THE GEOMETRY OF RIEMANNIAN SPACES*

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The primary purpose of this paper is to expose, in as simple and clear a form as is possible, the fundamentals of the geometric structure of a Riemannian space.

It is a general truth that the methods which pierce most deeply into the heart of a geometric theory are invariant methods, that is, methods which are independent of the choice of the coordinates in terms of which the theory is expressed analytically. In the case of Riemannian geometry, these are the methods of tensor analysis.

As important, perhaps, as the use of invariant methods is the expression of the analytic theory, so far as possible, in terms of invariant quantities alone. For it is in this form that the theory becomes most illuminating and suggestive. But, in ordinary tensor analysis, the components of a tensor are not invariants. A first step toward our goal will be, then, to introduce for Riemannian geometry an intrinsic tensor analysis, that is, a form of tensor analysis in which the components of all tensors are invariants.

Any theory of the geometry of a Riemannian space presupposes that the space is referred to a certain ennuple of congruences of curves. In the ordinary theory, this ennuple consists of the parametric curves. In the intrinsic theory, it is an ennuple E, whose choice, as will presently be evident, is entirely arbitrary.

The ordinary components of a tensor, that is, the components in the ordinary theory, are referred to the differentials of the coordinates x^i pertaining to the ennuple of the parametric curves. The intrinsic components are referred to the differentials of arc, ds^i, of the curves of the ennuple E.

There is no need in the intrinsic theory of actual coordinates pertaining to the ennuple E; the differentials of arc ds^i suffice. Accordingly, E can be chosen arbitrarily; it does not have to be a parametric ennuple, that is, an ennuple with which coordinates can be associated. It may be, and we shall ordinarily take it to be, an ennuple of general type, and hence, of course, not necessarily orthogonal.

Intrinsic components, referred to E, of the covariant derivative of a tensor are made possible by the introduction of invariant Christoffel symbols in

* An invited paper, presented to the Society, June 21, 1933, under a different title; received by the editors February 12, 1934.

place of the ordinary ones and the use of directional differentiation along the curves of E instead of partial differentiation with respect to the coordinates x^i.

It is perhaps well to emphasize the fact that we are not introducing a new tensor analysis or a new covariant differentiation, but simply new components for the usual tensors and their customary covariant derivatives.*

The fact that second directional derivatives are not, in general, independent of the order of differentiation has two consequences. On the one hand, it necessitates for directional differentiation conditions of integrability involving a set of invariants, $B_{ij}{}^k$, depending on three indices. On the other hand, it implies that the invariant Christoffel symbols, for example, those of the second kind, $C_{ij}{}^k$, are not symmetric in i and j. Actually, it turns out that $C_{ij}{}^k - C_{ji}{}^k = B_{ij}{}^k$.

In a previous paper,† the author discussed and compared two concepts bearing on two families of curves on a two-dimensional surface, namely, the concept of distantial spread, a measure of the deviation from equidistance of one of the families of curves with respect to the other, and the concept of angular spread, or associate curvature, a measure of the deviation from parallelism, in the sense of Levi-Civita, of the one family with respect to the other. These concepts, when generalized so as to apply to two congruences of curves in Riemannian space, give rise to a "distantial spread vector" of the two congruences, taken in a given order, and an "angular spread vector" of each

* The intrinsic absolute calculus which we employ may be described, with reference to the literature, from two different points of view. In the first place, it is a generalization to the case of an arbitrary ennuple of the intrinsic absolute calculus with respect to an orthogonal ennuple which gradually grew out of Ricci's theory of an orthogonal ennuple. The invariants associated with a tensor with respect to the orthogonal ennuple came to be known as the orthogonal coordinates of the tensor with respect to the ennuple and corresponding components of the covariant derivative of a tensor, based on Ricci's coefficients of rotation, were eventually introduced. The first complete account of this intrinsic absolute calculus with respect to an orthogonal ennuple seems to be in Berwald, *Differentialinvarianten in der Geometrie, Riemannsche Mannigfaltigkeiten und ihre Verallgemeinerungen*, Encyklopädie der Mathematischen Wissenschaften, III, D, 11 (1923), pp. 141–143.

From another point of view, our intrinsic absolute calculus may be described as the result of employing the arcs of the curves of the arbitrary ennuple E as so-called nonholonomic parameters. Though he does not use the term, G. Hessenberg, in his *Vektorielle Begründung der Differentialgeometrie*, Mathematische Annalen, vol. 78 (1917), pp. 187–217, appears to be the first to employ the method of nonholonomic parameters. The Pfaffians in his theory are not differentials of arc, whereas it is essential for our purpose that they should be. Of recent years, nonholonomic parameters have been used by Cartan, Schouten, Vranceanu, Horák, and others, particularly in the study of affine and more general connections and of nonholonomic manifolds. For the formal results in the case of a general linear connection, see Z. Horák, *Die Formeln für allgemeine lineare Übertragung bei Benutzung von nichtholonomen Parametern*, Nieuw Archief, vol. 15 (1928), pp. 193–201.

† Graustein, *Parallelism and equidistance in classical differential geometry*, these Transactions, vol. 34 (1932), pp. 557–593.

congruence with respect to the other. The latter vector is the same as the associate curvature vector of Bianchi and becomes, when the two congruences are identical, the curvature vector of the single congruence.

The intrinsic contravariant components of the distantial spread vector of the ith and jth congruences of the ennuple E are precisely $B_{ij}{}^k$, and those of the angular spread vector of the ith congruence with respect to the jth are $C_{ij}{}^k$. Thus, the intrinsic Christoffel symbols and the invariants $B_{ij}{}^k$ have geometric meanings of the first order of importance.

The results thus far described are given in §§1–5. In §6 are to be found applications of distantial spread vectors to questions of equidistance and to the problem of the inclusion of r linearly independent congruences of curves in a family of r-dimensional surfaces. Thereby further geometrical interpretations of the invariants $B_{ij}{}^k$ are obtained.

In §7 special types of ennuples are discussed: parametric ennuples; particular parametric ennuples, designated as ennuples of Tchebycheff and characterized by the fact that the differentials of arc ds^i are all exact; and, finally, Cartesian ennuples, that is, Tchebycheff ennuples whose angles are all constant. The next section treats of ennuples Cartesian at a point and their relationship to coordinates geodesic at a point.

A digression is made in §10 to apply the methods previously developed to spaces with general metric connections. Geometric interpretations of the intrinsic components of the tensor of torsion are found, in terms of torsion vectors closely allied to the torsion vector of Cartan, and relations between these torsion vectors and the distantial and angular spread vectors are established. Application of the results is made to spaces admitting absolute parallelism.

In §11 the transformation from one ennuple of congruences to a second is discussed. Of course, the transformation from the intrinsic components of a tensor, referred to the one ennuple, to those of the same tensor, referred to the other ennuple, is found to obey the formal laws of tensor analysis. Moreover, the relations between the intrinsic Christoffel symbols for the two ennuples are patterned precisely after the equations of Christoffel. But these Christoffel symbols may be interpreted in terms of the curvature and associate curvature vectors of the congruences of the two ennuples, as already noted. Thus, Christoffel's equations, expressed in terms of invariants, are simply a generalization to Riemannian geometry of the fundamental formula of Liouville for geodesic curvatures on a two-dimensional surface.

In §12 the general problem of the determination of the family of surfaces of lowest dimensionality in which lie all the congruences of an arbitrarily chosen set of congruences of curves is discussed. The problem is, of course, identical with that of the determination of the maximum number of functionally independent integrals of a system of linear homogeneous partial differential equations of the first order. It is considered here from a geometric point of view and is shown to depend, for its solution, on the consideration of a sequence of sets of vectors such that the vectors of each set are distantial spread vectors of congruences determined by the vectors of the preceding sets. Application of the results is made to nonholonomic manifolds.

1. **Oblique ennuple of congruences.** Let there be given in a Riemannian space V_n, referred to coordinates $(x^1, x^2, \cdots, x^n)$, an arbitrarily chosen ennuple, E, consisting of n ordered linearly independent congruences of directed curves, and let the contravariant components of the field of unit vectors tangent to the curves C_i of the ith congruence, and directed in the same senses as these curves, be $\bar{a}_{i|}{}^j, j=1, 2, \cdots, n$.

Suppose that $\bar{a}^i{}_j$ is the cofactor of $\bar{a}_{i|}{}^j$ in the determinant $|\bar{a}_{i|}{}^j|$, divided by the determinant: $\bar{a}_{k|}{}^j\bar{a}^i{}_j=\delta^i_k$, $\bar{a}_{j|}{}^i\bar{a}^j{}_k=\delta^i_k$. Then $\bar{a}^i{}_j$, for i fixed and $j=1, 2, \cdots, n$, are the covariant components of a vector-field, or, more simply, a vector, which is perpendicular to the tangent vectors of all n congruences except that of the curves C_i. The n vectors thus determined are known as the vectors conjugate (or reciprocal) to the n unit vectors tangent to the curves of E.

If $\partial/\partial s^i$ denotes directional differentiation in the positive direction of an arbitrary curve C_i,

$$\bar{a}_{i|}{}^j = \frac{\partial x^j}{\partial s^i}. \tag{1a}$$

Suppose that we write also, purely as a matter of notation,

$$\bar{a}^i{}_j = \frac{\partial s^i}{\partial x^j}. \tag{1b}$$

Then the relations between $\bar{a}_{i|}{}^j$ and $\bar{a}^i{}_j$ become

$$\frac{\partial x^j}{\partial s^k}\frac{\partial s^i}{\partial x^j} = \delta^i_k, \quad \frac{\partial x^i}{\partial s^j}\frac{\partial s^j}{\partial x^k} = \delta^i_k \quad (i, k = 1, 2, \cdots, n). \tag{2}$$

The first set of these equations says that the Pfaffian $\bar{a}^i{}_j dx^j$ has the value zero for every curve of E except a curve C_i and for a curve C_i is equal to the differential of arc of the curve, measured in the positive direction along it. Thus, the relations between the differentials of arc ds^i of the curves C_i and the differentials dx^i are

$$ds^i = \frac{\partial s^i}{\partial x^j}dx^j, \quad dx^i = \frac{\partial x^i}{\partial s^j}ds^j \quad (i = 1, 2, \cdots, n). \tag{3}$$

The relations between the directional derivatives $\partial/\partial s^i$ and the partial derivatives $\partial/\partial x^i$ are obviously

$$(4)\qquad \frac{\partial f}{\partial s^i} = \frac{\partial x^j}{\partial s^i}\frac{\partial f}{\partial x^j}, \qquad \frac{\partial f}{\partial x^i} = \frac{\partial s^j}{\partial x^i}\frac{\partial f}{\partial s^j} \qquad (i = 1, 2, \cdots, n).$$

From (2), (3), and (4) it follows that

$$(5)\qquad df = \frac{\partial f}{\partial s^i} ds^i.$$

Conditions of integrability. The fundamental relations

$$\frac{\partial}{\partial x^q}\frac{\partial f}{\partial x^p} - \frac{\partial}{\partial x^p}\frac{\partial f}{\partial x^q} = 0 \qquad (p, q = 1, 2, \cdots, n),$$

when expressed in terms of directional derivatives, take the form

$$(6)\qquad \frac{\partial}{\partial s^j}\frac{\partial f}{\partial s^i} - \frac{\partial}{\partial s^i}\frac{\partial f}{\partial s^j} = B_{ij}{}^k \frac{\partial f}{\partial s^k} \qquad (i, j = 1, 2, \cdots, n),$$

where

$$(7a)\qquad B_{ij}{}^k = \frac{\partial x^r}{\partial s^j}\frac{\partial}{\partial s^i}\frac{\partial s^k}{\partial x^r} - \frac{\partial s^r}{\partial s^i}\frac{\partial}{\partial s^j}\frac{\partial s^k}{\partial x^r},$$

or

$$(7b)\qquad B_{ij}{}^k = \frac{\partial s^k}{\partial x^r}\left(\frac{\partial}{\partial s^j}\frac{\partial x^r}{\partial s^i} - \frac{\partial}{\partial s^i}\frac{\partial x^r}{\partial s^j}\right).$$

The expression in (7b) follows from that in (7a) by virtue of the relations obtained by directional differentiation of the first set of equations (2).

THEOREM 1. *A necessary and sufficient condition that $f_i ds^i$, where $f_i = f_i(x^1, x^2, \cdots, x^n)$, be an exact differential is that*

$$(8)\qquad \frac{\partial f_i}{\partial s^j} - \frac{\partial f_j}{\partial s^i} = B_{ij}{}^k f_k \qquad (i, j = 1, 2, \cdots, n).$$

The theorem follows directly from (6) inasmuch as, according to (5), $f_i ds^i$ is an exact differential if and only if there exists a function f such that $\partial f/\partial s^i = f_i$, $i = 1, 2, \cdots, n$.

2. **Intrinsic tensor analysis.** The geometric basis, or system of reference, for ordinary tensor analysis is the system of parametric hypersurfaces $x^i = c_i$, $i = 1, 2, \cdots, n$, or the corresponding ennuple of congruences of parametric curves. This ennuple is evidently of very special type.

As the system of reference for our intrinsic tensor analysis, we take the arbitrarily chosen ennuple E of the preceding section.

In the ordinary theory, the basic differentials are the differentials dx^i and the basic derivatives are the partial derivatives $\partial/\partial x^i$. In the intrinsic theory it is the differentials of arc, ds^i, and the directional derivatives, $\partial/\partial s^i$, which are fundamental.

Whereas the ordinary components of a tensor, that is, the components in the ordinary theory, are referred to dx^i and $\partial/\partial x^i$, the intrinsic components are to be referred to ds^i and $\partial/\partial s^i$. For example, if $\bar{b}_{ij}^{kl}$ are the ordinary components of a tensor of the fourth order, that is, if $\bar{b}_{ij}^{kl}dx^i\delta x^j(\partial f/\partial s^k)(\partial\phi/\partial s^l)$, where f and ϕ are invariant functions, is an invariant, the intrinsic components, b_{ij}^{kl}, of the tensor are to be such that $b_{ij}^{kl}ds^i\delta s^j(\partial f/\partial s^k)(\partial\phi/\partial s^l)$ is the new form of this invariant:

$$\bar{b}_{ij}^{kl}dx^i\delta x^j\frac{\partial f}{\partial x^k}\frac{\partial\phi}{\partial x^l} = b_{ij}^{kl}ds^i\delta s^j\frac{\partial f}{\partial s^k}\frac{\partial\phi}{\partial s^l}. \tag{9}$$

To obtain the transformation from the ordinary to the intrinsic components of a tensor, we should substitute for $dx^i, \delta x^i, \cdots, \partial f/\partial x^i, \partial\phi/\partial x^i, \cdots$ in an equation such as (9) their values in terms of $ds^i, \delta s^i, \cdots, \partial f/\partial s^i, \partial\phi/\partial s^i, \cdots$, as given by (3) and (4). But equations such as (9) and the transformations (3) and (4) have the same form as the analogous equations and transformations associated with a change from the coordinates x^i to new coordinates y^i. Hence, *the transformation from the ordinary to the intrinsic components of a tensor obeys the standard formal laws of tensor analysis.* If, in the transformation of the components of a tensor which is the result of a change from the coordinates x^i to coordinates y^i, y^i is replaced by s^i, the transformation becomes that from the ordinary components to the intrinsic components.

Thus, if $\bar{g}_{ij}$ and $\bar{g}^{ij}$ are the ordinary covariant and contravariant components, and g_{ij} and g^{ij} the corresponding intrinsic components, of the fundamental tensor, we have

$$g_{ij} = \bar{g}_{kl}\frac{\partial x^k}{\partial s^i}\frac{\partial x^l}{\partial s^j}, \qquad \bar{g}_{ij} = g_{kl}\frac{\partial s^k}{\partial x^i}\frac{\partial s^l}{\partial x^j}, \tag{10a}$$

$$g^{ij} = \bar{g}^{kl}\frac{\partial s^i}{\partial x^k}\frac{\partial s^j}{\partial x^l}, \qquad \bar{g}^{ij} = g^{kl}\frac{\partial x^i}{\partial s^k}\frac{\partial x^j}{\partial s^l}. \tag{10b}$$

The invariant form of the linear element, $ds^2=\bar{g}_{ij}\,dx^i\,dx^j$, is

$$ds^2 = g_{ij}ds^ids^j. \tag{11}$$

Since $\bar{g}_{ij}$ and $\bar{g}^{ij}$ are symmetric, so also are g_{ij} and g^{ij}.

Again, the relations between the ordinary contravariant and covariant components, $\bar{a}^i$ and $\bar{a}_i$, of a vector and the corresponding intrinsic components, a^i and a_i, are

$$(12a)\qquad a^i = \bar{a}^j\frac{\partial s^i}{\partial x^j}, \qquad \bar{a}^i = a^j\frac{\partial x^i}{\partial s^j},$$

$$(12b)\qquad a_i = \bar{a}_j\frac{\partial x^j}{\partial s^i}, \qquad \bar{a}_i = a_j\frac{\partial s^j}{\partial x^i}.$$

Inasmuch as $\partial x^j/\partial s^i$ and $\partial s^i/\partial x^j$, for i fixed and $j=1, 2, \cdots, n$, are respectively the contravariant and covariant components of vectors, namely, the ith tangent and the ith conjugate vector associated with E, it follows from equations such as the first sets in (10) and (12) that *the intrinsic components of a tensor are actually invariants.**

Components of the vectors pertaining to E. If $\bar{a}_h|^i$ and $\bar{a}_h|_i$ are the ordinary contravariant and covariant components, and $a_h|^i$ and $a_h|_i$ the corresponding intrinsic components, of the field of unit vectors tangent to the curves C_h of the hth congruence of E, we have

$$(13a)\qquad \bar{a}_h|^i = \frac{\partial x^i}{\partial s^h}, \qquad \bar{a}_h|_i = g_{hj}\frac{\partial s^j}{\partial x^i},$$

$$(13b)\qquad a_h|^i = \delta_h^i, \qquad a_h|_i = g_{hi}.$$

Formulas (13b) follow from (13a) by means of (12) and (2), and the second equation in (13a) follows from the first by virtue of (10a) and (2).

Denoting the ordinary covariant and contravariant components of the hth conjugate vector-field by $\bar{a}^h|_i$ and $\bar{a}^h|^i$, and the corresponding intrinsic components by $a^h|_i$ and $a^h|^i$, we have†

$$(14a)\qquad \bar{a}^h|_i = \frac{\partial s^h}{\partial x^i}, \qquad \bar{a}^h|^i = g^{hj}\frac{\partial x^i}{\partial s^j},$$

$$(14b)\qquad a^h|_i = \delta_i^h, \qquad a^h|^i = g^{hi}.$$

Geometric interpretations. The first of the formulas (10a) says that

$$(15)\qquad g_{ii} = 1, \quad g_{ij} = \cos\omega_{ij} \qquad (i, j = 1, 2, \cdots, n),$$

where ω_{ij} is the angle‡ at $P:(x^1, x^2, \cdots, x^n)$ between the directed curves C_i and C_j which pass through P.

* From the usual point of view, the first sets of equations in (10) and (12) define invariants pertaining to the given tensors with respect to E, and the second sets express the ordinary components of the tensors in terms of these invariants and the components of the vectors pertaining to E. See the long footnote in the introduction and, for example, Eisenhart, *Riemannian Geometry*, p. 97.

† It is to be noted, from (13a), (14a), and (11), that $\bar{a}_i|^j = \bar{a}_i{}^j$ and $\bar{a}^i|_j = \bar{a}^i{}_j$.

‡ The angle ϕ between two vectors at a point shall be restricted to lie in the interval $0 \leqq \phi \leqq \pi$.

Inasmuch as

$$\sum_i a^h|_i \, a^h|^i = \sum_i \delta^h_i \, \delta^i_h = 1,$$

we conclude that *the length of the conjugate vector* $a^h|$ *at the point* $P:(x^i)$ *is equal to* $\sec \theta_h$, *where* θ_h *is the angle at* P *between the vector* $a^h|$ *and the tangent vector* $a_h|$ *at* P. It follows that $0 \leqq \theta_h < \pi/2$.

The geometric meaning of the first of the formulas (10b) is now clear:

$$(16) \qquad g^{ii} = \sec^2 \theta_i, \quad g^{ij} = \cos \Omega_{ij} \sec \theta_i \sec \theta_j \qquad (i, j = 1, 2, \cdots, n),$$

where Ω_{ij} *is the angle at* P *between the conjugate vectors* $a^i|$ *and* $a^j|$.

We now introduce in the flat space, S_n, tangent to V_n at P, the Cartesian coordinate system with respect to which the intrinsic contravariant components X^i of an arbitrary vector V at P are the coordinates $(X^1, X^2, \cdots, X^n)$ of the "terminal point" Q of V. The axes of this system, which we shall call the intrinsic contravariant coordinate system at P, are the directed tangents to the curves of E which pass through P. Furthermore, inasmuch as the tangent vector $a_h|^i = \delta^i_h$ is of unit length, the unit of measure on each axis, relative to measurement in V_n, is actually unity.

We also introduce in S_n the intrinsic covariant coordinate system, with respect to which the intrinsic covariant components X_i of the vector V are the coordinates $(X_1, X_2, \cdots, X_n)$ of the point Q. The axes of this system coincide in direction and sense with the conjugate vectors at P. The unit of measurement on the hth axis is not unity, but $\sec \theta_h$; for the hth conjugate vector $a^h|_i = \delta^h_i$ is of length $\sec \theta_h$.

THEOREM 2. *The intrinsic covariant (contravariant) components of a vector* V *at a point* P *are, on the one hand, the orthogonal projections of* V *on the axes of the intrinsic contravariant (covariant) system of coordinates at* P, *and, on the other hand, the parallel projections of* V *on the axes of the covariant (contravariant) system of coordinates at* P.

The second part of the theorem amounts to the previous identification of the components of V as Cartesian coordinates. The first part follows from the relations $a_h|^i X_i = X_h$, $a^h|_i X^i = X^h$. In particular, we have in $a_h|_i = g_{hi}$ and $a^h|^i = g^{hi}$ new interpretations of g_{hi} and g^{hi}.

Case of an orthogonal ennuple. If each two congruences of E cut at right angles, the tangent vectors $a_h|$ form an orthogonal ennuple of vectors, the conjugate vectors $a^h|$ become unit vectors coincident with the corresponding tangent vectors, the intrinsic contravariant and covariant coordinate systems of Theorem 2 coincide in a rectangular system, parallel and orthogonal projections on the axes of this system are identical, and the intrinsic contravariant and covariant components of a vector are the same. Furthermore, inasmuch as now $g_{ii}=g^{ii}=1$, $g_{ij}=g^{ij}=0$, $i\neq j$, any two tensors which are associate to one another in that they are obtainable from one another by raising subscripts or lowering superscripts by means of the fundamental tensor, have the same components.

3. **Intrinsic covariant differentiation.** We now introduce, for the ennuple E, invariant Christoffel symbols, C_{ijk} and $C_{ij}{}^k$, to take the place of the ordinary Christoffel symbols,

$$\overline{C}_{ijk} = [ij, k], \quad \overline{C}_{ij}{}^k = \left\{ \begin{matrix} k \\ ij \end{matrix} \right\}.$$

Inasmuch as we shall assume that $C_{ijh}=g_{hk}C_{ij}{}^k$, it suffices to define $C_{ij}{}^k$. We first write the formula for the transformation of $\overline{C}_{ij}{}^k$ induced by a change from the coordinates x^i to coordinates y^i. In this formula we replace the first partial derivatives of x^i with respect to y^i by the corresponding directional derivatives and the single second partial derivative by a specific one of the two corresponding directional derivatives. Thus we get

$$C_{ij}{}^k \frac{\partial x^t}{\partial s^k} = \overline{C}_{pq}{}^t \frac{\partial x^p}{\partial s^i} \frac{\partial x^q}{\partial s^j} + \frac{\partial}{\partial s^j} \frac{\partial x^t}{\partial s^i}. \tag{17}$$

Since second directional derivatives are not, in general, independent of the order of differentiation, $C_{ij}{}^k$ is not, in general, symmetric in i and j. In fact, we have, from (17) and (7a), that

$$C_{ij}{}^k - C_{ji}{}^k = B_{ij}{}^k. \tag{18}$$

Intrinsic covariant differentiation. Formula (17) enables us to find the intrinsic components of the covariant derivative of any tensor in terms of the intrinsic components of the tensor.

Consider, for example, the vector of equations (12). The ordinary components of the covariant derivative of this vector are

$$\bar{a}_{i,j} = \frac{\partial \bar{a}_i}{\partial x^j} - \bar{a}_h \overline{C}_{ij}{}^h, \quad \bar{a}^i{}_{,j} = \frac{\partial \bar{a}^i}{\partial x^j} + \bar{a}^h \overline{C}_{hj}{}^i. \tag{19}$$

The intrinsic components, which we shall denote by $a_{i,j}$ and $a^i{}_{,j}$ respectively, are expressible in terms of the ordinary components, according to the definitions of §2, by the formulas

$$a_{i,j} = \bar{a}_{k,l} \frac{\partial x^k}{\partial s^i} \frac{\partial x^l}{\partial s^j}, \quad a^i{}_{,j} = \bar{a}^k{}_{,l} \frac{\partial s^i}{\partial x^k} \frac{\partial x^l}{\partial s^j}. \tag{20}$$

When the values of $\bar{a}_{i,j}$ and $\bar{a}^i{}_{,j}$, rewritten in terms of directional derivatives and the intrinsic components a_i and a^i of the vector, are substituted in (20), these formulas take on, by virtue of (17), the following desired forms*:

$$(21)\qquad a_{i,j} = \frac{\partial a_i}{\partial s^j} - a_h C_{ij}{}^h, \quad a^i{}_{,j} = \frac{\partial a^i}{\partial s^j} + a^h C_{hj}{}^i.$$

In comparing (21) with (19), it must be borne in mind that $C_{ij}{}^k$, unlike $\overline{C}_{ij}{}^k$, is not symmetric in the two subscripts. In (21) and, in fact, in all similar formulas for the intrinsic components of covariant derivatives of tensors, it will be found that the second subscript on the C indicates the component of the covariant derivative in question.† In the corresponding formulas for the ordinary components of covariant derivatives, the second subscript on the $\overline{C}$ may be, and usually is, made to play the same role. The two sets of formulas have, then, the same forms.

It is evident that, if the ordinary components of a tensor are all zero, so also are the intrinsic components. Thus, since $\bar{g}_{ij,k}=0$ and $\bar{g}^{ij}{}_{,k}=0$, it follows that $g_{ij,k}=0$ and $g^{ij}{}_{,k}=0$.

Similarly, it follows that, if $f_{,ij}$ are the intrinsic components of the covariant derivative of the gradient, $f_{,i}=\partial f/\partial s^i$, of an invariant function f, then $f_{,ij}=f_{,ji}$. Hence, a necessary and sufficient condition that the vector with the intrinsic covariant components a_i be the gradient of a function is that $a_{i,j}$ be a symmetric tensor.

A little consideration shows that this last proposition should be simply a restatement of Theorem 1. As a matter of fact, it is readily proved that

$$(22)\qquad a_{i,j} - a_{j,i} \equiv \frac{\partial a_i}{\partial s^j} - \frac{\partial a_j}{\partial s^i} - B_{ij}{}^k a_k.$$

Invariant form of C_{ijk}. Setting

$$(23)\qquad C_{ijk} = g_{kh}C_{ij}{}^h, \quad B_{ijk} = g_{kh}B_{ij}{}^h,$$

we get, from (18),

$$(24)\qquad C_{ijk} - C_{jik} = B_{ijk}.$$

Since $g_{ik,j}=0$, we also have

$$(25)\qquad C_{ijk} + C_{kji} = \frac{\partial g_{ik}}{\partial s^j}.$$

* We might have started with these forms, with $C_{ij}{}^k$ unknown, and then derived (17) from them.

† This corresponds to the fact that in (17) the second subscript in $C_{ij}{}^k$ indicates the second differentiation in the formation of the last term.

Equations (24) and (25) are n^3 in number and yield a unique solution for C_{ijk}, namely*

$$C_{ijk} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial s^j} + \frac{\partial g_{jk}}{\partial s^i} - \frac{\partial g_{ij}}{\partial s^k} + B_{ijk} + B_{jki} - B_{kij}\right]. \tag{26}$$

The corresponding expression for $C_{ij}{}^k$ is readily obtained.

4. Geometric interpretations of invariant Christoffel symbols. These have to do with the curvature and associate curvature vectors of the congruences of the given ennuple.

If a^i are the intrinsic contravariant components of the field of unit vectors tangent to the directed curves C of a given congruence, the vector

$$c^r = a^r{}_{,j}a^j = \left(\frac{\partial a^r}{\partial s^j} + C_{ij}{}^r a^i\right)a^j, \tag{27a}$$

is the curvature vector of the curves C and its identical vanishing is the condition that the curves C be geodesics.

The vector

$$c^r = b^r{}_{,j}a^j = \left(\frac{\partial b^r}{\partial s^j} + C_{ij}{}^r b^i\right)a^j, \tag{27b}$$

where b^i are the intrinsic components of an arbitrary field of unit vectors, is known as the associate curvature vector of this vector field with respect to the curves C. It is identically the null vector when and only when the vectors of the field are parallel, in the sense of Levi-Civita, with respect to the curves C.

If the vector-field b^i originates as the field of unit vectors tangent to the directed curves K of a certain congruence, we call the vector (27b) the associate curvature vector of the curves K with respect to the curves C and say that the curves K are parallel with respect to the curves C when and only when it is identically null.

Curvature vectors of the given congruences. For the curves C_h of the ennuple E, $a^r = a_h|^r = \delta_h^r$. Hence (27a) becomes $c^r = C_{hh}{}^r$.

* The corresponding formula in Hessenberg, loc. cit., p. 211, has the same form, though Hessenberg employs, instead of ds^i and $\partial/\partial s^i$, the Pfaffians $du^i = ds^i/\rho_i$ and the corresponding derivatives $\partial/\partial u^i = \rho_i(\partial/\partial s^i)$, where ρ_i are invariants. This is, of course, to be expected, inasmuch as equations (24) and (25) are obviously unchanged by this change of Pfaffians.

The change of Pfaffians still leaves the space referred to the ennuple E. The transformations which it effects on the fundamental quantities are readily found to be $g^*_{ij} = \rho_i\rho_j g_{ij}$ and

$$\rho_k B^*_{ij}{}^k = \rho_i\rho_j B_{ij}{}^k + \rho_j\delta_i^k\frac{\partial \rho_i}{\partial s^j} - \rho_i\delta_j^k\frac{\partial \rho_j}{\partial s^i}, \quad \rho_k C^*_{ij}{}^k = \rho_i\rho_j C_{ij}{}^k + \rho_j\delta_i^k\frac{\partial \rho_i}{\partial s^j}.$$

It will be clear from these relations, after the perusal of the next two sections, why it is essential that we use ds^i as the basic Pfaffians.

THEOREM 3. *The intrinsic components, $c_{hh}|^i$ and $c_{hh}|_i$, of the curvature vector of the curves C are respectively $C_{hh}{}^i$ and C_{hhi}:**

$$c_{hh}|^i = C_{hh}{}^i, \quad c_{hh}|_i = C_{hhi} \qquad (i = 1, 2, \cdots, n). \tag{28}$$

According to Theorem 2, C_{hhi} and $C_{hh}{}^i$ are respectively the orthogonal and parallel (parallel and orthogonal) projections of the curvature vector of the curves C_h on the ith axis of the intrinsic contravariant (covariant) coordinate system at $P:(x^i)$. Since the curvature vector of a curve is perpendicular to the curve, C_{hhh} should be zero and this is the case.

COROLLARY. *A necessary and sufficient condition that the curves C_h be geodesics is that $C_{hh}{}^i=0$ or $C_{hhi}=0$, $i=1, 2, \cdots, n$.*

Associate curvature vectors. When we set $b^r=a_h|^r$ and $a^j=a_k|^j$, (27b) becomes $c^r=C_{hk}{}^r$.

THEOREM 4. *The intrinsic components, $c_{hk}|^i$ and $c_{hk}|_i$, of the associate curvature vector of the curves C_h with respect to the curves C_k are respectively $C_{hk}{}^i$ and C_{hki}:*

$$c_{hk}|^i = C_{hk}{}^i, \quad c_{hk}|_i = C_{hki} \qquad (i = 1, 2, \cdots, n). \tag{29}$$

In the sense of Theorem 2, C_{hki} and $C_{hk}{}^i$ are respectively the orthogonal and parallel projections, on the curves C_i, of the associate curvature vector of the curves C_h with respect to the curves C_k. In particular, $C_{hkh}=0$; this vector is perpendicular to C_h.

COROLLARY. *The curves C_h are parallel with respect to the curves C_k if and only if $C_{hk}{}^i=0$ or $C_{hki}=0$, $i=1, 2, \cdots, n$.*

Geometric interpretation of intrinsic covariant differentiation. The geometric significance of the first of formulas (21) is now clear.

THEOREM 5. *The (i, j)th intrinsic component of the covariant derivative of a covariant tensor is equal to the directional derivative, along C_j, of the ith intrinsic component of the vector, minus the scalar product of the vector with the associate curvature vector of the curves C_i with respect to the curves C_j.*

In particular, the (i, j)th component reduces to $\partial a_i/\partial s^j$ if and only if the vector is always perpendicular to the associate curvature vector in question.

We conclude also: (a) for every vector a, but for a fixed i, $a_{i,j}$ reduces to $\partial a_i/\partial s^j$ for $j=1, 2, \cdots, n$ when and only when the curves C_i are parallel with respect to all the curves C_j, $j=1, 2, \cdots, n$; and (b) for every vector a,

* For the case of an orthogonal ennuple, this theorem is known; see Levi-Civita, *The Absolute Differential Calculus*, p. 275.

but for a fixed j, $a_{i,j}$ reduces to $\partial a_i/\partial s^j$ for $i=1, 2, \cdots, n$ if and only if all the curves C_i, $i=1, 2, \cdots, n$, are parallel with respect to the curves C_j. In particular, the curves C_i in case (a), and the curves C_j in case (b), must be geodesics.

From the geometric interpretation of $a_{i,j}$ we may pass to one for $a^i{}_{,j}$ by means of the relation $a^i{}_{,j}=g^{ir}a_{r,j}$.

Geodesics and parallelism. Suppose that there is given a directed curve C: $x^i=x^i(s)$, expressed parametrically in terms of the arc s, and let $\bar{a}^i$ and a^i be the ordinary and intrinsic contravariant components of the unit vector tangent to C at an arbitrary point P of C.

For the curve C, $dx^i=\bar{a}^i ds$. But $dx^i=(\partial x^i/\partial s^j)ds^j$. Thus, $(\partial x^i/\partial s^j)ds^j=\bar{a}^i ds$. Hence, since $\bar{a}^i=a^j(\partial x^i/\partial s^j)$, it follows that $ds^i=a^i ds$. For the curve C, we have, then,

$$\bar{a}^i = \frac{dx^i}{ds}, \quad a^i = \frac{ds^i}{ds} \qquad (i = 1, 2, \cdots, n). \tag{30}$$

Applying these results to the right-hand side of (27a), set equal to zero, we obtain, as the conditions that the curve C be a geodesic:

$$\frac{d^2s^r}{ds^2} + C_{ij}{}^r \frac{ds^i}{ds}\frac{ds^j}{ds} = 0 \qquad (r = 1, 2, \cdots, n),$$

where d^2s^r/ds^2 stands for $(d/ds)(ds^r/ds)$.

From (27b) we obtain similar conditions that unit vectors b in the points of the curve C be parallel with respect to the curve C.

Coefficients of rotation. The coefficients of rotation of the ennuple E, patterned after Ricci's coefficients of rotation for an orthogonal ennuple, are of two kinds, namely,

$$\gamma_{lhk} = a_l|_{i,j}a_h|^i a_k|^j, \quad \gamma^l{}_{hk} = a^l|_{i,j}a_h|^i a_k|^j.$$

It is readily shown that

$$\gamma_{lhk} = -C_{hkl} + \frac{\partial g_{hl}}{\partial s^k} = C_{lkh}, \quad \gamma^l{}_{hk} = -C_{hk}{}^l,$$

and hence, in case E is an orthogonal ennuple, that

$$\gamma_{lhk} = \gamma^l{}_{hk} = -C_{hkl} = -C_{hk}{}^l.$$

Thus, the coefficients of rotation of any ennuple E are identical, essentially, with the invariant Christoffel symbols formed for the ennuple. Theorems 3 and 4 furnish, then, simple geometric interpretations of the coefficients of rotation.

5. **The distantial spread vector.** The associate curvature vector of the congruence of curves C_h with respect to the congruence of curves C_k is based on angle. Since its length is a measure of the deviation from parallelism of the first congruence with respect to the second, we may call it, in accordance with a terminology employed in a previous paper,* the angular spread vector of the curves C_h with respect to the curves C_k.

We shall now introduce for the two congruences of curves a "spread vector" which is based on distance and which we shall call the *distantial spread vector* of the two congruences, in a given order.

Let P be an arbitrarily chosen point in V_n and let C_h and C_k be the curves of the two congruences which pass through P. Mark on C_h the point P_1 at the directed distance Δs^h from P and on C_k the point P_2 at the directed distance Δs^k from P. On the curve of the kth congruence through P_1 mark the point Q_1 at the directed distance Δs^k from P_1, and on the curve of the hth congruence through P_2 mark the point Q_2 at the directed distance Δs^h from P_2, and draw the vector $\overline{Q_1Q_2}$ joining Q_1 to Q_2. Then, the limit of the ratio $\overline{Q_1Q_2}/(\Delta s^h \Delta s^k)$, when Δs^h and Δs^k approach zero, exists and is defined as the distantial spread vector, at P, of the congruences of curves C_h and C_k, in this order.

If $\bar{b}_{hk}|^i$ are the ordinary contravariant components of this vector, the definition says that

$$\bar{b}_{hk}|^i = \lim_{\Delta s^h, \Delta s^k \to 0} \frac{w^i - z^i}{\Delta s^h \Delta s^k},$$

where (z^i) and (w^i) are respectively the coordinates of Q_1 and Q_2.

It is readily shown that

$$z^i = x^i + \frac{\partial x^i}{\partial s^h} h + \frac{\partial x^i}{\partial s^k} k + \frac{1}{2!}\left(\frac{\partial^2 x^i}{\partial s^{h2}} h^2 + 2 \frac{\partial}{\partial s^h} \frac{\partial x^i}{\partial s^k} hk + \frac{\partial^2 x^i}{\partial s^{k2}} k^2 \right) + \cdots,$$

where $h = \Delta s^h$ and $k = \Delta s^k$ and the coefficients are evaluated for P. From these coordinates for Q_1 we obtain the coordinates (w^i) of Q_2 by interchanging h and k and $\partial/\partial s^h$ and $\partial/\partial s^k$. Thus we find that

$$\bar{b}_{hk}|^i = \frac{\partial}{\partial s^k} \frac{\partial x^i}{\partial s^h} - \frac{\partial}{\partial s^h} \frac{\partial x^i}{\partial s^k}.$$

It follows from this result and (7b) that $b_{hk}|^i = B_{hk}{}^i$, where $b_{hk}|^i$ are the intrinsic components of the distantial spread vector.

THEOREM 6. *The intrinsic components, $b_{hk}|^i$ and $b_{hk}|_i$, of the distantial spread vector of the curves C_h and the curves C_k are respectively $B_{hk}{}^i$ and B_{hki}:*

$$b_{hk}|^i = B_{hk}{}^i, \quad b_{hk}|_i = B_{hki}. \tag{31}$$

* Graustein, loc. cit., p. 559.

We thus have simple geometric interpretations of the invariants B.

It is evident that $b_{kh}|^i = -b_{hk}|^i$. In particular, $b_{hh}|^i = 0$.

By virtue of (28), (29), and (31), relations (18) and (24) become

$$c_{hk}|^i - c_{kh}|^i = b_{hk}|^i, \qquad c_{hk}|_i - c_{kh}|_i = b_{hk}|_i. \tag{32}$$

Thus, the difference between the angular spread vector of the curves C_h with respect to the curves C_k and that of the curves C_k with respect to the curves C_h is equal to the distantial spread vector of the curves C_h and C_k. In particular:

THEOREM 7. *If two of the three spread vectors of two congruences are null, the third is also.*

Further interpretations of the distantial spread vector are given in §§6, 12.

Some identities. We note, for future use, the identical relations

$$\frac{\partial}{\partial s^i} B_{jk}{}^m + \frac{\partial}{\partial s^j} B_{ki}{}^m + \frac{\partial}{\partial s^k} B_{ij}{}^m = B_{ir}{}^m B_{jk}{}^r + B_{jr}{}^m B_{ki}{}^r + B_{kr}{}^m B_{ij}{}^r, \tag{33}$$

which are readily established by substituting for the B's their values from (7a) and making use of the integrability conditions (6).

The corresponding relations for the B_{ijk} are obtained from these by multiplying by g_{pm}, summing over m, and applying (25). They are

$$\frac{\partial}{\partial s^i} B_{jkp} + \frac{\partial}{\partial s^j} B_{kip} + \frac{\partial}{\partial s^k} B_{ijp} = D_{irp} B_{jk}{}^r + D_{jrp} B_{ki}{}^r + D_{krp} B_{ij}{}^r, \tag{34}$$

where

$$D_{irp} = B_{irp} + \frac{\partial g_{ip}}{\partial s^r} = C_{irp} + C_{pir}.$$

6. **Applications and interpretations of distantial spread vectors.** In this connection we shall first discuss the question of the inclusion of the curves of two or more congruences in subspaces of V_n, and show that it finds its answer in conditions on the distantial spread vectors of the congruences.

A family of r-dimensional surfaces ($r = 2, \cdots, n-1$) consists of the ∞^{n-r} r-dimensional surfaces defined by $n-r$ equations of the form $\phi^i(x^1, x^2, \cdots, x^n) = c_i$, $i = 1, 2, \cdots, n-r$, where $\phi^1, \phi^2, \cdots, \phi^{n-r}$ are functionally independent and $c_1, c_2, \cdots, c_{n-r}$ are arbitrary constants. A family of $(n-1)$-dimensional surfaces is called a family of hypersurfaces.

A family of r-dimensional surfaces is said to contain a congruence of curves, or the congruence is said to lie in it, if each surface of the family contains ∞^{r-1} curves of the congruence.

THEOREM 8. *The family of hypersurfaces* $\phi(x^1, x^2, \cdots, x^n) = \text{const.}$ *contains a given congruence of curves if and only if the directional derivative of* ϕ *along the curves of the congruences is identically zero.*

The theorem is self-evident.

THEOREM 9. *The curves of r linearly independent congruences of curves lie in a family of r-dimensional surfaces if and only if the distantial spread vectors of the congruences, taken in pairs, are linear combinations of the r tangent vectors of the congruences.**

Without loss of generality, we may assume that the given congruences are the first r congruences of the ennuple E. According to Theorem 8, the family of hypersurfaces $\phi = \text{const.}$ contains these r congruences if and only if $\partial\phi/\partial s^i = 0$, $i = 1, 2, \cdots, r$. Hence, the r congruences lie in a family of r-dimensional surfaces if and only if the system of differential equations

$$\frac{\partial \phi}{\partial s^i} = 0 \qquad (i = 1, 2, \cdots, r)$$

is completely integrable. But the conditions of integrability, as obtained from (6), reduce to

$$\sum_{k>r} B_{ij}{}^k \frac{\partial \phi}{\partial s^k} = 0 \qquad (i, j = 1, 2, \cdots, r),$$

and hence are identically satisfied when and only when

$$B_{ij}{}^k = 0 \quad (i, j = 1, 2, \cdots, r; k = r + 1, \cdots, n).$$

But, by (31) and (13b), these equations constitute necessary and sufficient conditions that the distantial spread vectors of the first r congruences, taken in pairs, are linearly dependent on the tangent vectors of these congruences.

The theorem is of the greatest interest in the cases $r = 2$ and $r = n - 1$. We shall discuss these cases in detail, with the purpose of bringing out the bearing of distantial spread vectors on equidistance. In this discussion, indices $a, b, c, \cdots$ are fixed, whereas indices $i, j, k, \cdots$ vary from 1 to n, except as otherwise stated.

Case $r = 2$. A typical instance of this case is the following.

THEOREM 10. *The two congruences consisting of the curves* C_a *and* C_b *of the ennuple E lie in a family of two-dimensional surfaces if and only if*

$$B_{ab}{}^k = 0 \qquad (k \neq a, b). \tag{35}$$

* For this theorem, expressed in terms of associate curvature vectors, see Struik, *Grundzüge der mehrdimensionalen Differentialgeometrie*, p. 53.

The pair of differential equations $\partial\phi/\partial s^a=0$, $\partial\phi/\partial s^b=0$ have, then, $n-2$ functionally independent solutions, ϕ^k, $k\neq a$, b, and the $n-2$ equations $\phi^k=c_k$, $k\neq a$, b, define the family of two-dimensional surfaces containing the curves C_a and C_b.

The individual equations $\partial\phi/\partial s^a=0$ and $\partial\phi/\partial s^b=0$ each have $n-1$ independent solutions, $n-2$ of which may be taken in each case as ϕ^k, $k\neq a$, b. Let the $(n-1)$st be ϕ^a in the case of the equation $\partial\phi/\partial s^b=0$, and ϕ^b, in the case of the equation $\partial\phi/\partial s^a=0$. Then, the congruence of curves C_a is represented by the equations $\phi^k=c_k$, $k\neq a$, and that of the curves C by the equations $\phi^k=c_k$, $k\neq b$.

The conditions of Theorem 10 demand the vanishing of all the components of the distantial spread vector of the congruences of curves C_a and C_b except $B_{ab}{}^a$ and $B_{ab}{}^b$. These two components have geometric interpretations in terms of the concept of the distantial spread of the one congruence with respect to the other formulated by R. M. Peters* as a generalization of a corresponding concept for the case $n=2$.†

To define this concept, we note first that on an arbitrary but fixed surface, S_2, of the family of two-dimensional surfaces containing the given congruences, there are ∞^1 curves C_a, defined by the equation $\phi^b=\text{const.}$, and ∞^1 curves C_b, defined by the equation $\phi^a=\text{const.}$ Restricting ourselves for the moment to these curves C_a and C_b, we form the logarithmic directional derivative in the positive direction of the curve C_a, of the distance, measured along an arbitrary curve C_b, between the curve C_a: $\phi^b=\phi_0^b$ and a neighboring curve C_a: $\phi^b=\phi_0+\Delta\phi^b$. Then the limit of this derivative, when $\Delta\phi^b$ approaches zero, namely,

$$-\frac{\partial}{\partial s^a}\log\left|\frac{\partial\phi^b}{\partial s^b}\right|,$$

is the *distantial spread of the congruence of curves C_a with respect to the congruence of curves C_b*. It vanishes identically when and only when on each surface S_2 the curves C_a are equidistant with respect to the curves C_b in that each two of them cut segments of equal length from the curves C_b.

The components $B_{ab}{}^a$ and $B_{ab}{}^b$ of the distantial spread vector of the congruences of curves C_a and C_b are essentially the distantial spreads of the two congruences with respect to one another. For we conclude from the identity

$$\frac{\partial}{\partial s^b}\frac{\partial\phi^b}{\partial s^a}-\frac{\partial}{\partial s^a}\frac{\partial\phi^b}{\partial s^b}=B_{ab}{}^k\frac{\partial\phi^b}{\partial s^k},$$

* Peters, *Parallelism and equidistance in Riemannian geometry*, offered to the American Journal of Mathematics.

† Graustein, loc. cit., p. 561.

inasmuch as $\partial\phi^b/\partial s^k=0,\ k\neq b$, that

$$B_{ab}{}^b = -\frac{\partial}{\partial s^a}\log\left|\frac{\partial\phi^b}{\partial s^b}\right|, \tag{36a}$$

and, in a similar fashion, find that

$$B_{ab}{}^a = \frac{\partial}{\partial s^b}\log\left|\frac{\partial\phi^a}{\partial s^a}\right|. \tag{36b}$$

These results, together with the conclusions they imply, are summarized, in general form, in the following theorems.

THEOREM 11. *Two congruences of curves C and K lie in a family of two-dimensional surfaces, S_2, if and only if their distantial spread vector lies always in the plane of their tangent vectors. The component, then, in the direction of the curves C, of the distantial spread vector of the curves C and K (in this order) is the negative of the distantial spread of the curves K with respect to the curves C, and the component in the direction of the curves K is equal to the distantial spread of the curves C with respect to the curves K. A necessary and sufficient condition that on each of the surfaces S_2 the curves of one congruence be equidistant with respect to those of the second is that the distantial spread vector of the congruences lie along the tangent vector of the first-named congruence.*

THEOREM 12. *The distantial spread vector of two congruences is identically the null vector if and only if* (a) *the two congruences lie in a family of two-dimensional surfaces S_2, and* (b) *on each surface S_2 the curves of each congruence are equidistant with respect to those of the other, that is, clothe the surface in the sense of Tchebycheff.*

Case $r=n-1$. Here we have the following typical result.

THEOREM 13. *A necessary and sufficient condition that the $n-1$ congruences of curves C_i, $i\neq a$, of the ennuple E lie in a family of hypersurfaces is that*

$$B_{ij}{}^a = 0 \qquad (i, j \neq a). \tag{37}$$

The $n-1$ differential equations $\partial\phi/\partial s^i=0$, $i\neq a$, have, then, a solution, ϕ^a, other than a constant, and the equation $\phi^a=\text{const.}$ defines the hypersurfaces, S_{n-1}, in which the $n-1$ congruences lie.

Since each curve C_a of the ath congruence meets each of the hypersurfaces S_{n-1} in just one point, ϕ^a is a parameter common to all the curves C_a. In terms of this parameter, the differential of arc of curves C_a has the value

$$ds^a = \left(\frac{\partial\phi^a}{\partial s^a}\right)^{-1} d\phi^a. \tag{38}$$

For, inasmuch as $\partial\phi^a/\partial s^i = 0$, $i \neq a$,

$$\frac{\partial\phi^a}{\partial s^i} = \frac{\partial\phi^a}{\partial s^a}\delta^i_a \qquad (i = 1, 2, \cdots, n), \tag{39}$$

and hence

$$d\phi^a = \frac{\partial\phi^a}{\partial s^a}ds^a. \tag{40}$$

In this case we shall find useful another concept developed by Peters,* namely that of the *distantial spread, in the direction of the curves* C_i, *of the hypersurfaces* S_{n-1} *with respect to the curves* C_a, $i \neq a$. If d is the distance, measured along an arbitrary curve C_a, between the hypersurface $\phi^a = \phi^a_0$ and a neighboring hypersurface $\phi^a = \phi^a_0 + \Delta\phi^a$, this distantial spread is the limit, when $\Delta\phi^a$ approaches zero, of the logarithmic derivative of d in the positive direction of the curves C_i, $i \neq a$. Its value, as obtained from (38), is found to be

$$-\frac{\partial}{\partial s^i}\log\left|\frac{\partial\phi^a}{\partial s^a}\right|.$$

As noted by Peters, a necessary and sufficient condition that the hypersurfaces S_{n-1} be equidistant with respect to the curves C_a in that each two of them cut segments of equal length from all the curves C_a is that the distantial spreads of the hypersurfaces S_{n-1} with respect to the curves C_a in the directions of the curves C_i, $i \neq a$, all vanish.

To express the fact that the distantial spread, in the direction of the curves C_b, of the hypersurfaces S_{n-1} with respect to the curves C_a vanishes, we shall say that the hypersurfaces are equidistant with respect to the curves C_a in the direction of the curves C_b.

The conditions of Theorem 13 demand the vanishing of the ath components of the distantial spread vectors of each two of the $n-1$ given congruences. The ath components of the distantial spread vectors of each of these congruences taken with the congruence of curves C_a are precisely the distantial spreads just defined. For, employing the method of the preceding case, we readily find that

$$B_{ia}{}^a = -\frac{\partial}{\partial s^i}\log\left|\frac{\partial\phi^a}{\partial s^a}\right| \qquad (i \neq a). \tag{41}$$

The results we have obtained may be summarized as follows.

* Loc. cit.

THEOREM 14. *The curves of $n-1$ linearly independent congruences lie in a family of hypersurfaces S_{n-1} if and only if the distantial spread vectors of each two of the congruences are linearly dependent on the tangent vectors of the congruences. If the congruences are those of the ennuple E other than that of the curves C_a, the component, in the direction of the curves C_a, of the distantial spread vector of the ith and ath congruences ($i \neq a$) is the distantial spread, in the direction of the curves C_i, of the hypersurfaces S_{n-1} with respect to the curves C_a. A necessary and sufficient condition that this component vanish is that the hypersurfaces S_{n-1} be equidistant with respect to the curves C_a, in the direction of the curves C_i.*

THEOREM 15. *The distantial spread vectors of each two congruences of an ennuple are linear combinations of the tangent vectors of $n-1$ of the congruences if and only if these $n-1$ congruences lie in a family of hypersurfaces and the hypersurfaces are equidistant with respect to the curves of the nth congruence.**

Returning now to the analytic discussion, we note the equivalence of equations (39) and (40), and hence conclude:

THEOREM 16. *The differential of arc ds^a of the curves C_a of an ennuple E possesses an integrating factor if and only if the other curves of E lie in a family of hypersurfaces. If $\phi^a=$*const. *is an equation of these hypersurfaces, then $\partial\phi^a/\partial s^a$ is an integrating factor of ds^a.*

It will be instructive to look at this question from another point of view. According to Theorem 1, Ids is an exact differential if and only if $B_{ij}{}^a=0$, $i, j \neq a$, and

$$\frac{\partial \log I}{\partial s^i} = - B_{ia}{}^a \qquad (i \neq a). \tag{42}$$

But Theorem 16 and equation (37) guarantee that the conditions $B_{ij}{}^a=0$, $i, j\neq a$, are sufficient that ds^a possess an integrating factor. Hence, the differential equations (42) must be compatible. Now, on the one hand, equations (42) are equivalent, by Theorem (16), to equations (41), and, on the other hand, the conditions for their complete integrability are

$$\frac{\partial}{\partial s^j} B_{ia}{}^a - \frac{\partial}{\partial s^i} B_{ja}{}^a = B_{ij}{}^r B_{ra}{}^a \qquad (i, j \neq a),$$

and these equations, by virtue of (37), are special cases of the identity (33).

7. **Parametric, Tchebycheff, and Cartesian ennuples of congruences.** *Parametric ennuples.* If the n congruences of curves of an ennuple are the intersections of n families of hypersurfaces, taken $n-1$ at a time, we shall call the

* For applications of distantial spreads in the case of a parametric ennuple, see Peters, loc. cit.

ennuple parametric. If $\phi^i = c_i$ is the equation of the ith family of hypersurfaces, S_{n-1}^i, and it is the curves C_i of the ith congruence which do not lie in the hypersurfaces S_{n-1}^i, $i = 1, 2, \cdots, n$, then ϕ^i is a parameter for the curves C_i and $\phi^1, \phi^2, \cdots, \phi^n$ may be used as parameters (coordinates) in V_n.

It is evident geometrically that an ennuple is parametric when and only when each $n-1$ congruences belonging to it lie in a corresponding family of hypersurfaces, or, what amounts to the same thing, according to Theorem 16, if and only if the differential of arc of the curves of each congruence has an integrating factor. Theorem 13 and the subsequent developments lead, then, to the following conclusions.

THEOREM 17. *A necessary and sufficient condition that the ennuple E be parametric is that*

$$B_{ij}{}^k = 0 \quad (k \neq i, j;\ i, j, k = 1, 2, \cdots, n). \tag{43}$$

If, then, ϕ^i *is a parameter for the curves* C_i,

$$B_{ik}{}^k = -\frac{\partial}{\partial s^i} \log \left| \frac{\partial \phi^k}{\partial s^k} \right| \qquad (k \neq i;\ i, k = 1, 2, \cdots, n). \tag{44}$$

Returning to Theorem 10, we note that equations (35), when a and b, as well as k, vary from 1 to n, are identical with (43). Hence:

THEOREM 18. *An ennuple is parametric if and only if each two of its congruences lie in a corresponding family of two-dimensional surfaces.**

From Theorems 11 and 14 we get two interpretations of the quantities $B_{ik}{}^k$ of (44):

THEOREM 19. *If the ennuple E is parametric, then* $B_{ik}{}^k$ *is equal to the distantial spread of the curves* C_i *with respect to the curves* C_k, *and also to the distantial spread, in the direction of the curves* C_i, *of the hypersurfaces* S_{n-1}^k *with respect to the curves* C_k.

From (43) and (18) we conclude:

THEOREM 20. *A necessary and sufficient condition that the ennuple E be parametric is that* $C_{ij}{}^k = C_{ji}{}^k$, $k \neq i, j$, *that is, that the angular spread vectors of each two congruences of E with respect to one another have the same intrinsic contravariant components external to the plane of the tangent vectors of the two congruences.*

In case E is an orthogonal ennuple, it is readily shown that the equations

* This condition, expressed in analytic form, is to be found in Bortolotti, *Reti di Cebiceff e sistemi coniugati nelle* V_n *riemanniane*, Rendiconti della Accademia dei Lincei, (6), vol. 5 (1927), pp. 741–745.

$C_{ij}{}^k = C_{ji}{}^k$, $k \neq i, j$, are equivalent, by virtue of (25), to the equations $C_{ij}{}^k = 0$ $(i, j, k \neq)$.

COROLLARY. *An orthogonal ennuple is parametric, that is, each of its congruences is normal, if and only if the angular spread vectors of each two of its congruences with respect to one another lie in the plane of the tangent vectors of the two congruences.*

Inasmuch as $B_{ij}{}^k = 0$, $k \neq i, j$, it follows that, if we set

$$\frac{\nabla_j F}{\nabla s^i} = \frac{\partial F}{\partial s^i} + B_{ij}{}^j F,$$

the conditions of integrability (6) take the form

$$\frac{\nabla_i}{\nabla s^j} \frac{\partial f}{\partial s^i} = \frac{\nabla_j}{\nabla s^i} \frac{\partial f}{\partial s^j} \qquad (i, j = 1, 2, \cdots, n)$$

and the conditions (8) that $f_i ds^i$ be an exact differential become $\nabla_i f_i / \nabla s^j = \nabla_j f_j / \nabla s^i$, $i, j = 1, 2, \cdots, n$.

The expression $\nabla_j F / \nabla s^i$ may be called *the modified directional derivative of F in the direction of the curves C_i with respect to the curves C_j*. It is a generalization of the modified directional derivative employed, to good effect, in the theory of ordinary surfaces.* In comparison with the covariant derivative, it has the advantage that it involves, besides F, only the B's. On the other hand, it does not have tensor character, and cannot be extended to apply to an arbitrary ennuple of congruences.

Tchebycheff ennuples. If each differential of arc, ds^k, of the ennuple E is an exact differential of a function s^k of the x's, E is a parametric ennuple with the variables s^k as parameters. The linear element, referred to these parameters, is

$$ds^2 = g_{ij} ds^i ds^j, \qquad g_{ij} = \cos \omega_{ij}. \tag{45}$$

Inasmuch as s^k is the common arc of all the curves C_k, $k = 1, 2, \cdots, n$, the ennuple is a generalization of a Tchebycheff system of curves clothing a two-dimensional surface and may appropriately be called a Tchebycheff ennuple.

According to Theorem 1, ds^k is an exact differential if and only if $B_{ij}{}^k = 0$, $i, j = 1, 2, \cdots, n$. Hence:

THEOREM 21. *An ennuple of congruences is a Tchebycheff ennuple if, and only if, for each two congruences of the ennuple, the distantial spread vector is a null vector, or the angular spread vector of the one with respect to the other is identical with that of the second with respect to the first.*

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* Graustein, loc. cit., p. 575. The generalization was first discovered by Ruth M. Peters, in another connection.

It is to be noted that a parametric ennuple is, in particular, a Tchebycheff ennuple if and only if the kth family of hypersurfaces is an equidistant family with respect to the kth congruence of curves, $k=1, 2, \cdots, n$; see Theorem 15.

From (24) and (25) we conclude, inasmuch as $C_{hkh}=0$, that

$$C_{hhk} = \frac{\partial g_{hk}}{\partial s^h} - B_{khh}.$$

From these relations we may obtain interesting conditions under which the curves C_h are geodesics. In particular, we have

THEOREM 22. *The curves of a Tchebycheff ennuple are all geodesics if and only if the angle between the curves of each two congruences is constant along the curves of both congruences.**

It follows that the linear element (45), where s^i are actual coordinates and g_{ij} is independent of s^i and s^j, $i, j=1, 2, \cdots, n$, is characteristic of a space which contains a Tchebycheff ennuple of geodesics.

Cartesian ennuples. When E is a Tchebycheff ennuple and g_{ij} are constants, the linear element (45) characterizes V_n as a euclidean space referred to the ennuple of congruences of coordinate curves of a Cartesian coordinate system in which the unit of measure for each coordinate is the unit distance of the space. Accordingly, we shall call a Tchebycheff ennuple for which the the g_{ij} are constant a Cartesian ennuple. We may then state

THEOREM 23. *A necessary and sufficient condition that V_n be euclidean is that it contain a Cartesian ennuple of congruences.*

This type of ennuple may be characterized analytically and geometrically as follows.

THEOREM 24. *An ennuple of congruences is a Cartesian ennuple if and only if*

$$B_{ij}{}^k = 0, \quad \frac{\partial g_{ij}}{\partial s^k} = 0 \qquad (i, j, k = 1, 2, \cdots, n), \tag{46}$$

or

$$C_{ij}{}^k = 0 \qquad (i, j, k = 1, 2, \cdots, n), \tag{47}$$

* The counterpart of this theorem, to the effect that the curves of each congruence of a Tchebycheff ennuple are parallel with respect to those of every other congruence if and only if the angle between the curves of each two congruences is constant along the curves of the remaining congruences, is given by Bortolotti, loc. cit., p. 741. It is to be noted that Bortolotti's conception of a Tchebycheff ennuple differs from the one here used.

that is, if and only if it is a Tchebycheff ennuple each two of whose congruences intersect under a constant angle or has the property that the curves of each congruence are geodesics and are parallel with respect to the curves of every other congruence.

The equivalence of equations (46) and (47) follows from previous relations. If $\partial g_{ij}/\partial s^k=0$ and $B_{ijk}=0$, then (26) says that $C_{ijk}=0$; and, if $C_{ijk}=0$, (25) and (24) tell us that $\partial g_{ij}/\partial s^k=0$ and $B_{ijk}=0$. But the equations $B_{ijk}=0$ and $B_{ij}{}^k=0$ are equivalent, and also the equations $C_{ijk}=0$ and $C_{ij}{}^k=0$.

8. **Ennuples Cartesian at a point.** We shall say that an ennuple of congruences is Cartesian at a particular point P if it behaves at P like a Cartesian ennuple, that is, if it has at P the analytic or geometric characteristics described in Theorem 23.

THEOREM 25. *If x^i are geodesic coordinates at P, the ennuple E is Cartesian at P if and only if the ordinary components, $\bar{a}_h|^i$, of the unit tangent vectors of E behave like constants at P.*

Since x^i are geodesic coordinates at P, $\bar{C}_{ij}{}^k=0$ at P. Hence, according to (17), $C_{ij}{}^k=0$ at P if and only if $\bar{a}_h|^i$ behave like constants at P. Thus, the theorem is proved.

Inasmuch as, when x^i are geodesic coordinates at P, $\bar{g}_{ij}$ behave like constants at P, it follows that, if the ordinary components of a vector field behave like constants at P, so also do the ordinary components of the corresponding field of unit vectors. Hence, we conclude:

THEOREM 26. *There exist infinitely many ennuples which are Cartesian at a given point P and have at P prescribed tangent vectors.*

If $\bar{a}_h|^i$ behave like constants at P, so also do $\bar{a}^h|_i$. Hence, if x^i are geodesic coordinates at P: $(0, 0, \cdots, 0)$ and the ennuple E is Cartesian at P,

$$ds^h = \bar{a}^h|_j dx^j = (\bar{a}^h|_j)_0 dx^j + c^h{}_{jrs}x^r x^s dx^j + \cdots .$$

Consequently, if terms in x^i of the second degree and higher are neglected, the differentials of arc ds^h become exact differentials and we may write $s^h=(\bar{a}^h|_j)_0 x^j$.

Comparison with geodesic coordinates. We assume now that the ennuple E is a parametric ennuple and inquire whether, if parameters for E are geodesic at a point P, E is Cartesian at P, and conversely.

We may assume that E is the parametric ennuple corresponding to the basic coordinates x^i. According to Theorem 17, we have, then,

$$B_{ij}{}^i = \frac{\partial}{\partial s^j}\log\left|\frac{\partial x^i}{\partial s^i}\right| \qquad (i\neq j;\ i,j=1,2,\cdots,n), \tag{48}$$

whereas $B_{ij}^k = 0$ for $k \neq i, j$.

Since in this case

$$\frac{\partial x^i}{\partial s^j} = 0 \qquad (i \neq j;\, i, j = 1, 2, \cdots, n),$$

and hence

$$\frac{\partial s^i}{\partial x^j} = 0, \quad \frac{\partial s^i}{\partial x^i} = \left(\frac{\partial x^i}{\partial s^i}\right)^{-1} \qquad (i \neq j;\, i, j = 1, 2, \cdots, n).$$

equation (17) becomes

$$C_{ij}^k = \overline{C}_{ij}^k \frac{\partial x^i}{\partial s^i} \frac{\partial x^j}{\partial s^j} \frac{\partial s^k}{\partial x^k} + \delta_i^k \frac{\partial}{\partial s^j} \log \left| \frac{\partial x^i}{\partial s^i} \right|, \tag{49a}$$

whence

$$C_{ii}^i = \overline{C}_{ii}^i \frac{\partial x^i}{\partial s^i} + \frac{\partial}{\partial s^i} \log \left| \frac{\partial x^i}{\partial s^i} \right|. \tag{49b}$$

Furthermore, we have, from (10a), since $g_{ii} = 1$,

$$\bar{g}_{ii} = \left(\frac{\partial x^i}{\partial s^i}\right)^{-2}.$$

If the coordinates x^i are geodesic at P, $\overline{C}_{ij}^k = 0$ at P; and $\bar{g}_{ii}$, and hence $\partial x^i/\partial s^i$, act like constants at P. It follows, then, from (49a), that $C_{ij}^k = 0$ at P, that is, that E is Cartesian at P.

If, conversely, E is Cartesian at P, $C_{ij}^k = 0$ and $B_{ij}^k = 0$ at P. Hence, from (49a) and (48), we find that $\overline{C}_{ij}^k = 0$ at P except when $i = j = k$, whereas from (49b) we get

$$\frac{\partial x^i}{\partial s^i} \overline{C}_{ii}^i + \frac{\partial}{\partial s^i} \log \left| \frac{\partial x^i}{\partial s^i} \right| = 0 \text{ at } P.$$

Thus, the coordinates x^i are not necessarily geodesic at P; they are geodesic if and only if $\partial^2 x^i/\partial s^{i2} = 0$ at P. However, it is evident that there exist coordinates X^i for E which are geodesic at P; we have merely to choose $X^i = X^i(x^i)$, $i = 1, 2, \cdots, n$, so that $\partial^2 X^i/\partial s^{i2} = 0$ at P.*

We have now answered the proposed question.

THEOREM 27. *There exist, for a parametric ennuple, coordinates which are geodesic at a given point P if and only if the ennuple is Cartesian at P.*

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* It would appear from this discussion that a completely geometric characterization of geodesic coordinates is impossible. In this connection, see Levi-Civita, *The Absolute Differential Calculus*, p. 168.

It is evident from this discussion that the concept of an ennuple Cartesian at a point is more fundamental than the concept of geodesic coordinates. Furthermore it is also more general, in that an ennuple Cartesian at a point does not need to be a parametric ennuple at all.

That an ennuple Cartesian at a point serves all the purposes for which geodesic coordinates are ordinarily employed is guaranteed by the following proposition.

THEOREM 28. *When the ennuple of reference is Cartesian at a point P, then at P intrinsic covariant differentiation becomes directional differentiation and second cross directional derivatives are independent of the order of differentiation.*

From (6) and Theorem 21 it follows that second cross directional derivatives are independent of the order of differentiation at a given point P if and only if the ennuple of reference behaves like a Tchebycheff ennuple at P.

9. **Intrinsic components of the Riemann tensors.** It may be shown in various ways that the intrinsic components, referred to the ennuple E, of the Riemann tensor with the ordinary components

$$(50)\qquad \overline{R}^l{}_{ijk} = \frac{\partial}{\partial x^j}\overline{C}_{ik}{}^l - \frac{\partial}{\partial x^k}\overline{C}_{ij}{}^l + \overline{C}_{ik}{}^m\overline{C}_{mj}{}^l - \overline{C}_{ij}{}^m\overline{C}_{mk}{}^l,$$

are

$$(51)\qquad R^l{}_{ijk} = \frac{\partial}{\partial s^j}C_{ik}{}^l - \frac{\partial}{\partial s^k}C_{ij}{}^l + C_{ik}{}^m C_{mj}{}^l - C_{ij}{}^m C_{mk}{}^l + C_{im}{}^l B_{jk}{}^m.$$

From $R_{hijk} = g_{hl}R^l{}_{ijk}$, it follows that

$$(52)\qquad R_{hijk} = \frac{\partial}{\partial s^j}C_{ikh} - \frac{\partial}{\partial s^k}C_{ijh} + C_{ij}{}^m C_{hkm} - C_{ik}{}^m C_{hjm} + C_{imh}B_{jk}{}^m$$

are the intrinsic components of the Riemann tensor whose ordinary components are $\overline{R}_{hijk} = \bar{g}_{hl}\overline{R}_{lijk}$.

The identities satisfied by the covariant Riemann tensor have, of course, the same form in terms of its intrinsic components as in terms of its ordinary components. In this connection, it is of interest to note that the identities $R_{hijk}+R_{hjki}+R_{hkij}=0$ are equivalent to the identities (34).*

The conditions of integrability for covariant differentiation also have the same forms in terms of intrinsic components as in terms of ordinary components. Thus, if a_i, $a_{i,j}$, and $a_{i,jk}$ are respectively the intrinsic components of a covariant vector and its first and second covariant derivatives, then

* See Dei, *Sulle relazioni differenziali che legano i coefficienti di rotazione del Ricci*, Rendiconti della Accademia dei Lincei, (5), vol. 32 (1923), pp. 474–478, where this conclusion is reached in essence, though not in form, in the case of an orthogonal ennuple.

$$a_{i,jk} - a_{i,kj} = a_l R^l{}_{ijk}. \tag{53}$$

Again, in the case of a contravariant vector, we have

$$a^i{}_{,jk} - a^i{}_{,kj} = - a^l R^i{}_{ljk}. \tag{54}$$

Setting $a^i = a_h|^i = \delta^i_h$ in (54) and $a_i = a_h|_i = g_{hi}$ in (53), we obtain the following expressions for $R^h{}_{ijk}$ and R_{hijk}:

$$R^h{}_{ijk} = a_i|^h{}_{,kj} - a_i|^h{}_{,jk}, \qquad R_{hijk} = a_h|_{i,jk} - a_h|_{i,kj},$$

in terms of the derivatives of the unit vectors tangent to the curves of E. On the other hand, formulas (51) and (52), in light of (29) and (31), furnish expressions for $R^h{}_{ijk}$ and R_{hijk} in terms of the curvature and angular and distantial spread vectors of E, namely,

$$R^h{}_{ijk} = c_{ik}|^h{}_{,j} - c_{ij}|^h{}_{,k} + c_{im}|^h b_{jk}|^m,$$

$$R_{hijk} = c_{ik}|_{h,j} - c_{ij}|_{h,k} + c_{im}|_h b_{jk}|^m.$$

To these relations may be adjoined the simple expressions

$$c_{hk}|^i = a_h|^i{}_{,k}, \qquad c_{hk}|_i = a_h|_{i,k}$$

for the curvature and angular spread vectors of E in terms of the tangent vectors.*

The Ricci tensor. From (50) we find as the components of the Ricci tensor, $R_{ij} = R^k{}_{ijk}$:

$$R_{ij} = \frac{\partial}{\partial s^j} C_{ik}{}^k - \frac{\partial}{\partial s^k} C_{ij}{}^k + C_{ik}{}^m C_{mj}{}^k - C_{ij}{}^m C_{mk}{}^k + C_{imk} B_{jk}{}^m.$$

Adding to the right-hand side of this equation the expression

$$\frac{1}{2}\left(\frac{\partial}{\partial s^i} B_{jk}{}^k - \frac{\partial}{\partial s^j} B_{ik}{}^k + \frac{\partial}{\partial s^k} B_{ij}{}^k + B_{ij}{}^m B_{mk}{}^k\right),$$

whose value is readily shown to be zero by (33), and making use of (18) and the relation $\partial(\log g^{1/2})/\partial s^i = C_{ki}{}^k$, we find the following symmetric expression for R_{ij}:

$$R_{ij} = (\log g^{1/2})_{,ij} + \frac{1}{2}\,\frac{\partial}{\partial s^i} B_{jk}{}^k + \frac{1}{2}\,\frac{\partial}{\partial s^j} B_{ik}{}^k - \frac{1}{2}\,\frac{\partial}{\partial s^k}(C_{ij}{}^k + C_{ji}{}^k)$$
$$- \tfrac{1}{2}(C_{ij}{}^k + C_{ji}{}^k) B_{km}{}^m + C_{ik}{}^m C_{jm}{}^k.$$

Geometric interpretations. We note, without going into detail, that the

* It is to be noted that none of the relations in this paragraph are invariant in form with respect to a change from intrinsic to ordinary components.

intrinsic components of the Riemann and Ricci tensors, especially in case E is an orthogonal ennuple, have interesting geometrical interpretations in terms of the Riemannian and mean curvatures.

10. **Metric connections. The torsion vector.** The most general connection which possesses a metric based on the tensor $\bar{g}_{ij}$ is obtained by employing, instead of the Christoffel symbols $\overline{C}_{ij}{}^k$, arbitrary coefficients of connection, $\mathbf{\Gamma}_{ij}{}^k$, such that

$$\bar{g}_{ij;k} \equiv \frac{\partial \bar{g}_{ij}}{\partial x^k} - \bar{g}_{ir}\mathbf{\Gamma}_{jk}{}^r - \bar{g}_{rj}\mathbf{\Gamma}_{ik}{}^r = 0 \qquad (i, j, k = 1, 2, \cdots, n). \tag{55}$$

The skew-symmetric part of $\mathbf{\Gamma}_{ij}{}^k$, namely

$$\overline{S}_{ij}{}^k = \tfrac{1}{2}(\mathbf{\Gamma}_{ij}{}^k - \mathbf{\Gamma}_{ji}{}^k), \tag{56}$$

has tensor character. It is called the *torsion tensor* of the connection.

When the torsion tensor $\overline{S}_{ij}{}^k$ is known, the connection $\mathbf{\Gamma}_{ij}{}^k$ is completely determined. For it is readily found, from (55) and (56), that

$$\mathbf{\Gamma}_{ijk} = \overline{C}_{ijk} + (\overline{S}_{ijk} + \overline{S}_{jki} - \overline{S}_{kij}), \tag{57}$$

where, for example, $\mathbf{\Gamma}_{ijk} = \bar{g}_{hk}\mathbf{\Gamma}_{ij}{}^h$. In particular, the connection is Riemannian if and only if the torsion tensor is null.

By replacing $C_{ij}{}^k$ and $\overline{C}_{pq}{}^t$ in (17) by $\Gamma_{ij}{}^k$ and $\mathbf{\Gamma}_{pq}{}^t$, we obtain equations which define the invariant coefficients of connection, $\Gamma_{ij}{}^k$, referred to the ennuple E. In the same way, we obtain from (19) and (21) the formulas for covariant differentiation with respect to the connection, and, from (27), the expressions for the new angular spread or curvature vectors. From the latter, it follows that $\Gamma_{hk}{}^i$ are the intrinsic contravariant components of the angular spread vector, $\gamma_{hk}|^i$, of the congruence of the curves C_h with respect to the congruence of curves C_k. On the other hand, the components $B_{hk}{}^i$ of the distantial spread vector, $b_{hk}|^i$, of the curves C_h and C_k remain the same as before; this vector is dependent only on the components, $\bar{g}_{ij}$, of the metric.

From the laws of transformation of $\mathbf{\Gamma}_{ij}{}^k$ and $\overline{C}_{ij}{}^k$ into $\Gamma_{ij}{}^k$ and $C_{ij}{}^k$, it follows that $\mathbf{\Gamma}_{ij}{}^k - \overline{C}_{ij}{}^k$ is a tensor whose intrinsic components are $\Gamma_{ij}{}^k - C_{ij}{}^k$. Hence, we conclude from (57) that

$$\Gamma_{ijk} = C_{ijk} + (S_{ijk} + S_{jki} - S_{kij}), \tag{58}$$

where S_{ijk} are the intrinsic components of the tensor $\overline{S}_{ijk}$.

From these relations we obtain, by virtue of (24), the following expressions for the intrinsic components of the torsion tensor*:

* An equation of the same form as this holds for the general linear connection; see Horák, loc. cit., p. 197.

$$2S_{ij}{}^k = (\Gamma_{ij}{}^k - \Gamma_{ji}{}^k) - B_{ij}{}^k. \tag{59}$$

Torsion vectors. In a given oriented planar element at a point P choose two ordered infinitesimal vectors $\overline{PP_1}$ and $\overline{PP_2}$ such that the direction of rotation about P from the first vector to the second is that of the given orientation. Transport each vector along the other by the displacement determined by $\Gamma_{ij}{}^k$, thus obtaining the new vectors $\overline{P_1Q_1}$ and $\overline{P_2Q_2}$. Then the limit of the ratio of the vector $\overline{Q_1Q_2}$ to the area of the parallelogram determined by the vectors $\overline{PP_1}$ and $\overline{PP_2}$ is a vector at P which is independent of these vectors, provided merely that they are chosen as described, and so pertains simply to the given oriented planar element. This vector is due to Cartan* and is called by him *the torsion vector at P for the given oriented planar element.*

If the vectors $\overline{PP_1}$ and $\overline{PP_2}$ are respectively d_hx^i and d_kx^i, the coordinates of Q_1 are, to within terms of higher order,

$$x^r + d_hx^r + d_kx^r + d_hd_kx^r + \Gamma_{ji}{}^r d_hx^i d_kx^j.$$

Hence, the torsion vector at P for the given oriented planar element has the components

$$2 \csc \phi \bar{S}_{ij}{}^r \bar{a}_h|^i \bar{a}_k|^j,$$

where $\bar{a}_h|^i$ and $\bar{a}_k|^j$ are the unit vectors in the directions of $\overline{PP_1}$ and $\overline{PP_2}$ and ϕ is the angle between them.

We shall find it convenient to employ, instead of the torsion vector of Cartan, *a torsion vector for two ordered directions at a point.* This we define by the equations

$$\bar{s}_{hk}|^r = -2\bar{S}_{ij}{}^r \bar{a}_h|^i \bar{a}_k|^j, \tag{60}$$

where $\bar{a}_h|^i$ and $\bar{a}_k|^i$ are the unit vectors in the two directions. In particular, if $\bar{a}_h|^i$ and $\bar{a}_k|^i$ are the fields of unit vectors tangent to the curves of the hth and kth congruences of the ennuple E, we shall call $\bar{s}_{hk}|^r$ the torsion vector of these congruences, in the order given.

A geometric interpretation of the torsion vector (60) is obtained by rephrasing the definition of the torsion vector of Cartan. A second interpretation, more useful to us, is the following: *If, in the definition of the distantial spread vector of the congruences of curves C_h and C_k, we redefine Q_1 and Q_2 as the terminal points of the vectors at P_1 and P_2 which are parallel respectively, according to the connection $\Gamma_{ij}{}^k$, to the vectors $\overline{PP_2}$ and $\overline{PP_1}$, the definition becomes a description of the torsion vector of the congruences, in the order given.*

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* *Sur les variétés à connexion affine et la théorie de la relativité généralisée,* Annales de l'École Normale, (3), vol. 40 (1923), pp. 325–412.

To establish this interpretation, it suffices to note that the coordinates of the new point Q_1 are, to within terms of higher order,

$$x^r + d_h x^r + d_k x^r - \Gamma_{ji}{}^r d_h x^i d_k x^j.$$

It follows from the interpretation that a sufficient condition that the torsion vector of two congruences be identical with their distantial spread vector is that the curves of each congruence be parallel, according to the connection $\Gamma_{ij}{}^k$, with respect to those of the other.* This is not a necessary condition, as we shall see shortly.

Whereas the distantial spread vector of the two congruences depends only on the metric, the torsion vector depends also on the connection. If the connection is Riemannian, the torsion vector is always null.

In terms of intrinsic components, referred to the ennuple E, (60) becomes

$$s_{hk}|^r = -2S_{ij}{}^r a_h|^i a_k|^j = -2S_{hk}{}^r.$$

Thus we have simple geometric interpretations of the intrinsic components of the tensor of torsion.

THEOREM 29. *The intrinsic components, $s_{hk}|^r$ and $s_{hk}|_r$, of the torsion vector of the curves C_h and C_k are respectively $-2S_{hk}{}^r$ and $-2S_{hkr}$:*

$$s_{hk}|^r = -2S_{hk}{}^r, \quad s_{hk}|_r = -2S_{hkr}.$$

We may now interpret the important relations (59), by rewriting them in the form

$$s_{ij}|^r = b_{ij}|^r - (\gamma_{ij}|^r - \gamma_{ji}|^r). \tag{61}$$

THEOREM 30. *The difference between the distantial spread vector and the torsion vector of two ordered congruences is equal to the difference between the angular spread vectors of the two congruences with respect to one another.*

It is now clear that a necessary and sufficient condition that the torsion vector of two ordered congruences coincide with their distantial spread vector is that the angular spread vectors of the congruences with respect to one another be identical.

We shall say that the connection $\Gamma_{ij}{}^k$ is symmetric with respect to the ennuple E if $\Gamma_{ij}{}^k = \Gamma_{ji}{}^k$ for $i, j, k = 1, 2, \cdots, n$. From (59) or (61) we infer, then, the following proposition:

THEOREM 31. *The distantial spread and torsion vectors of each two congruences of the ennuple E are identical if and only if the connection is symmetric with respect to E.*

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* The analytic content of this statement, in a different form, is to be found in Bortolotti, *Parallelismi assoluti nelle V_n riemanniane*, Atti del Reale Istituto Veneto, vol. 86 (1927), pp. 455–465.

For a given metric with the fundamental tensor $\bar{g}_{ij}$, there is a unique metric connection which is symmetric with respect to E. For it follows from $g_{ij;k}=0$, or from (58), (59), and (26), that, if $\Gamma_{ij}{}^k=\Gamma_{ji}{}^k$, then

$$\Gamma_{ij}{}^k = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial s^j}+\frac{\partial g_{jk}}{\partial s^i}-\frac{\partial g_{ij}}{\partial s^k}\right). \tag{62}$$

*Spaces admitting absolute parallelism.** The given space is said to admit (complete) absolute parallelism if there exist n linearly independent fields of absolutely parallel vectors, or, what amounts to the same thing, if there exists an ennuple of congruences such that the curves of each congruence are parallel with respect to all n congruences. It is evident from the geometric interpretation of $\Gamma_{ij}{}^k$ that this is true of the ennuple E if and only if $\Gamma_{ij}{}^k=0$ for $i, j, k=1, 2, \cdots, n$, and hence, according to (62), if and only if $\Gamma_{ij}{}^k$ is symmetric in i, j and g_{ij} are constants.

THEOREM 32. *A metric connection admits absolute parallelism if and only if there exists an ennuple of congruences, E, which has constant angles and with respect to which the connection is symmetric.*

Since $\Gamma_{ij}{}^k=0$, it follows that the intrinsic components, referred to E, of the covariant derivative of a tensor are simply the directional derivatives, along the curves of E, of the intrinsic components of the tensor.

The intrinsic components of the curvature tensor of the connection are obtained by replacing $C_{ij}{}^k$ by $\Gamma_{ij}{}^k$ in (51). Hence, when the connection admits absolute parallelism, the curvature tensor is actually zero, as it should be.

11. Transformation from one ennuple of congruences to a second. We return now to the study of Riemannian space, and assume that there is given, in addition to E, a second ennuple, E', consisting of the congruences of directed curves C_i', $i=1, 2, \cdots, n$.

We shall distinguish by primes symbols referred to the ennuple E'. For example, we shall denote by g'_{ij} the (intrinsic) covariant components of the fundamental tensor, referred to E'.

The components, referred to E, of the tangent and conjugate vectors and the distantial and angular spread vectors of the congruences of E we have denoted by $a_h|$, $a^h|$, $b_{hk}|$, $c_{hk}|$. The components, referred to E', of the corresponding fundamental vectors connected with the congruences of E' we shall designate by $\alpha'_h|$, $\alpha'^h|$, $\beta'_{hk}|$, $\gamma'_{hk}|$. According to the foregoing convention, $\alpha_h|$, $\alpha^h|$, $\beta_{hk}|$, $\gamma_{hk}|$ then denote the components, referred to E, of the

* For a review of this subject, see Eisenhart, *Spaces admitting complete absolute parallelism*, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 217–226.

fundamental vectors pertaining to E', and a_h', $a'^h|$, $b_{hk}'|$, $c_{hk}'|$ the components, referred to E', of the fundamental vectors pertaining to E.

In §4 we noted that

$$\bar{a}^i = \frac{dx^i}{ds}, \qquad a^i = \frac{ds^i}{ds} \tag{63}$$

are respectively the ordinary components and the components referred to E of the unit contravariant vector tangent to the directed curve C. It follows, then, that

$$\frac{dx^i}{ds} = \frac{ds^j}{ds}\frac{\partial x^i}{\partial s^j}, \quad \frac{ds^i}{ds} = \frac{dx^j}{ds}\frac{\partial s^i}{\partial x^j}, \quad \frac{df}{ds} = \frac{\partial f}{\partial s^j}\frac{ds^j}{ds}. \tag{64}$$

Similar formulas hold when C is referred to E' instead of to E.

The second of formulas (63) suggests that we denote by $\partial s^i/\partial s'^h$ the contravariant components, referred to E, of the unit vector tangent to the general curve C_h' of the hth congruence of E':

$$\alpha_h|^i = \frac{\partial s^i}{\partial s'^h}, \tag{65}$$

and by $\partial s'^i/\partial s^h$ the contravariant components, referred to E', of the unit vector tangent to the general curve C_h of the hth congruence of E:

$$a_h'|^i = \frac{\partial s'^i}{\partial s^h}. \tag{66}$$

From the first two formulas in (64) and the corresponding formulas mentioned in connection with them, we have

$$\begin{aligned} \frac{\partial x^i}{\partial s'^h} &= \frac{\partial s^j}{\partial s'^h}\frac{\partial x^i}{\partial s^j}, & \frac{\partial s^i}{\partial s'^h} &= \frac{\partial x^j}{\partial s'^h}\frac{\partial s^i}{\partial x^j}, \\ \frac{\partial x^i}{\partial s^h} &= \frac{\partial s'^j}{\partial s^h}\frac{\partial x^i}{\partial s'^j}, & \frac{\partial s'^i}{\partial s^h} &= \frac{\partial x^j}{\partial s^h}\frac{\partial s'^i}{\partial x^j}, \end{aligned} \tag{67}$$

where $\partial s'^i/\partial x^j$ has the same significance with respect to E' as has $\partial s^i/\partial x^j$ with respect to E.

Using (67) in conjunction with (2) and similar equations in $\partial x^j/\partial s'^i$ and $\partial s'^i/\partial x^j$, we readily establish the fundamental relations

$$\frac{\partial s^j}{\partial s'^k}\frac{\partial s'^i}{\partial s^j} = \delta_k^i, \qquad \frac{\partial s^i}{\partial s'^j}\frac{\partial s'^j}{\partial s^k} = \delta_k^i, \tag{68}$$

which state that the systems of quantities $\partial s^j/\partial s'^i$ and $\partial s'^i/\partial s^j$ are conjugate to one another.

According to the first of these equations, the Pfaffian $(\partial s'^i/\partial s^j)ds^j$ is zero for every curve of the ennuple E' except a curve C_i' and for a curve C_i' is equal to the differential of arc, ds'^i, of the curve. Thus we obtain, as the relations between ds^i and ds'^i:

$$ds'^i = \frac{\partial s'^i}{\partial s^j} ds^j, \quad ds^i = \frac{\partial s^i}{\partial s'^j} ds'^j. \tag{69}$$

Applying the last of the equations (64), we find

$$\frac{\partial f}{\partial s'^i} = \frac{\partial f}{\partial s^j}\frac{\partial s^j}{\partial s'^i}, \quad \frac{\partial f}{\partial s^i} = \frac{\partial f}{\partial s'^j}\frac{\partial s'^j}{\partial s^i}, \tag{70}$$

as the relations between the directional derivatives in the positive directions of the curves C_i and those in the positive directions of the curves C_i'.

Formulas (69) and (70) guarantee that *the transformation from the components of a tensor, referred to E, to the components of the tensor, referred to E', obeys the formal laws of tensor analysis*. The relations between the two sets of components for the fundamental tensor are, for example,

$$\begin{aligned} g'_{ij} &= g_{kl}\frac{\partial s^k}{\partial s'^i}\frac{\partial s^l}{\partial s'^j}, & g_{ij} &= g'_{kl}\frac{\partial s'^k}{\partial s^i}\frac{\partial s'^l}{\partial s^j}, \\ g'^{ij} &= g^{kl}\frac{\partial s'^i}{\partial s^k}\frac{\partial s'^j}{\partial s^l}, & g^{ij} &= g'^{kl}\frac{\partial s^i}{\partial s'^k}\frac{\partial s^j}{\partial s'^l}, \end{aligned} \tag{71}$$

and the corresponding relations in the case of an arbitrary vector are

$$\begin{aligned} a'^i &= a^j\frac{\partial s'^i}{\partial s^j}, & a^i &= a'^j\frac{\partial s^i}{\partial s'^j}, \\ a'_i &= a_j\frac{\partial s^j}{\partial s'^i}, & a_i &= a'_j\frac{\partial s'^j}{\partial s^i}. \end{aligned} \tag{72}$$

Tangent and conjugate vector-fields of E and E'. The components, referred respectively to E and E', of the field of unit vectors tangent to the curves C_h of the ennuple E are

$$a_h\big|^i = \delta_h^i, \quad a_h'\big|^i = \frac{\partial s'^i}{\partial s^h}, \tag{73a}$$

$$a_h\big|_i = g_{hi}, \quad a_h'\big|_i = g_{hj}\frac{\partial s^j}{\partial s'^i} = g'_{ji}\frac{\partial s'^j}{\partial s^h}. \tag{73b}$$

The formulas on the left are identical with (16b); those on the right follow from them by means of (72), (71), and (68).

Similarly, we find, as the components of the hth field of conjugate vectors associated with the ennuple E,

$$a^h|_i = \delta^h_i, \qquad a'^h|_i = \frac{\partial s^h}{\partial s'^i}, \tag{74a}$$

$$a^h|^i = g^{hi}, \qquad a'^h|^i = g^{hj}\frac{\partial s'^i}{\partial s^j} = g'^{ji}\frac{\partial s^h}{\partial s'^j}. \tag{74b}$$

The components, referred respectively to E' and E, of the field of unit vectors tangent to the curves C_h' of the ennuple E' are

$$\alpha'_h|^i = \delta^i_h, \qquad \alpha_h|^i = \frac{\partial s^i}{\partial s'^h}, \tag{75a}$$

$$\alpha'_h|_i = g'_{hi}, \qquad \alpha_h|_i = g'_{hj}\frac{\partial s'^j}{\partial s^i} = g_{ji}\frac{\partial s^j}{\partial s'^h}, \tag{75b}$$

while those of the hth field of conjugate vectors associated with E' are

$$\alpha'^h|_i = \delta^h_i, \qquad \alpha^h|_i = \frac{\partial s'^h}{\partial s^i}, \tag{76a}$$

$$\alpha'^h|^i = g'^{hi}, \qquad \alpha^h|^i = g'^{hj}\frac{\partial s^i}{\partial s'^j} = g^{ji}\frac{\partial s'^h}{\partial s^j}. \tag{76b}$$

From the relations

$$\alpha_i|^j = a'^j|_i = \frac{\partial s^j}{\partial s'^i}, \qquad \alpha^i|_j = a_j'|^i = \frac{\partial s'^i}{\partial s^j}, \tag{77}$$

it follows that, when i is fixed and $j=1, 2, \cdots, n$, $\partial s^j/\partial s'^i$ and $\partial s'^i/\partial s^j$ are respectively components, referred to E, of the ith tangent and conjugate vectors associated with E', whereas, when j is fixed and $i=1, 2, \cdots, n$, they are respectively components, referred to E', of the jth conjugate and tangent vectors associated with E.

Interpretations in terms of angles. If ϕ_{hi} is the angle which the hth tangent vector of E' makes with the ith conjugate vector of E and ϕ'_{hi} is the angle which the hth tangent vector of E makes with the ith conjugate vector of E', it follows, either directly or by virtue of Theorem 2, that

$$\alpha_h|^i = \frac{\partial s^i}{\partial s'^h} = \sec\theta_i \cos\phi_{hi}, \qquad a_h'|^i = \frac{\partial s'^i}{\partial s^h} = \sec\theta_i' \cos\phi'_{hi}, \tag{78}$$

where θ_i is the angle between the ith tangent and conjugate vectors of E and θ_i' is the angle between the ith tangent and conjugate vectors of E'.

If E is an orthogonal ennuple, the angles θ_i are all zero, and $\alpha_h|^i = \partial s^i/\partial s'^h = \cos\phi_{hi}$, so that $\partial s^i/\partial s'^h$ are direction cosines. If E' is also orthogonal, then $\phi_{ij} = \phi'_{ji}$ and hence $\partial s^j/\partial s'^i = \partial s'^i/\partial s^j$ for all i, j.

Returning to the general case, we remark that, inasmuch as we now have interpretations in terms of angles of $\partial s^j/\partial s'^i$, $\partial s'^i/\partial s^j$, and g_{ij}, g^{ij}, g'_{ij}, g'^{ij} (see §2), we may write all of the formulas (71) and (73)–(76) in terms of angles. For example, if ψ_{hk} is the angle between the hth tangent vector of E and the kth tangent vector of E', the second formulas in (73b) and (75b) both become, by application of Theorem 2,

$$\cos\psi_{hk} = \sum_j \cos\omega_{hj}\sec\theta_j\cos\phi_{kj} = \sum_j \cos\omega'_{kj}\sec\theta'_j\cos\phi'_{hj}. \tag{79}$$

Similarly, if χ_{hk} is the angle between the hth conjugate vector of E and the kth conjugate vector of E', we obtain, from the second formula in either (74b) or (76b),

$$\cos\chi_{hk} = \sum_j \cos\Omega_{hj}\sec\theta_j\cos\phi'_{jk} = \sum_j \cos\Omega'_{kj}\sec\theta'_j\cos\phi_{jh}.$$

Here, ω_{ij} and Ω_{ij} are the angles defined in §2 for E, and ω'_{ij} and Ω'_{ij} are the corresponding angles for E'.

Transformation of Christoffel symbols and the B's. From equations (17) and similar equations for the ennuple E', we readily obtain the equations of the transformation,

$$C'_{ij}{}^k \frac{\partial s^r}{\partial s'^k} = C_{pq}{}^r \frac{\partial s^p}{\partial s'^i}\frac{\partial s^q}{\partial s'^j} + \frac{\partial}{\partial s'^j}\frac{\partial s^r}{\partial s'^i}, \tag{80}$$

of the Christoffel symbols $C_{ij}{}^k$ for the ennuple E into the Christoffel symbols $C'_{ij}{}^k$ for the ennuple E'.

From these equations follow directly those of the transformation of the symbols $B_{ij}{}^k$ for E into the symbols $B'_{ij}{}^k$ for E', namely

$$B'_{ij}{}^k \frac{\partial s^r}{\partial s'^k} = B_{pq}{}^r \frac{\partial s^p}{\partial s'^i}\frac{\partial s^q}{\partial s'^j} + \left(\frac{\partial}{\partial s'^j}\frac{\partial s^r}{\partial s'^i} - \frac{\partial}{\partial s'^i}\frac{\partial s^r}{\partial s'^j}\right). \tag{81}$$

If we apply the result of differentiating the first of equations (68) to the last term in (81), we find that (81) may be rewritten in the form

$$B'_{ij}{}^k = \left(\frac{\partial}{\partial s^p}\frac{\partial s'^k}{\partial s^q} - \frac{\partial}{\partial s^q}\frac{\partial s'^k}{\partial s^p} + B_{pq}{}^r\frac{\partial s'^k}{\partial s^r}\right)\frac{\partial s^p}{\partial s'^i}\frac{\partial s^q}{\partial s'^j},$$

and hence, by virtue of (22) and the relations $\partial s'^k/\partial s^q = \alpha^k|_q$ and $\alpha'^k|_j = \delta^k_j$, in the form

$$\alpha'^k\big|_{i,j} - \alpha'^k\big|_{j,i} = (\alpha^k\big|_{p,q} - \alpha^k\big|_{q,p})\frac{\partial s^p}{\partial s'^i}\frac{\partial s^q}{\partial s'^j},$$

where the components of the covariant derivatives are, as indicated, referred to E'.*

Transformation of angular and distantial spread vectors. Appealing to the equations for E' corresponding to (28) and (29), we conclude that the components, referred to E, of the angular spread and curvature vectors of the congruences of E' are $\gamma_{ij}|^r = C'^{\,k}_{ij}(\partial s^r/\partial s'^k)$. It follows, then, from (80), that the equations of the transformation from the angular spread and curvature vectors of E to those of E', expressed in terms of components referred to E, are

$$\gamma_{ij}|^r = c_{pq}|^r\frac{\partial s^p}{\partial s'^i}\frac{\partial s^q}{\partial s'^j} + \frac{\partial}{\partial s'^j}\frac{\partial s^r}{\partial s'^i}. \tag{82}$$

The inverse transformation, when expressed in terms of the components referred to E', that is, $\gamma'_{ij}|^k$ and $c'_{ij}|^k$, has the same form.

Equations (82) *constitute the generalizations to Riemannian geometry of the most general form of the fundamental relation of Liouville for geodesic curvatures on a two-dimensional surface.*† But equations (82) are Christoffel's equations in invariant form. Thus, we have a striking geometric interpretation of Christoffel's famous formulas.

By means of (31) and the corresponding equations for E', we readily deduce from (81) the equations of the transformation from the distantial spread vectors of the pairs of congruences of E to those of the pairs of congruences of E'. Written in terms of components referred to E', they are

$$\beta_{ij}|^r = b_{pq}|^r\frac{\partial s^p}{\partial s'^i}\frac{\partial s^q}{\partial s'^j} + \frac{\partial}{\partial s'^j}\frac{\partial s^r}{\partial s'^i} - \frac{\partial}{\partial s'^i}\frac{\partial s^r}{\partial s'^j}. \tag{83}$$

Since $\partial s^r/\partial s'^i = \alpha_i|^r$, we infer from these equations the following theorem.‡

THEOREM 33. *If E is an ennuple of Tchebycheff, a necessary and sufficient condition that E' be an ennuple of Tchebycheff is that*

$$\frac{\partial \alpha_i|^r}{\partial s'^j} = \frac{\partial \alpha_j|^r}{\partial s'^i} \qquad (i, j, r = 1, 2, \cdots, n).$$

* The first of these equations has a simple geometric interpretation. According to Theorem 21, $B'^{\,k}_{ij}=0$ characterizes ds'^k as an exact differential; but, by Theorem 1, the vanishing of the quantities in the parenthesis is precisely the condition that $(\partial s'^k/\partial s^r)\,ds^r = ds'^k$ be exact.

† Graustein, loc. cit., p. 570.

‡ A generalization of Theorem 20 in Graustein, loc. cit., p. 580. This theorem is for the case $n=2$ and assumes that E is an orthogonal ennuple.

An obvious solution of these equations is $\alpha_i|^r$ constant. But $\alpha_i|^r$ must, in any case, satisfy the n equations $\bar{g}_{kl}\alpha_i|^k\alpha_i|^l=1$, and these equations cannot be satisfied by constants, in general.

Applications to a sheaf of congruences. A totality of ∞^{n-1} congruences which has the property that each two congruences cut under a constant angle we shall call a sheaf of congruences.

THEOREM 34. *If E is an ennuple from a sheaf of congruences, E' is an ennuple from the sheaf if and only if the n^2 quantities $\partial s^i/\partial s'^j$ are constants.*

Since E belongs to the sheaf, the angles ω_{ij}, θ_i, Ω_{ij} pertaining to E are all constant. Evidently, the ith congruence of E' belongs to the sheaf if and only if the angles ψ_{ri} $(r=1, 2, \cdots, n)$ which it makes with the n congruences of E are constant. But this is the case, according to (79), if and only if the angles ϕ_{ir} $(r=1, 2, \cdots, n)$ are constants, and hence, by (78), if and only if $\partial s_r/\partial s'^i$ $(r=1, 2, \cdots, n)$ are constants. Thus, the theorem is proved.

It follows that, if E and E' are ennuples from the same sheaf, formulas (82) and (83) become

$$\gamma_{ij}|^r = c_{pq}|^r \frac{\partial s^p}{\partial s'^i}\frac{\partial s^q}{\partial s'^j}, \qquad \beta_{ij}|^r = b_{pq}|^r \frac{\partial s^p}{\partial s'^i}\frac{\partial s^q}{\partial s'^j}. \tag{84}$$

These relations between the curvature and spread vectors of E and those of E' reflect the fact, evident from (80), that in a transformation from one ennuple of a sheaf to a second the Christoffel symbols behave like the components of tensors.

The relations embody various results. To begin with, we note

THEOREM 35. *If one ennuple from a sheaf of congruences has constant distantial spread vectors, and hence constant angular spread and curvature vectors, so has every ennuple from the sheaf.*

In particular, if one ennuple is an ennuple of Tchebycheff and therefore Cartesian, so is every ennuple.

From (84) and (71) we conclude:

$$g'^{ij}\gamma_{ij}|^r = g^{ij}c_{ij}|^r.$$

Hence the vector $g^{ij}c_{ij}|^r$ is the same for every ennuple of the sheaf and is, then, in this sense, an invariant vector of the sheaf. In particular, if E is orthogonal,

$$g^{ij}c_{ij}|^r = \sum_i c_{ii}|^r.$$

Thus, the sum of the curvature vectors of an orthogonal ennuple of a sheaf is the same for every orthogonal ennuple of the sheaf.*

Employing (84) and (71), we can construct other invariants of the sheaf. For example, the tensors

$$g^{ik}g^{jl}c_{ij}|^r c_{kl}|^t, \quad g^{ik}g^{jl}b_{ij}|^r b_{kl}|^t$$

are the same for every ennuple of the sheaf.

When we multiply each of these tensors by g_{rt} and sum over r, t, we obtain two scalar invariants of the sheaf. The values of these scalars for an orthogonal ennuple are $\sum(1/c_{ij}^2)$, $\sum(1/b_{ij}^2)$, where $1/c_{ij}$ and $1/b_{ij}$ are the lengths of the vectors $c_{ij}|$ and $b_{ij}|$. Thus, the sum of the squares of the lengths of all the curvature and angular spread vectors,† and the sum of the squares of the lengths of all the distantial spread vectors, of an orthogonal ennuple of a sheaf are the same for every orthogonal ennuple of the sheaf.

12. **Inclusion of congruences of curves in families of surfaces.** In §6 we found the conditions under which r linearly independent congruences lie in a family of r-dimensional surfaces. The purpose of this section is to treat the most general problem of this type, namely that of determining the family of surfaces of lowest dimensionality in which lie all the congruences of an arbitrarily chosen set of congruences.

The problem is not a simple one and much preliminary work is needed. To begin with, we shall show, by means of the following theorem, that we may restrict ourselves to sets of linearly independent congruences.

THEOREM 36. *If in a set of congruences there are r, and no more than r, linearly independent congruences and an arbitrarily chosen, but fixed, subset of r linearly independent congruences lies in a family of k-dimensional surfaces S_k and in no family of surfaces of lower dimensionality, all the congruences of the set lie in the surfaces S_k and in no family of surfaces of lower dimensionality.*

To establish the theorem, it suffices to prove that, if r linearly independent congruences lie in a family of k-dimensional surfaces S_k, any congruence which is a linear combination of them lies in the surfaces S_k. But this proposition is easily established.

The solution of the proposed problem is going to depend, not only on the distantial spread vectors of the given congruences, but also on the distantial spread vectors of the congruences determined by the distantial spread vectors

* Bortolotti, *Stelle di congruenze e parallelismo assoluto*, Rendiconti della Accademia dei Lincei, (6), vol. 9 (1929), pp. 530–538, gives this theorem. He approaches the subject indirectly, through the study of a metric connection with absolute parallelism, and is concerned with invariants which are the same simply for every orthogonal ennuple, not for every ennuple, of the sheaf.

† This result is also to be found in the paper by Bortolotti just cited.

of the given congruences. Accordingly, we shall find it convenient to introduce the following terminology.

DEFINITION 1. *The distantial spread vector of the two congruences determined by two ordered vector-fields shall be called the distantial spread vector of the two vector-fields, in the given order.*

We next prove a theorem for the distantial spread vectors of a given set of vector-fields analogous to Theorem 36 for congruences.

THEOREM 37. *If in a set of vector-fields there are r, and no more than r, linearly independent vector-fields, say $V_1, V_2, \cdots, V_r$, and if $V_1, V_2, \cdots, V_r$ and their distantial spread vectors are linearly dependent on k linearly independent vector-fields $V_1, V_2, \cdots, V_r, V_{r+1}, \cdots, V_k$, then the distantial spread vector of any two vector-fields of the given set is a linear combination of $V_1, V_2, \cdots, V_k$.*

Let the vector-fields $V_1, V_2, \cdots, V_r$ serve as the first r tangent vector-fields of an ennuple E, let r linearly independent linear combinations of $V_1, V_2, \cdots, V_r$ serve as the first r tangent vector-fields of a second ennuple E', and assume that the remaining tangent vector fields of E and E' are identical. Then

$$\alpha_h\big|^i = \frac{\partial s^i}{\partial s'^h} = 0 \qquad (h = 1, 2, \cdots, r;\ i = r+1, \cdots, n).$$

Hence, by (83),

$$\beta_{ij}\big|^t = \sum_{p,q}^{1-r} b_{pq}\big|^t \frac{\partial s^p}{\partial s'^i}\,\frac{\partial s^q}{\partial s'^j} \qquad (i, j = 1, 2, \cdots, r;\ t = r+1, \cdots, n). \tag{85}$$

Without loss of generality we may assume that $V_1, V_2, \cdots, V_k$ are the first k tangent vector-fields of E. It follows, then, by hypothesis, that $b_{pq}|^t = 0$ for $p, q = 1, 2, \cdots, r$; $t = k+1, \cdots, n$. Hence, $\beta_{ij}|^t$ is a linear combination of $V_1, V_2, \cdots, V_k$ and the theorem follows.

Suppose, now, that we have given a set, T_0, of linearly independent vectors, that is, vector-fields. In solving the proposed problem for the congruences determined by the vectors of T_0, we shall have to consider all the vectors obtainable from those of T_0 by repeated application of the process of finding distantial spread vectors. To systematize the repetition of this process, we adopt the following definition.

DEFINITION 2. *A distantial spread vector of T_0 of order $k(\geqq 1)$ shall be a distantial spread vector of two distantial spread vectors of T_0 of orders lower than k, at least one of which is of order $k-1$. A distantial spread vector of T_0 of order zero shall be a vector of T_0.*

The totality of distantial spread vectors of T_0 of order i we shall denote by D_i and the totality of those of orders $0, 1, \cdots, i$ we shall designate by T_i. Evidently,

$$T_i = T_{i-1} + D_i = D_0 + D_1 + \cdots + D_i \qquad (i \geqq 1).$$

THEOREM 38. *The set of vectors $T_i (i \geqq 1)$ consists of the vectors of T_0 and the distantial spread vectors of the vectors of T_{i-1}.*

The theorem follows directly from the definition.

For our purposes, the essential aspect of the set of vectors T_i is the *maximum number of linearly independent vectors contained in T_i*. We shall call this *the dimension number of T_i* and denote it by n_i. Inasmuch as T_i contains T_{i-1}, it is clear that $n_i \geqq n_{i-1}$, $i \geqq 1$.

The sequence $n_0, n_1, n_2, \cdots$ we shall refer to as *the sequence of dimension numbers of T_0*. Corresponding to it we may choose, in various ways, a sequence of sets of vectors

$$V_1, \cdots, V_{n_0}; V_{n_0+1}, \cdots, V_{n_1}; V_{n_1+1}, \cdots, V_{n_2}; \cdots, \tag{86}$$

such that $V_1, \cdots, V_{n_i}$ are n_i linearly independent vectors in T_i. From the definition of n_i, it is evident that all the vectors of T_i are linear combinations of $V_1, \cdots, V_{n_i}$, and that the vectors, $V_{n_{i-1}+1}, \cdots, V_{n_i}$, of the $(i+1)$st set of the sequence belong to D_i.

Since n_i can never exceed n, the sequence (86) is finite. We can, however, say more than this. It is true that, if a certain group in the sequence is empty, all succeeding groups are empty. In other words:

THEOREM 39. *If two consecutive numbers in the sequence of dimension numbers of T_0 are equal, all subsequent ones are equal to them*:

$$n_0 < n_1 < \cdots < n_{k-1} < n_k = n_{k+l} \leqq n \qquad (l = 1, 2, \cdots).$$

We are to prove that, if $n_{k+1} = n_k$, then $n_{k+2} = n_k$. Since $n_{k+1} = n_k$, the vectors of T_{k+1}, as well as the vectors of T_k, are linearly dependent on $V_1, \cdots, V_{n_k}$. The distantial spread vectors of $V_1, \cdots, V_{n_k}$, since they belong, by Theorem 38, to T_{k+1}, are then linearly dependent on $V_1, \cdots, V_{n_k}$. Hence, by Theorem 37, the distantial spread vectors of all the vectors of T_{k+1} are linearly dependent on $V_1, \cdots, V_{n_k}$. But these distantial spread vectors, together with the vectors of T_0, are precisely the vectors of T_{k+2}. Thus the vectors of T_{k+2} are linear combinations of $V_1, \cdots, V_{n_k}$, and therefore $n_{k+2} = n_k$.

By a *reduced set* of a given set of vectors we shall mean a set of linearly independent vectors of the given set on which all the vectors of the given set are linearly dependent; and by a *reduced sequence* of the sequence D_0,

$D_1, D_2, \cdots$ of successive distantial spread vectors of T_0 we shall mean a sequence of sets of vectors $D_0', D_1', D_2', \cdots$ such that the vectors of T_i', where $T_i' = T_{i-1}' + D_i'$, constitute a reduced set of the set of vectors T_i.

The sequence of groups of vectors (86) is a reduced sequence of the sequence $D_0, D_1, D_2, \cdots$. But this sequence is perhaps lacking in an important property of the sequence $D_0, D_1, D_2, \cdots$, namely, the property that every vector in the set D_i is a distantial spread vector of two vectors belonging to preceding sets. We shall call this the *property of cohesion*.

It is conceivable that a sequence of successive distantial spread vectors of T_0 cannot be rendered both reduced and cohesive. That this is not the case we shall prove by actually defining a sequence $D_0', D_1', D_2', \cdots$ which has both properties.

DEFINITION 3. *The vectors of D_0' shall be identical with those of D_0 (or T_0). The vectors of D_i' $(i \geqq 1)$ shall be chosen from the distantial spread vectors of the vectors of T_{i-1}' so that the vectors of T_i' constitute a reduced set for the vectors of T_0 and the distantial spread vectors of the vectors of T_{i-1}'.*

It is evident that the sequence $D_0', D_1', D_2', \cdots$ thus defined is cohesive. To show that it is a reduced sequence of $D_0, D_1, D_2, \cdots$, we must prove (a) that D_i' is a subset of D_i, and (b) that the dimension number of T_i' is the same as that of T_i.

A. *D_i' is a subset of D_i.* It is obvious from the definition that the theorem is true for $i = 0, 1$. Suppose that $i \geqq 2$. It follows from the definition that the distantial spread vectors of the vectors of T_{i-1}', from which the vectors of D_i' are chosen, are distantial spread vectors of T_0 of orders not greater than i. Hence, if a vector of D_i' does not belong to D_i, it is a distantial spread vector of T_0 of order less than i, and so is included among the distantial spread vectors of vectors of T_{i-2}'. But these are, by definition, linear combinations of the vectors of T_{i-1}'. Thus, the vector of D_i' in question is linearly dependent on the vectors of T_{i-1}', and this contradicts the demand that the vectors of $T_i' = T_{i-1}' + D_i'$ be linearly independent.

Since D_i' is a subset of D_i, T_i' is a subset of T_i, $i = 0, 1, 2, \cdots$. According to the definition, the vectors of T_i' are linearly dependent. It remains then to prove that the number of them, n_i', is equal to the maximum number, n_i, of linearly independent vectors in T_i.

B. $n_i' = n_i$. Inasmuch as $n_0' = n_0$, it suffices to show that, if $n_i' = n_i$, then $n_{i+1}' = n_{i+1}$. Since $n_i' = n_i$, the vectors of T_i are linear combinations of the vectors of T_i'. But the distantial spread vectors of the vectors of T_i' are linearly dependent on the vectors of $T_{i+1}' = T_i' + D_{i+1}'$. Hence, by Theorem 37, the distantial spread vectors of the vectors of T_i are linearly dependent on the vectors of T'_{i+1}. But this means, according to Theorem 38, that the vectors of T_{i+1} are linear combinations of the vectors of T'_{i+1}, and hence $n'_{i+1}=n_{i+1}$.

The result thus established may be stated as follows.

THEOREM 40. *Any sequence of groups of vectors formed as described in Definition 3 is a cohesive, reduced sequence of the sequence $D_0, D_1, D_2, \cdots$ of the successive distantial spread vectors of T_0.*

We return now to the proposed problem, restricting ourselves, as is permitted by Theorem 36, to a set of linearly independent congruences.

THEOREM 41. *If m is the largest number in the sequence of dimension numbers of the set, T_0, of vector-fields tangent to r linearly independent congruences, the r congruences lie in a family of m-dimensional surfaces and in no family of surfaces of lower dimensionality.*

By hypothesis,

$$r = n_0 < n_1 < n_2 < \cdots < n_{p-1} < n_p = m = n_{p+l} \qquad (l = 1, 2, \cdots). \tag{87}$$

Without loss of generality, we may take the given congruences as the first n_0 congruences of an ennuple E. The congruences lie, then, in the family of hypersurfaces $\phi=$const. if and only if

$$\frac{\partial\phi}{\partial s^i} = 0 \qquad (i = 1, 2, \cdots, n_0), \tag{88}$$

and this system of equations is compatible if and only if

$$\sum_{k=n_0+1}^{n} B_{ij}{}^k \frac{\partial\phi}{\partial s^k} = 0 \quad (i \neq j;\, i, j = 1, 2, \cdots, n_0). \tag{89}$$

The vectors $b_{ij}|^k=B_{ij}^k$, $i, j=1, 2, \cdots, n_0$, are the distantial spread vectors of the vectors of T_0, or T'_0. From them we choose, according to the prescriptions of Definition 3, the vectors of D'_1. In $T'_1=T'_0+D'_1$ we have then n_1 linearly independent vectors, on which all the distantial spread vectors of T'_0 are linearly dependent.

The vectors of T'_0 are the first n_0 tangent vectors of E and those of D'_1 may be thought of as the next n_1-n_0 tangent vectors of E. Equations (89) are, then, equivalent to the equations $\partial\phi/\partial s^k=0$, $k=n_0+1, \cdots, n_1$, and, when adjoined to equations (88), give rise to the extended system of equations

$$\frac{\partial\phi}{\partial s^i} = 0 \qquad (i = 1, 2, \cdots, n_1). \tag{90}$$

The procedure now repeats itself. The conditions of compatibility of (90), namely,

$$\sum_{k=n_1+1}^{n} B_{ij}{}^k \frac{\partial \phi}{\partial s^k} = 0 \quad (i \neq j;\, i, j = 1, 2, \cdots, n_1), \tag{91}$$

involve the distantial spread vectors of the vectors of T_1'. From these we choose, following Definition 3, the vectors of D_2', thus obtaining the n_2 linearly independent vectors, $T_2' = T_1' + D_2'$, on which the vectors involved in (91) are linearly dependent. Assuming that the vectors of D_2' are the "next" $n_2 - n_1$ tangent vectors of E, we find, then, that equations (90) and (91) may be replaced by the equivalent system $\partial\phi/\partial s^i = 0$, $i = 1, 2, \cdots, n_2$.

After this procedure has been carried out p times, we obtain the system of equations

$$\frac{\partial \phi}{\partial s^i} = 0 \qquad (i = 1, 2, \cdots, n_p), \tag{92}$$

as necessary and sufficient condition that the family of hypersurfaces $\phi = \text{const.}$ contain the r given congruences. Since, by (87), $n_p = m$, this system of equations is completely integrable. For the distantial spread vectors of the vectors of T_p' are, by hypothesis, linear combinations of the vectors of T_p', and since these are the first n_p vectors of E, $B_{ij}{}^k = 0$ for $i, j = 1, 2, \cdots, n_p$, $k = n_{p+1}, \cdots, n$, so that the conditions of integrability are identically satisfied.

It follows, now, that the r given congruences lie in the family of m-dimensional surfaces defined by $n-m$ functionally independent solutions of the system of equations (92), and in no family of surfaces of lower dimensionality.

The fact that equations (92) are completely integrable means that the first n_p $(=m)$ congruences of E lie in a family of m-dimensional surfaces. The tangent vectors of these congruences are precisely the vectors of T_p' and, by hypothesis, T_p' is a reduced set of vectors for the set consisting of T_0 and the distantial spread vectors of T_0 of all orders. Hence:

THEOREM 42. *A necessary and sufficient condition that r linearly independent congruences lie in a family of m-dimensional surfaces and in no family of surfaces of lower dimensionality is that m be the minimum number of linearly independent vectors on which the tangent vectors of the congruences and their distantial spread vectors of all orders are linearly dependent. The curves of m congruences whose tangent vectors constitute such a set of linearly independent vectors lie, then, in a family of m-dimensional surfaces, and it is this family of surfaces in which the given congruences are contained.*

*Application to nonholonomic spaces.** Let there be given in V_n a system of $n-r$ $(r>1)$ linearly independent total differential equations

$$\overline{A}^i{}_j dx^j = 0 \qquad (i = r+1, \cdots, n), \tag{93}$$

where $\overline{A}^i{}_j$ are functions of $x^1, x^2, \cdots, x^n$.

If the system has no integral whatsoever, it is said to represent in V_n a single r-dimensional nonholonomic space V_n^r.

In case the system has precisely $n-m$ $(m \geqq r)$ independent integrals, we shall say that it represents a nonholonomic manifold V_n which lies in a family of m-dimensional (metric) spaces. Actually, the system represents in this case ∞^{n-m} single nonholonomic spaces V_m^r, one in each of the m-dimensional (metric) spaces determined by the $n-m$ integrals. In particular, if the system is completely integrable $(m=r)$, the ∞^{n-r} nonholonomic spaces are the ∞^{n-r} (metric) space themselves.

We proceed to show how the discussion of the nonholonomic manifold (93) may be brought within the scope of our general theory and to deduce geometric conditions that the manifold lie in a family of m-dimensional (metric) spaces.

We think of the coefficients $\overline{A}^i{}_j$ in (93) as defining $n-r$ linearly independent covariant vector-fields $\overline{A}^i{}_j$, $i=r+1, \cdots, n$, and select r other covariant vector-fields $\overline{A}^i{}_j$, $i=1, 2, \cdots, r$, so that the n fields $\overline{A}^i{}_j$, $i=1, 2, \cdots, n$, are linearly independent. We then set $\bar{a}^i{}_j = \rho^i \overline{A}^i{}_j$, choosing the n functions ρ^i so that the n fields of contravariant vectors $\bar{a}_i{}^j$ which are conjugate to the n fields of covariant vectors $\bar{a}^i{}_j$ consist of unit vectors. Thereby, we obtain in V_n an ennuple of congruences, E, with reference to which the equations (93) of the nonholonomic manifold take the form

$$ds^i \equiv \bar{a}^i{}_j dx^j = 0 \qquad (i = r+1, \cdots, n). \tag{94}$$

It follows that the congruences of curves which lie in the nonholonomic manifold are precisely the congruences which are linearly dependent on the first r congruences of the ennuple E. Furthermore, ϕ is an integral of (94) if and only if $\partial\phi/\partial s^i = 0$, $i=1, 2, \cdots, r$, that is, if and only if the family of hypersurfaces $\phi = \text{const.}$ contains the first r congruences of E. Hence, the conditions of Theorem 42, applied to these r congruences, are precisely the conditions under which the nonholonomic manifold lies in a family of m-dimensional (metric) spaces.

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* For an extended treatment of nonholonomic spaces, see Vranceanu, *Studio geometrico dei sistemi anolonomi*, Annali di Matematica, (4), vol. 6 (1928), pp. 9–43.

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