initial point for $\sigma = 0$.*

In the first place, $t^{(2)}$ cannot be conjugate to $t^{(1)}$ on g. For if it were, there would be a normal* secondary extremal $(\bar{\eta}) \not\equiv (0)$ vanishing at $t^{(2)}$ and $t^{(1)}$.† This curve $(\bar{\eta})$, with the set $(u) = (0)$, would make $I(\eta, u, 0)$ vanish. Now by Theorem 6, $I(\eta, u, 0)$ is positive or zero for all admissible sets (u, η) and so $(\bar{\eta})$ with the set $(u) = (0)$ would minimize $I(\eta, u, 0)$ among admissible sets (u, η). Hence $(\bar{\eta})$ would have to satisfy conditions (3.8) and so be a characteristic solution for $\sigma = 0$. Since $(\bar{\eta})$ is non-tangential, this is contrary to hypothesis.

Next suppose that $\bar{t} \neq t^{(2)}$ were conjugate to $t^{(1)}$ on g. Then there would exist a normal secondary extremal $(\bar{\eta}) \not\equiv (0)$ vanishing at $t^{(1)}$ and $\bar{t}$. The expression $I(\eta, u, 0)$ would be zero if evaluated along the broken secondary extremal (η) consisting of $(\bar{\eta})$ in the interval $t^{(1)}\bar{t}$ and of the t-axis in the remainder of the interval $t^{(1)}t^{(2)}$. The curve (η) would actually have a corner at $\bar{t}$, because the only normal secondary extremal through a point on the t-axis in the direction of the t-axis is $(\eta) \equiv (0)$.‡

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* A *normal* secondary extremal is one which satisfies the relation

$$\bar{x}_i' \eta_i = 0 \qquad (i = 1, \cdots, n).$$

† See Bliss, loc. cit., p. 200.

‡ See Bliss, loc. cit., p. 199.

The arc (η) with the set $(u)=(0)$ would minimize $I(\eta, u, 0)$ among admissible sets (u, η), and so would have to satisfy the corner conditions

$$\left[\frac{\partial\omega}{\partial\dot{\eta}_i}\right]_{\bar{t}-}^{\bar{t}+} \equiv \frac{\partial^2 F}{\partial\dot{x}_i\partial\dot{x}_j}[\dot{\eta}_j]_{\bar{t}-}^{\bar{t}+} = 0 \qquad (i, j = 1, \cdots, n).$$

From this it would follow, due to the actual presence of a corner at $\bar{t}$, that

$$[\dot{\eta}_j]_{\bar{t}-}^{\bar{t}+} = k\bar{x}_j'(t) \qquad k \neq 0 \quad (j = 1, \cdots, n). \tag{5.6}$$

Hence

$$\bar{\eta}_j'(\bar{t}) = -k\bar{x}_j'(\bar{t}) \qquad (j = 1, \cdots, n). \tag{5.7}$$

But this is impossible; for along the normal secondary extremal $(\bar{\eta})$ we have

$$\bar{x}_j'\bar{\eta}_j = 0 \qquad (j = 1, \cdots, n), \tag{5.8}$$

and hence, by differentiation,

$$\bar{x}_j'(\bar{t})\bar{\eta}_j'(\bar{t}) = 0 \qquad (j = 1, \cdots, n), \tag{5.9}$$

which contradicts (5.7).

Thus there is no point on g conjugate to $t^{(1)}$.

We come now to the final theorem. The arc g and the set $(\alpha)=(0)$ shall be said to furnish a proper, strong, relative minimum to J if there exist a neighborhood N of g and a neighborhood M of $(\alpha)=(0)$ such that the value of J is less when evaluated for g and $(\alpha)=(0)$ than when evaluated for any other admissible arc in N with ends determined by a set (α) in M.

THEOREM 8. *In order that the extremal g, without multiple points, and the set* $(\alpha)=(0)$ *afford a proper strong relative minimum to J it is sufficient that the transversality conditions* (2.6) *be satisfied, that the Legendre and Weierstrass sufficient conditions hold, that there be no negative characteristic roots, and that there be no characteristic solutions for* $\sigma=0$ *except the tangential solutions vanishing at both ends.*

Under the hypotheses of this theorem, Theorem 7 tells us that the end points of g are not conjugate, and so we can set up the function $\bar{J}(\alpha, 0)$, and hence the quadratic form $H(u, 0)$. According to Theorem 4, $H(u, 0)$ is equal to $I(\eta, u, 0)$, where (η) is any secondary extremal with ends determined by (u) through (3.9). By Theorem 6, $H(u, 0)\geqq 0$.

Now if $H(u, 0)$ were 0 for some $(u)\neq(0)$, then $I(\eta, u, 0)$ would be zero if evaluated for (u) and any secondary extremal $(\bar{\eta})$ with ends determined by (u). Hence $(\bar{\eta})$ would minimize $I(\eta, u, 0)$ and so would satisfy (3.8) and be a characteristic solution for $\sigma=0$ not vanishing at both ends. This contradicts the hypotheses. Thus $H(u, 0)$ is positive definite.

Now the Legendre and Weierstrass sufficient conditions are assumed to hold along g. Also, by Theorem 7, there is no point on g conjugate to its initial point. Hence g furnishes a minimum to J in the fixed end point problem. Furthermore, there exists a neighborhood N of g such that if an extremal E determined by a set (α) lies in N, then, if (α) is sufficiently near (0), E will afford a minimum to J in the fixed end point problem, with respect to admissible arcs in N joining the end points of E.*

Let g' be any admissible arc in N, its end points being given by a certain set (α). Then if (α) is near enough to (0) the extremal determined by (α) will lie in N, and

$$J_{g',\alpha} \geqq J(\alpha). \tag{5.10}$$

But $H(\alpha, 0)$ gives the terms of second order in $J(\alpha)$, so that for (α) sufficiently near (0) we have

$$J(\alpha) \geqq J(0), \tag{5.11}$$

the equality holding only if $(\alpha) = (0)$.

Hence for g' sufficiently near g and (α) sufficiently near (0),

$$J_{g',\alpha} \geqq J(0). \tag{5.12}$$

This inequality becomes an equality only if g' is identical with g.

Thus the theorem is proved.

6. **The fixed end point problem.** This is the case that $r=0$ and $\theta=$ constant, the end conditions being

$$x_i^s = \text{constant} \qquad (i = 1, \cdots, n;\ s = 1, 2).$$

The expression $I(\eta, u, \sigma)$ is replaced by

$$I(\eta, \sigma) = 2\int_{t^{(1)}}^{t^{(2)}} \Omega dt,$$

and the accessory boundary problem has the form

$$\frac{d}{dt}\left[\frac{\partial\Omega}{\partial\dot{\eta}_i}\right] - \frac{\partial\Omega}{\partial\eta_i} = 0 \qquad (i = 1, \cdots, n),$$

$$\eta_i^s = 0 \qquad (i = 1, \cdots, n;\ s = 1, 2).$$

The necessary condition of Theorem 3 holds as stated.

To prove Theorem 8 in the fixed end point case, we shall prove that under the hypotheses of the theorem there is no point on g conjugate to its

* Cf. Morse, loc. cit., p. 535, and Bliss, Annals of Mathematics, April, 1932, p. 267, Lemma 1.

initial point for $\sigma=0$. This will follow if we can prove Theorem 7, which in turn is based on Theorem 6.

The first two paragraphs in the proof of Theorem 6 hold as before. Next, Lemma 4 shows us that there is no point on g conjugate to its initial point for $\sigma=\sigma_0$, and Lemma 5 extends this property to values of σ slightly greater than σ_0. Hence $(\eta)\equiv(0)$ furnishes a proper minimum to $I(\eta, \sigma)$ (see proof of Lemma 3) for these values of σ, and so $I(\eta, \sigma)>0$ for these values of σ for $(\eta)\not\equiv(0)$. This contradicts the hypothesis that σ_0 was the least upper bound of the values of σ for which $I(\eta, \sigma)$ is positive for all admissible sets $(\eta)\not\equiv(0)$. Theorem 6 follows, and hence Theorems 7 and 8.

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