ON A CERTAIN CORRESPONDENCE BETWEEN SURFACES IN HYPERSPACE*

BY

V. G. GROVE

1. INTRODUCTION

Consider a surface S and a point x on S . Let the parametric vector equation of S be

$$(1) \quad x = x(u, v).$$

The ambient space of the osculating planes at the point x to all of the curves through x is a certain space $S(2, 0)$ called *the two-osculating space of S at x* . This space is determined by the six points

$$(2) \quad x, x_u, x_v, x_{uu}, x_{uv}, x_{vv}.$$

It is the purpose of this paper to find all surfaces $\bar{S}$ in one-to-one point correspondence with S , such that the two-osculating space $\bar{S}(2, 0)$ of $\bar{S}$ coincides with the two-osculating space $S(2, 0)$ of S at corresponding points. We shall find that the surface S is not arbitrary, but that the functions x satisfy certain third-order partial differential equations studied by Lane† and by Bompiani.‡ A similar statement holds for the surface $\bar{S}$.

Let the surfaces S and $\bar{S}$ be in one-to-one point correspondence so that the corresponding points have the same curvilinear coordinates.

In order that $\bar{S}(2, 0)$ at $\bar{x}$ coincide with $S(2, 0)$ at x , it is necessary and sufficient that the functions

$$(3) \quad \bar{x}, \bar{x}_u, \bar{x}_v, \bar{x}_{uu}, \bar{x}_{uv}, \bar{x}_{vv}$$

be expressible as linear, homogeneous functions of the functions (2). The parametric vector equation of $\bar{S}$ will therefore be of the form

$$(4) \quad \bar{x} = \bar{x}(u, v) = Ax_{uu} + Bx_{uv} + Cx_{vv} + \alpha x_u + \beta x_v + \gamma x.$$

We shall call the case in which $S(2, 0)$ is a space of five dimensions and in which the coefficients A, B, C of (4) satisfy the inequality

* Presented to the Society, April 7, 1934; received by the editors February 20, 1934.

† E. P. Lane, *Integral surfaces of pairs of partial differential equations of the third order*, these Transactions, vol. 32 (1930), pp. 782–793. Hereafter referred to as Lane, *Surfaces*.

‡ E. Bompiani, *Determinazione delle superficie integrali d'un sistema di equazioni a derivate parziali lineari ed omogenee*, Rendiconti del Reale Istituto Lombardo di Scienze e Lettere, vol. 52 (1919), pp. 820–830. Hereafter referred to as Bompiani, *Surfaces*.

$$(5) \quad B^2 - 4AC \neq 0$$

the non-parabolic case, and the case in which $S(2, 0)$ is a space of five dimensions and in which

$$(6) \quad B^2 - 4AC = 0$$

the parabolic case. By proper choice of ϕ , ψ , and λ in the transformation

$$\bar{u} = \phi(u, v), \quad \bar{v} = \psi(u, v), \quad \bar{x} = \lambda \bar{x}',$$

in the non-parabolic case, we may write (4) in the form

$$(7) \quad \bar{x} = x_{uv} + \alpha x_u + \beta x_v + \gamma x;$$

and in the parabolic case in the form

$$(8) \quad \bar{x} = x_{uu} + \alpha x_u + \beta x_v + \gamma x.$$

We shall denote by $S(3, 0)$ the ambient space of the three-dimensional spaces osculating all of the curves on S through x . The space $S(3, 0)$ is determined by the six points (2) and the points

$$(9) \quad x_{uuu}, x_{uuv}, x_{uvv}, x_{vvv}.$$

2. THE NON-PARABOLIC CASE

If we differentiate $\bar{x}$ defined by (7) with respect to u and v we obtain the following expressions:

$$(10) \quad \begin{aligned} \bar{x}_u &= x_{uuu} + \alpha x_{uu} + \beta x_{uv} + (\alpha_u + \gamma)x_u + \beta_u x_v + \gamma_u x, \\ \bar{x}_v &= x_{uvv} + \alpha x_{uv} + \beta x_{vv} + \alpha_v x_u + (\beta_v + \gamma)x_v + \gamma_v x. \end{aligned}$$

The points $\bar{x}_u, \bar{x}_v$ are in the space $S(2, 0)$ if, and only if, the functions x defining the surface S satisfy a system of differential equations of the form

$$(11) \quad \begin{aligned} x_{uuv} &= ax_{uu} + hx_{uv} + bx_{vv} + lx_u + mx_v + dx, \\ x_{uvv} &= a'x_{uu} + h'x_{uv} + b'x_{vv} + l'x_u + m'x_v + d'x. \end{aligned}$$

It follows therefore that in the non-parabolic case $S(3, 0)$ is of dimensions no higher than seven.

Subcase a. Suppose that $S(3, 0)$ is a space of seven dimensions. It follows that the functions x satisfy the equations (11) and no other third-order differential equations. Under these conditions some of the integrability conditions* of system (11) are

$$a' = b = 0, \quad ah' + l' - a^2 - a_v = 0, \quad b'h + m - b'^2 - b'_u = 0.$$

Equations (10) may be written in the form

* Bompiani, *Surfaces*, p. 632.

$$(12) \quad \begin{aligned} \bar{x}_u &= (a + \alpha)x_{uu} + (h + \beta)x_{uv} + (l + \alpha_u + \gamma)x_u + (m + \beta_u)x_v + (d + \gamma_u)x, \\ \bar{x}_v &= (h' + \alpha)x_{uv} + (b' + \beta)x_{vv} + (l' + \alpha_v)x_u + (m' + \beta_v + \gamma)x_v + (d' + \gamma_v)x. \end{aligned}$$

From (12) we see that the points $\bar{x}_{uu}$, $\bar{x}_{uv}$, $\bar{x}_{vv}$ lie in $S(2, 0)$ if, and only if,

$$(13) \quad \alpha + a = 0, \quad \beta + b' = 0.$$

Therefore the point $\bar{x}$ defined by the expression

$$(14) \quad \bar{x} = x_{uv} - ax_u - b'x_v + \gamma x$$

generates a surface $\bar{S}$ whose two-osculating space $\bar{S}(2, 0)$ at $\bar{x}$ coincides with the two-osculating space $S(2, 0)$ at x .

From (12) and (14) we find that the expressions for $\bar{x}_u$ and $\bar{x}_v$ may be written in the form

$$(15) \quad \begin{aligned} \bar{x}_u &= [a(h - b') + l - a_u + \gamma]x_u + [d + \gamma_u + \gamma(h - b')]x + (h - b')\bar{x}, \\ \bar{x}_v &= [b'(h' - a) + m' - b'_v + \gamma]x_v + [d' + \gamma_v + \gamma(h' - a)]x + (h' - a)\bar{x}. \end{aligned}$$

Therefore the lines g joining corresponding points x and $\bar{x}$ of S and $\bar{S}$ form a congruence G , and the surfaces S and $\bar{S}$ sustain C nets* in relation C ; the developables of G intersect S and $\bar{S}$ in these C nets. Conversely if two nets are in relation C their sustaining surfaces have coincident two-osculating spaces at corresponding points.

Subcase b. Suppose that $S(3, 0)$ is of six dimensions. By proper choice of the notation, the functions x satisfy a system of differential equations of the form

$$(16) \quad \begin{aligned} x_{uuu} &= ax_{uu} + hx_{uv} + bx_{vv} + lx_u + mx_v + dx, \\ x_{uuv} &= a'x_{uu} + h'x_{uv} + b'x_{vv} + l'x_u + m'x_v + d'x \\ x_{uuu} &= Ax_{vvv} + a''x_{uu} + h''x_{uv} + b''x_{vv} + l''x_u + m''x_v + d''x, \end{aligned}$$

but no other third-order differential equations.

From (7) we find that

$$(17) \quad \begin{aligned} \bar{x}_u &= (a + \alpha)x_{uu} + (h + \beta)x_{uv} + bx_{vv} + (l + \alpha_u + \gamma)x_u + (m + \beta_u)x_v \\ &\quad + (d + \gamma_u)x, \\ \bar{x}_v &= a'x_{uu} + (h' + \alpha)x_{uv} + (\beta + b')x_{vv} + (l' + \alpha_v)x_u + (m' + \beta_v + \gamma)x_v \\ &\quad + (d' + \gamma_v)x. \end{aligned}$$

It follows from (17) and (16) that the points $\bar{x}_{uu}$, $\bar{x}_{uv}$, $\bar{x}_{vv}$ lie in $S(2, 0)$ if, and only if,

$$(18) \quad \beta + b' = 0, \quad b = 0, \quad A(a + \alpha) = 0.$$

* V. G. Grove, *The transformation C of nets in hyperspace*, these Transactions, vol. 33 (1931), pp. 733-741.

If we use (18) we may write equations (17) in the form

$$\begin{aligned}
 \bar{x}_u &= (a + \alpha)x_{uu} + [(l + \alpha_u + \gamma) - \alpha(h - b')]x_u + [m - b'_u + b'(h - b')]x_v \\
 &\quad + [d + \gamma_u - \gamma(h - b')]x + (h - b')\bar{x}, \\
 (19) \quad \bar{x}_v &= a'x_{uv} + [l' + \alpha_v - \alpha(h' + \alpha)]x_u + [m' - b'_v + \gamma + b'(h' + \alpha)]x_v \\
 &\quad + [d' + \gamma_v - \gamma(h' + \alpha)]x + (h' + \alpha)\bar{x}.
 \end{aligned}$$

Some of the integrability conditions of system (16) with $b=0$ are

$$\begin{aligned}
 (20) \quad Aa' &= 0, \quad a^2 + a'h + a_v = a'a'' + ah' + a'b' + a'_u + l', \\
 &\quad b'(h - b') + m - b'_u = a'b''.
 \end{aligned}$$

A. Suppose first that $A \neq 0$, $a' = 0$. Under conditions (18) equations (19) may be written in the form

$$\begin{aligned}
 (21) \quad \bar{x}_u &= [l - a_u + \gamma + a(h - b')]x_u + [d + \gamma_u - \gamma(h - b')]x + (h - b')\bar{x}, \\
 \bar{x}_v &= [m' - b'_v + \gamma + b'(h' - a)]x_v + [d' + \gamma_v - \gamma(h' - a)]x + (h' - a)\bar{x}.
 \end{aligned}$$

It follows therefore that if $A \neq 0$, $a' = 0$, the surfaces S and $\bar{S}$ sustain C nets, and the lines g joining corresponding points x and $\bar{x}$ form a congruence G , the developables of G intersecting these surfaces in their C nets.

B. Suppose that $A = 0$. Under this condition another integrability condition of system (16) is $b'' = 0$. Equations (19) may now be written in the form

$$\begin{aligned}
 (22) \quad \bar{x}_u &= (a + \alpha)x_{uu} + [l + \alpha_u + \gamma - \alpha(h - b')]x_u + [d + \gamma_u - \gamma(h - b')]x \\
 &\quad + (h - b')\bar{x}, \\
 \bar{x}_v &= a'x_{uv} + [l' + \alpha_v - \alpha(h' + \alpha)]x_u + [m' - b'_v + \gamma + b'(h' + \alpha)]x_v \\
 &\quad + [d' + \gamma_v - \gamma(h' + \alpha)]x + (h' + \alpha)\bar{x}.
 \end{aligned}$$

It follows that the tangent to $v = \text{const.}$ on $\bar{S}$ intersects the osculating plane to $v = \text{const.}$ on S . The tangent planes to S and $\bar{S}$ at x and $\bar{x}$ respectively intersect in a point; they will intersect in a line if, and only if, $a' = a + \alpha = 0$, that is, if, and only if, the parametric nets on S and $\bar{S}$ are in relation C . In this latter case the lines joining corresponding points x and $\bar{x}$ form a congruence.

3. THE PARABOLIC CASE

Let us consider the parabolic case. If we differentiate $\bar{x}$ defined by (8) with respect to u and v , we obtain

$$\begin{aligned}
 (23) \quad \bar{x}_u &= x_{uuu} + \alpha x_{uu} + \beta x_{uv} + (\alpha_u + \gamma)x_u + \beta_u x_v + \gamma_u x, \\
 \bar{x}_v &= x_{uvv} + \alpha x_{uv} + \beta x_{vv} + \alpha_v x_u + (\beta_v + \gamma)x_v + \gamma_v x.
 \end{aligned}$$

It follows therefore that if the points $\bar{x}_u, \bar{x}_v$ lie in $S(2, 0)$ the functions x must satisfy a system of differential equations of the form

$$(24) \quad \begin{aligned} x_{uuu} &= ax_{uu} + hx_{uv} + bx_{vv} + lx_u + mx_v + dx, \\ x_{uuv} &= a'x_{uu} + h'x_{uv} + b'x_{vv} + l'x_u + m'x_v + d'x. \end{aligned}$$

It follows therefore that $S(3, 0)$ is of dimensions no higher than seven.

Subcase a. Suppose that $S(3, 0)$ is of seven dimensions, that is, that the functions x do not satisfy a third third-order differential equation.

The system (24) has the following integrability conditions*:

$$(25) \quad \begin{aligned} b &= 0, \quad h = b', \quad a_v = a'_u + a'h' + l', \\ h_v + ah' + l &= h'_u + a'h + h'^2 + m', \\ ab' + m &= b'_u + b'h', \quad l_v + al' = l'_u + a'l + h'l' + d', \\ m_v + am' + d &= m'_u + a'm + h'm', \\ d_v + ad' &= d'_u + a'd + d'h'. \end{aligned}$$

It follows from (23) and (24) that the functions $\bar{x}_u$ and $\bar{x}_v$ are defined by the expressions

$$(26) \quad \begin{aligned} \bar{x}_u &= (a + \alpha)x_{uu} + (h + \beta)x_{uv} + (l + \alpha_u + \gamma)x_u + (m + \beta_u)x_v + (d + \gamma_u)x, \\ \bar{x}_v &= a'x_{uu} + (h' + \alpha)x_{uv} + (b' + \beta)x_{vv} + (l' + \alpha_v)x_u + (m' + \beta_v + \gamma)x_v \\ &\quad + (d' + \gamma_v)x. \end{aligned}$$

From (26) we find that the points $\bar{x}_{uu}$, $\bar{x}_{uv}$, $\bar{x}_{vv}$ lie in the space $S(2, 0)$ if and only if

$$(27) \quad \alpha + h' = 0, \quad \beta + b' = 0.$$

Therefore the surface $\bar{S}$ generated by the point $\bar{x}$ defined by the expression

$$(28) \quad \bar{x} = x_{uu} - h'x_u - b'x_v + \gamma x$$

is such that the two-osculating space $\bar{S}(2, 0)$ at $\bar{x}$ coincides with the space $S(2, 0)$ at x for every choice of γ .

If we make use of equation (28) we may write equation (26) in the form

$$(29) \quad \begin{aligned} \bar{x}_u &= \mu x_u + f x + A \bar{x}, \\ \bar{x}_v &= r x_u + \mu x_v + g x + B \bar{x}, \end{aligned}$$

wherein

$$(30) \quad \begin{aligned} \mu &= h'(a - h') + l - h'_u + \gamma = a'b' + m' - b'_v + \gamma, \\ f &= d + \gamma_u - \gamma(a - h'), \quad A = a - h', \quad B = a', \\ g &= d' + \gamma_v - a'\gamma, \quad r = a'h' + l' - h'_v. \end{aligned}$$

* Lane, *Surfaces*, p. 792.

We may readily verify that as x ($\bar{x}$) moves along the curve $v = \text{const.}$ on S ($\bar{S}$) the point

$$y = \bar{x} - \mu x, \quad r \neq 0,$$

describes a curve whose tangent at y is the line g joining x to $\bar{x}$. Moreover there exists no other curve on S ($\bar{S}$) along which x ($\bar{x}$) may move so that the line g will generate a developable surface. We may readily verify that *the lines g generate a congruence G composed of the tangents to a one-parameter family of asymptotic curves on the surface generated by the point y* . However the point y defined by the expression

$$y = \bar{x} - \mu x, \quad r = 0,$$

is a fixed point, and the lines g form a bundle of lines through this fixed point.

Subcase b. Suppose that the space $S(3, 0)$ is of six dimensions.

A. The points x_{uv} , x_{vv} , as may be seen from (23), will lie in the space $S(2, 0)$ if

$$(31) \quad \beta = -h, \quad \alpha = -h',$$

and if x satisfies the equations (24) and a differential equation of the form

$$(32) \quad x_{vvv} = a''x_{uu} + h''x_{uv} + b''x_{vv} + l''x_u + m''x_v + d''x.$$

Some of the integrability conditions of the system composed of equations (24) and (32) are

$$b = 0, \quad h = b', \quad m + b'(a - h') - b'_u = 0.$$

We may readily verify that *the point $\bar{x}$ defined by*

$$(33) \quad \bar{x} = x_{uu} - h'x_u - b'x_v + \gamma x$$

generates a surface $\bar{S}$ whose two-osculating space $\bar{S}(2, 0)$ at $\bar{x}$ coincides with the two-osculating space $S(2, 0)$ of S at x . Moreover the tangent planes to S at x and $\bar{S}$ at $\bar{x}$ intersect in a line h . The projectivity determined on h by the pencils of tangent lines to S and $\bar{S}$ at x and $\bar{x}$ is parabolic. The lines g joining x to $\bar{x}$ form a congruence of tangents to a one-parameter family of asymptotic curves on a surface.

B. The space $\bar{S}(2, 0)$ of $\bar{S}$ at $\bar{x}$ will also coincide with the space $S(2, 0)$ at x if

$$\beta + b' = 0,$$

and if x satisfies equations (24) and a differential equation of the form

$$(34) \quad x_{uuv} = a''x_{uu} + h''x_{uv} + b''x_{vv} + l''x_u + m''x_v + d''x.$$

Two of the integrability conditions of such a system are

$$b = 0, \quad b' = 0.$$

It follows therefore that any point defined by the expression

$$\bar{x} = x_{uu} + \alpha x_u + \gamma x$$

(α and γ arbitrary) in the osculating plane to $v = \text{const.}$ on S at x generates a surface $\bar{S}$ whose two-osculating space $\bar{S}(2, 0)$ at $\bar{x}$ coincides with the space $S(2, 0)$ at x . The tangent planes to S and $\bar{S}$ at x and $\bar{x}$ intersect in a point.

Suppose that in the expression (4) $A = B = C = 0$. By a transformation of the curvilinear coordinates we may write (4) in the form

$$(35) \quad \bar{x} = x_u + \gamma x.$$

By repeated differentiations we find that $\bar{S}(2, 0)$ coincides with $S(2, 0)$ if, and only if, the functions x satisfy a system of differential equations composed of equations of the form (24) and (34). It follows that the space $S(3, 0)$ of S at x is of six dimensions. Conversely if the functions satisfy such a system, a point $\bar{x}$ defined by (35) generates a surface of the required type.

4. THE CONJUGATE CASE

Suppose now that S sustains a conjugate net. By proper choice of the parameters we may take this net to be the parametric net. The functions x therefore satisfy an equation of the Laplace type

$$(36) \quad x_{uv} = ax_u + bx_v + cx.$$

It follows from (36) that $S(2, 0)$ is a space of four dimensions and that $S(3, 0)$ is a space of not more than six dimensions.

Let the point $\bar{x}$ be defined by the expression

$$(37) \quad \bar{x} = Ax_{uu} + Cx_{vv} + \alpha x_u + \beta x_v + \gamma x,$$

wherein not both A and C are zero.

A. Suppose first that $\bar{S}(3, 0)$ is of six dimensions. We find readily that there exist no surfaces $\bar{S}$ distinct from S such that the spaces $\bar{S}(2, 0)$ and $S(2, 0)$ coincide.

B. Suppose that $S(3, 0)$ is of five dimensions. We find from (37) that

$$(38) \quad \begin{aligned} \bar{x}_u = & Ax_{uuu} + (A + \alpha)x_{uu} + (bC + C_u)x_{vv} + [C(a_v + a^2) + a\beta + \alpha_u + \gamma]x_u \\ & + [C(c + ab + b_v) + b\beta + \beta_u]x_v + [C(c_v + ac) + \beta c + \gamma_u]x. \end{aligned}$$

A symmetrical expression obtains for $\bar{x}_v$. It follows that if $A \neq 0$, the func-

tions x satisfy an equation of the form

$$(39) \quad x_{uuu} = a'x_{uu} + b'x_{vv} + l'x_u + m'x_v + d'x.$$

In order that $\bar{x}_v$ lie in the space $S(2, 0)$, and that $S(3, 0)$ be a space of five dimensions the coefficient C must be zero.

Some of the integrability conditions of the system composed of equations (36) and (39) are

$$b' = 0, \quad m' = 0, \quad a'_v - a_u = c + ab + a_u.$$

Hence the curves $v = \text{const.}$ on S are plane curves. With the expression for $\bar{x}_v$ and $C = 0$, we find that $\bar{x}_{vv}$ lies in $S(2, 0)$ if, and only if, $\beta = 0$. Hence $\bar{x}$ lies in the plane of the curve $v = \text{const.}$

If we set $A = 1$, we find that the points $\bar{x}_u, \bar{x}_v$ are defined by the expressions

$$(40) \quad \begin{aligned} \bar{x}_u &= [l' + \gamma + \alpha_u - \alpha(a' + \alpha)]x_u \\ &\quad + [d' + \gamma_u - \gamma(a' + \alpha)]x + (a' + \alpha)\bar{x}, \\ \bar{x}_v &= (c + ab + a_u + \alpha_v)x_u + (b^2 + b_u + \alpha b + \gamma)x_v \\ &\quad + (c_u + bc + \alpha c + \gamma_v - a\gamma)x + a\bar{x}. \end{aligned}$$

The tangent planes to S and $\bar{S}$ at x and $\bar{x}$ intersect in a line. Hence if $S(3, 0)$ is a space of five dimensions, and if S sustains a conjugate net, the point $\bar{x}$ defined by (37) will describe a surface $\bar{S}$ whose two-osculating space $S(2, 0)$ at $\bar{x}$ coincides with $S(2, 0)$ at x if and only if each curve of one of the component families of curves of the conjugate net is a plane curve, and the point $\bar{x}$ is a point in the plane of the curve. The lines g joining x and $\bar{x}$ form a congruence.

Suppose that $\bar{x}$ lies in the tangent plane of S at x , that is, suppose that in (37) $A = C = 0$. We readily verify that if $S(3, 0)$ is of six dimensions the space $\bar{S}(2, 0)$ at $\bar{x}$ cannot coincide with the space $S(2, 0)$ at x for distinct surfaces S and $\bar{S}$. If $S(3, 0)$ is a space of five dimensions, the point $\bar{x}$ must lie in the tangent to one of the curves of the conjugate net, and that family of curves is a family of plane curves.

5. THE ASYMPTOTIC CASE

Suppose that S sustains a one-parameter family of asymptotic curves. Let the notation be so chosen that the curves $v = \text{const.}$ are the asymptotics. It follows that the functions x defining S satisfy the differential equation

$$(41) \quad x_{uu} = ax_u + bx_v + cx.$$

It follows that the space $S(3, 0)$ is a space of six dimensions at most.

Let $\bar{x}$ be defined by an expression of the form

$$(42) \quad \bar{x} = Bx_{uv} + Cx_{vv} + \alpha x_u + \beta x_v + \gamma x,$$

wherein not both B and C are zero.

A. We may readily verify that if $S(3, 0)$ is a space of six dimensions, there exists no surface $\bar{S}$ distinct from S with the desired property.

B. Suppose therefore that $S(3, 0)$ is a space of five dimensions. It follows from (42) that the points $\bar{x}_u$ and $\bar{x}_v$ are in $S(2, 0)$ if, and only if, $C=0$, and the functions x satisfy a differential equation of the form

$$(43) \quad x_{uvv} = h'x_{uv} + b'x_{vv} + l'x_u + m'x_v + d'x.$$

Two of the integrability conditions of the system composed of equations (41) and (43) are

$$(44) \quad b = 0, \quad c - b_u' + b'(a - b') = 0.$$

It follows therefore that the surface S is ruled.

If in (42) we set $C=0$, $B=1$, we find that

$$(45) \quad \begin{aligned} \bar{x}_u &= (a + \beta)x_{uv} + (a_v + a\alpha + \alpha_u + \gamma)x_u \\ &\quad + (c + \beta_u)x_v + (c_v + \alpha c + \gamma_u)x, \\ \bar{x}_v &= (a' + \alpha)x_{uv} + (\beta + b')x_{vv} + (l' + \alpha_v)x_u \\ &\quad + (m' + \beta_v + \gamma)x_v + (d' + \gamma_v)x. \end{aligned}$$

The points $\bar{x}_{uu}$, $\bar{x}_{uv}$, $\bar{x}_{vv}$ lie in $S(2, 0)$ if, and only if, $\beta = -b'$. Equation (45) may be written in the form

$$(46) \quad \begin{aligned} \bar{x}_u &= [a_v + a\alpha + \alpha_u + \gamma - \alpha(a - b')]x_u + [c_v + \alpha c + \gamma_u - \gamma(a - b')]x \\ &\quad + (a - b')\bar{x}, \\ \bar{x}_v &= [l' + \alpha_v - \alpha(a' + \alpha)]x_u + [m' - b_v' + \gamma + b'(a' + \alpha)]x_v \\ &\quad + [d' + \gamma_v - \gamma(a' + \alpha)]x + (a' + \alpha)\bar{x}. \end{aligned}$$

The point $\bar{x}$ defined by the expression

$$\bar{x} = x_{uv} + \alpha x_u - b'x_v + \gamma x$$

for arbitrary values of α and γ generates a surface $\bar{S}$ whose two-osculating space $\bar{S}(2, 0)$ at $\bar{x}$ coincides with the two-osculating space $S(2, 0)$ of S at x .

The point r defined by the expression $r = x_u - b'x$ is readily characterized as the only point, on the generator through x of the ruled surface, describing a surface for which the osculating plane to the curve $u = \text{const.}$ at r lies in the space of three dimensions tangent to the ruled surface along the generator through x . We find that

$$r_v + \alpha r = \bar{x} - (\alpha b' + \gamma + b_v')x.$$

It follows that *the lines g joining x to $\bar{x}$ form a congruence. The line g passes through x and intersects the tangent line to the curve $u = \text{const.}$ on the surface generated by the point r .*

Suppose that $\bar{x}$ lies in the tangent plane to S at x . We readily verify that $\bar{S}(2, 0)$ at $\bar{x}$ will coincide with $S(2, 0)$ at x if and only if $\bar{x}$ lies in the tangent line of the asymptotic curve on S through x , and if the functions x defining the surface satisfy a differential equation of the form (43).

MICHIGAN STATE COLLEGE,
EAST LANSING, MICH.