ALMOST PERIODIC FUNCTIONS IN GROUPS, II*

BY

S. BOCHNER AND J. VON NEUMANN

The present paper is a continuation of the article by J. von Neumann on *Almost periodic functions in a group*, I [1].† Its main object is to extend the theory of almost periodicity to those functions having values which are not numbers but elements of a general linear space L. For functions of a real variable this extension was begun by Bochner [2], and then applied by him, see [3], to a problem concerning partial differential equations.

Bochner assumed L to be both complete and metric. In the present paper we shall admit more general linear spaces. We shall drop the metric but keep the completeness. Since the usual notion of completeness is based on the notion of metric, it was necessary to establish, for linear spaces, a notion of completeness independent of it. This was done in the preceding note of J. von Neumann [4]. The results of this note will be employed throughout, and we observe that, from the very beginning, we shall assume that L is linear with respect to arbitrary complex coefficients, see [4], Appendix I.

As in [1], the main difficulty to overcome was the definition and the establishment of a mean. This was done in Part I. The definition of a mean remained actually the same as in [1], but the proof of the existence of a mean necessitated a more elaborate argument, although, in broad lines, the argument does not differ essentially.

In Part II we deduce the existence and uniqueness of a Fourier expansion for any almost periodic function. It is worth pointing out that the representations occurring in the Fourier expansions of abstract almost periodic functions are the same as for numerical almost periodic functions, only the constant coefficients by which the representations are multiplied are abstract elements instead of numbers. (More than that, if in a linear manifold L different topologies are suitable for our purposes, then even the nature of the coefficients no longer determines the precise nature of abstractness of the almost periodic function.) Thus, roughly speaking, there are no more abstract almost periodic functions than numerical almost periodic functions. In particular, if a group admits of no other numerical almost periodic functions than the constant ones, there exists no non-constant abstract almost periodic function, no matter how general the range-space L may be.

* Presented to the Society, December 28, 1934; received by the editors June 7, 1934.

† Numbers in brackets refer to the bibliography at the end of this paper.

In Part III we deduce the approximation theorem. Moreover, what is new also for numerical functions, we deduce summation theorems, that is to say, theorems concerning the construction of an almost periodic function of which only the Fourier expansion is given. For functions of Bohr these theorems were established by Bochner [5].

Finally, in Part IV we consider the special case in which L is a Hilbert space, or the space of bounded linear transformations of a Hilbert space into itself. We particularly refer to Theorems 39 and 40. Theorem 39 treats one of the rare cases of a class of almost periodic functions for which the Fourier expansions may be completely characterized by direct properties. Theorem 40 implies a necessary and sufficient criterion for a locally compact separable group to be compact. We should also mention that for the Abelian addition group of all integers, R. H. Cameron, in an unpublished paper, has found a result which has some connection with our Theorem 39.

PART I. THE MEAN-VALUE

Let L be a convex topological space which we shall assume throughout to be topologically complete (cf. [4], Definitions 1, 2b, and 10); and let $\mathfrak{G}$ denote a fixed arbitrary group. The elements of $\mathfrak{G}$ will be denoted by $a, b, c, \cdots, x, y, z, \cdots$. *Whenever a function is not specified, it will be tacitly assumed to be an element of* $L_b^{\mathfrak{G}}$. ($L_b^{\mathfrak{G}}$ is the set of all bounded functions with the domain $\mathfrak{G}$ and a range $\subset L$, cf. [4], Definition 11. $L_b^{\mathfrak{G}}$ is a topologically complete convex topological set, cf. [4], Theorem 18.)

If $F \epsilon L_b^{\mathfrak{G}}$, the "translated" functions $r_aF = F(xa)$, $l_aF = F(ax)$ also belong to $L_b^{\mathfrak{G}}$ for any $a \epsilon \mathfrak{G}$. We shall denote by $\mathfrak{R}_F$ the set of all r_aF, and by $\mathfrak{L}_F$ the set of all $l_aF (a \epsilon \mathfrak{G})$.

DEFINITION 1. *A function F is almost periodic if both sets $\mathfrak{R}_F$ and $\mathfrak{L}_F$ are totally bounded* (cf. [4], Definition 6). *The set of all almost periodic functions will be denoted by Ap.*

THEOREM 1. *Every constant function is almost periodic.*

The proof is obvious.

DEFINITION 2. *Given groups $\mathfrak{G}_1, \mathfrak{G}_2, \cdots, \mathfrak{G}_p$; we denote by $\mathfrak{G}_1 \times \mathfrak{G}_2 \times \cdots \times \mathfrak{G}_p$ the group consisting of all p-tuples*

$$x = [x_1, \cdots, x_p] \qquad (x_1 \epsilon \mathfrak{G}_1, \cdots, x_p \epsilon \mathfrak{G}_p)$$

with the rules

$$[x_1, \cdots, x_p][y_1, \cdots, y_p] = [x_1y_1, \cdots, x_py_p],$$
$$[x_1, \cdots, x_p]^{-1} = [x_1^{-1}, \cdots, x_p^{-1}].$$

And we shall say that a function is "almost periodic in $x_1, \cdots, x_p$" if it is almost periodic in $[x_1, \cdots, x_p]$ in the group $\mathfrak{G}_1 \times \mathfrak{G}_2 \times \cdots \times \mathfrak{G}_p$. In case $\mathfrak{G}_1 = \mathfrak{G}_2 = \cdots = \mathfrak{G}_p = \mathfrak{G}$, we shall write $\mathfrak{G}^p$ for $\mathfrak{G}_1 \times \mathfrak{G}_2 \times \cdots \times \mathfrak{G}_p$.

THEOREM 2. *If F is almost periodic, the set of (x, y)-functions $F_a = F(xay)$ $(\epsilon L^{\mathfrak{G}^2})$ is totally bounded $(a \epsilon \mathfrak{G})$.*

Given $U \epsilon \mathfrak{U}$, we choose $V \epsilon \mathfrak{U}$, with $V + V - V \epsilon U$. We choose elements $b_1, \cdots, b_n$ such that to any y there corresponds an index $\nu = \nu(y)$ for which

$$F(zy) - F(zb_\nu) \epsilon V \qquad (z \epsilon \mathfrak{G}).$$

(Remember for this, and for all discussions below, [4], Definition 6.) Hence

$$F(xay) - F(xab_\nu) \epsilon V \qquad (x, a \epsilon \mathfrak{G}).$$

We consider the n functions $F_\nu(x) = F(xb_\nu)$. For each ν the set of x-functions $F_\nu(xa)$ is totally bounded in a. By a simple argument (compare [1], the corresponding part in the proof of Theorem 9) it follows that there exist elements $a_1, \cdots, a_m$, and to each a there corresponds a $\mu = \mu(a)$ such that

$$F_\nu(xa) - F_\nu(xa_\mu) \epsilon V \qquad (x \epsilon \mathfrak{G};\ \nu = 1, \cdots, n).$$

Hence to each a there corresponds a $\mu = \mu(a)$ such that, for all $x, y \epsilon \mathfrak{G}$,

$$\begin{aligned} F(xay) - F(xa_\mu y) &= (F(xay) - F(xab_\nu)) + (F(xab_\nu) - F(xa_\mu b_\nu)) \\ &\quad + (F(xa_\mu b_\nu) - F(xa_\mu b)) \epsilon V + V - V \subset U. \end{aligned}$$

THEOREM 3. *Let $F(z)$ be an almost periodic function. Let z be the product of positive or negative integer powers of elements $x', x'', \cdots, x^{(p)}, a', a'', \cdots, a^{(q)} \epsilon \mathfrak{G}$ in an arbitrary fixed order, and let $F(z)$ be considered as a function*

$$(1) \qquad F_{a', \cdots, a^{(q)}} = F_{a', \cdots, a^{(q)}}(x', \cdots, x^{(p)})$$

of $L^{\mathfrak{G}^p}$, depending on the parameters $a', \cdots, a^{(q)}$. The set of these functions is totally bounded in $a', \cdots, a^{(q)}$.

It is easily seen that in this proof we may replace in z any set of consecutive factors which are all powers of variables or all powers of parameters by a new variable or a new parameter respectively (this may increase the indices p and q). Thus we may assume that z has the form $a'x'a''x'' \cdots$ or $x'a'x''a'' \cdots$. We denote the number of factors in z by k. For $k = 2$, our theorem holds by Definition 1, and we are going to apply induction from k to $k+1$. Denoting the product of the first $k-1$ factors in z by ξ, we have to dispose of two cases: (i) $z = \xi xa$, (ii) $z = \xi ax$. A subscript to ξ shall indicate that special values have been assigned to all parameters occurring in ξ.

In Case i we know that to each $U \epsilon \mathfrak{U}$ there correspond quantities $\xi_1, \cdots, \xi_n$ with

$$F(\xi x)\in \mathop{\mathfrak{S}}_{\nu=1}^{n}(F(\xi_\nu x)+V'). \tag{2}$$

(Cf. [4], Definitions 6, 11.) Given $U\in\mathfrak{A}$, let $V+V\subset U$. Replacing x by xa in (2) we get

$$F(\xi xa)\in \mathop{\mathfrak{S}}_{\nu=1}^{n}(F(\xi_\nu xa)+V'). \tag{3}$$

We determine elements $a_1, \cdots, a_m$ such that

$$F(za)\in \mathop{\mathfrak{S}}_{\mu=1}^{m}(F(za_\mu)+V'). \tag{4}$$

Putting here $z=\xi_\nu x$ and substituting the result in (3), we obtain

$$F(\xi xa)\in \mathop{\mathfrak{S}}_{\mu=1}^{m}\mathop{\mathfrak{S}}_{\nu=1}^{n}(F(\xi_\nu xa_\mu)+V'+V')\subset \mathop{\mathfrak{S}}_{\mu=1}^{m}\mathop{\mathfrak{S}}_{\nu=1}^{n}(F(\xi_\nu xa_\mu)+U').$$

This proves Case i.

In Case ii we determine quantities $\xi_1, \cdots, \xi_n$, such that

$$F(\xi z)\in \mathop{\mathfrak{S}}_{\nu=1}^{n}(F(\xi_\nu z)+V');$$

hence

$$F(\xi ax)\in \mathop{\mathfrak{S}}_{\nu=1}^{n}(F(\xi_\nu ax)+V'). \tag{5}$$

By Theorem 2 we may determine elements $a_1, \cdots, a_m$ such that

$$F(yax)\in \mathop{\mathfrak{S}}_{\mu=1}^{m}(F(ya_\mu x)+V').$$

Putting here $y=\xi_\nu$ and substituting the result in (5), we obtain

$$F(\xi ax)\in \mathop{\mathfrak{S}}_{\mu=1}^{m}\mathop{\mathfrak{S}}_{\nu=1}^{n}(F(\xi_\nu a_\mu x)+U').$$

This proves Case ii.

COROLLARY. *Let $F(z)$ be an almost periodic function. Let z be the product of positive or negative integer powers of variables $x', x'', \cdots, x^{(p)}$, of parameters $a', a'', \cdots, a^{(q)}$, and of constant elements $c', c'', \cdots, c^{(r)}$, in an arbitrary fixed order. The set of functions $F(z)$ $(\in L_b^{\mathfrak{G}^p})$ is again totally bounded in a', $a'', \cdots, a^{(q)}$.*

If we consider also the elements $c', \cdots, c^{(r)}$ as parameters, we get a larger set of functions $\epsilon L^{\mathfrak{G}^r}$ which by Theorem 3 is totally bounded. But a subset of a totally bounded set is also totally bounded.

THEOREM 4. *The function* (1) *of Theorem* 3 *is almost periodic in* $x', \cdots, x^{(p)}$, *for any fixed values of the parameters* $a', \cdots, a^{(q)}$.

If we multiply the element $[x', \cdots, x^{(p)}]\epsilon\mathfrak{G}^p$ by an element $[b', \cdots, b^{(p)}]$ on the right or the left, the argument z in $F(z)$ goes over into a product of positive or negative integer powers of the variables x, the constants a, and the parameters b. Replacing in the corollary of Theorem 3 the letters a and c by b and a, we find that the resulting set of functions is totally bounded in the parameters b. By Definition 1, the function (1) is almost periodic.

THEOREM 5. *If* $F_1, \cdots, F_k$ *are almost periodic functions with values in linear spaces* $L_1, \cdots, L_k$ *respectively, if* S_κ $(\kappa=1, \cdots, k)$ *is the range of* F_κ, *and if* $\mathfrak{F}(f_1, \cdots, f_k)$ *is a function with the domain* $f_\kappa\epsilon S_\kappa$, $\kappa=1, \cdots, k$, *uniformly continuous in it, and with a range* $\subset L$, *then the function*

$$G(x)=\mathfrak{F}(F_1(x), \cdots, F_k(x))$$

is almost periodic.

Definition 1 is fulfilled for the function $G(x)$ on account of [4], Theorem 9, if this theorem is applied to the function $\mathfrak{F}(F_1, \cdots, F_k)$, considered as a function of the functions $F_1, \cdots, F_k$, with the L_κ, L of Theorem 9 in [4] put equal to $L_{\kappa b}^{\mathfrak{G}}$, $L_b^{\mathfrak{G}}$, and its ranges S_κ to the $\mathfrak{R}_{F_\kappa}$, $\mathfrak{L}_{F_\kappa}$ respectively $(\kappa=1, \cdots, k)$.

THEOREM 6. *If* $F, G\epsilon Ap$, *then* $F\pm G\epsilon Ap$. *If* $F\epsilon Ap$, *and* $\alpha(x)$ *is a numerical almost periodic function, then* $\alpha F\epsilon Ap$.

If $F\epsilon Ap$, the set of functions $r_aF=F(xa)$ is totally bounded; if we put $x=1$ we find that the range of F is totally bounded. Theorem 6 follows from Theorem 5, since the functions f_1+f_2, f_1f_2 are uniformly continuous if f_1 and f_2 run over totally bounded sets, the closure of such sets being compact and separable ([4], Theorem 11, Theorem 7, Definition 10, and Theorem 16).

THEOREM 7. *The set* Ap *is closed.*

Let F be a condensation point of Ap, $U\epsilon\mathfrak{U}$, $V+V\subset U$. There is a $G\epsilon Ap$ such that $F\epsilon G+V'$. Obviously $r_aF\epsilon r_aG+V'$. Choose elements $F_1, \cdots, F_n\epsilon Ap$, such that

$$\mathfrak{R}_G\subset\mathop{\mathfrak{S}}_{\nu=1}^{n}(F_\nu+V').$$

Then

$$\mathfrak{R}_F\subset\mathop{\mathfrak{S}}_{\nu=1}^{n}(F_\nu+V'+V')\subset\mathop{\mathfrak{S}}_{\nu=1}^{n}(F_\nu+U'),$$

which proves that $\mathfrak{R}_F$ is totally bounded. Similarly, $\mathfrak{L}_F$ is totally bounded.

COROLLARY. *If a sequence of almost periodic functions is uniformly convergent, the limit function is also almost periodic.*

THEOREM 8. *If $F \epsilon Ap$, the set $S = ((\mathfrak{R}_F)_{\text{conv}})_{\text{cl}}$ has the following properties:* (i) $S \subset Ap$, (ii) *S is compact and separable,* (iii) $S_{\text{conv}} = S$. *The same is true also for $T = ((\mathfrak{L}_F)_{\text{conv}})_{\text{cl}}$.*

(i) follows from Theorems 4, 6 and 7. (ii) follows from the fact that S is totally bounded ([4], Theorems 14 and 16). (iii) was proved in [4] Appendix I.

THEOREM 9. *If $G \epsilon ((\mathfrak{R}_F)_{\text{conv}})_{\text{cl}}$, then*

$$((\mathfrak{R}_G)_{\text{conv}})_{\text{cl}} \subset ((\mathfrak{R}_F)_{\text{conv}})_{\text{cl}}. \tag{6}$$

The same is true if we replace $\mathfrak{R}$ by $\mathfrak{L}$.

For any $U \epsilon \mathfrak{U}$, we have by assumption ([4], Theorem 5): $G \subset (\mathfrak{R}_F)_{\text{conv}} + U'$. This immediately leads to the result that $\mathfrak{R}_G \subset (\mathfrak{R}_F)_{\text{conv}} + U'$. Since U is arbitrary, it follows ([4], Theorem 5) that $\mathfrak{R}_G \subset ((\mathfrak{R}_F)_{\text{conv}})_{\text{cl}}$. The final relation (6) is now obvious by Theorem 8, (iii).

DEFINITION 3. *If two almost periodic functions G, F are in the relation* (6), *we shall write*

$$G \dashv F.$$

Remark. It follows from Theorem 9 that the relation $G \dashv F$ is equivalent to $G \epsilon ((\mathfrak{R}_F)_{\text{conv}})_{\text{cl}}$.

THEOREM 10. *$F \dashv G$ and $G \dashv H$ imply $F \dashv H$.*

This follows immediately from Theorem 9.

THEOREM 11. *If $F \epsilon Ap$, and $U \epsilon \mathfrak{U}$, there is a $G \dashv F$, and a number $\mu \geqq 0$, such that for $H \dashv G$ and $a \epsilon \mathfrak{G}$* (cf. [4], Definition 13),

$$\|H(a)\|_U^+ = \mu.$$

We consider, for $K \epsilon ((\mathfrak{R}_F)_{\text{conv}})_{\text{cl}}$, the numerical function

$$\|K\|_{U'}^+ = \underset{x \epsilon \mathfrak{G}}{\text{l.u.b.}} \|K(x)\|_U^+. \tag{7}$$

It is a continuous function; hence it assumes, on every compact separable set, a maximum and a minimum; and it is easily seen that, for $K_1 \dashv K_2$, $\|K_1\|_{U'}^+ \leqq \|K_2\|_{U'}^+$. Therefore, if G is an element for which (7) attains its minimum, and if μ denotes the minimum value, we have

$$\|K\|_{U'}^+ = \|H\|_{U'}^+ = \mu, \qquad \text{for } K \dashv H \dashv G.$$

Thus, for $a_1, \cdots, a_n \epsilon \mathfrak{G}; \alpha_1, \cdots, \alpha_n \geqq 0; \alpha_1 + \cdots + \alpha_n = 1$,

$$\underset{x\epsilon\mathfrak{G}}{\text{l.u.b.}} \left\|\alpha_1 H(a_1 x) + \cdots + \alpha_n H(a_n x)\right\|_U^+ = \underset{x\epsilon\mathfrak{G}}{\text{l.u.b.}} \left\|H(x)\right\|_U^+ = \mu.$$

Connecting this with

$$\mu \geqq \underset{x\epsilon\mathfrak{G}}{\text{l.u.b.}} \left\|H(x)\right\|_U^+ \geqq \underset{x\epsilon\mathfrak{G}}{\text{l.u.b.}} \sum_{\nu=1}^{n} \alpha_\nu \left\|H(a_\nu x)\right\|_U^+,$$

we find for the numerical function $\gamma(x) = \left\|H(x)\right\|_U^+$ the property

$$\underset{x\epsilon\mathfrak{G}}{\text{l.u.b.}}\ \gamma(x) = \underset{x\epsilon\mathfrak{G}}{\text{l.u.b.}}\ (\alpha_1\gamma(a_1 x) + \cdots + \alpha_n\gamma(a_n x)). \tag{8}$$

By Theorem 5, $\gamma(x)$ is almost periodic; and (8) proves that

$$M_x\gamma(x) = \underset{x\epsilon\mathfrak{G}}{\text{l.u.b.}}\ \gamma(x).$$

(Cf. the definition of M_x in [1], Definition 4.) Hence $\gamma(x)$ is constant (cf. [1], Theorem 7, (4), putting $f(x) = [\text{l.u.b.}_{x\epsilon\mathfrak{G}}\gamma(x)] - \gamma(x)$), and the proof of our theorem is completed.

COROLLARY. *If $F\epsilon Ap$, $U\epsilon\mathfrak{U}$, $f\epsilon L$, there is a $G \dashv F$, and a number $\mu \geqq 0$, such that for $H \dashv G$, $a\epsilon\mathfrak{G}$,*

$$\left\|H(a) - f\right\|_U^+ = \mu. \tag{9}$$

Apply Theorem 11 to $F_1(x) = F(x) - f$, and denote the resulting $G_1(x)$ by $G(x) - f$.

THEOREM 12. *If $F\epsilon Ap$, there is an $H \dashv F$ which is constant.*

We denote by S the range of $F(x)$, and by T the set $(S_{\text{conv}})_{\text{cl}}$. S is totally bounded (compare the proof of Theorem 6). Hence by [4], Theorems 11, 14, T is totally bounded too. If f (ϵL) is a value of a function $G \dashv F$, then f is a condensation point of elements of the form

$$\alpha_1 F(a_1 x) + \cdots + \alpha_n F(a_n x), \qquad \alpha_1, \cdots, \alpha_n \geqq 0; \alpha_1 + \cdots + \alpha_n = 1;$$

but an element of this form is contained in S_{conv}; thus $f\epsilon T$. Therefore there exists a compact separable set $T \subset L$ which contains the ranges of all $G \dashv F$.

Let $f_1, f_2, \cdots$ be a dense sequence in T, $W_1, W_2, W_3, \cdots$ a complete set of open neighborhoods of zero. Write all pairs $n, p = 1, 2, \cdots$ in a sequence n_k, p_k, $k = 1, 2, \cdots$, and define a sequence of functions $G_0, G_1, G_2, \cdots$ in this manner: $G_0 = F$; G_{k+1} is the G of the corollary to Theorem 11 if applied to $F = G_k$, $U = U_k = W_{p_k}$, $f = f_{n_k}$. Thus

$$F \dashv G_0 \dashv G_1 \dashv G_2 \dashv \cdots,$$

and for all $H \dashv G_k$,

$$\|H(x) - f_{n_k}\|^{+}_{U_k} = \mu_k, \tag{10}$$

where μ_k is a number depending on k.

The sets $((\mathfrak{R}_{G_k})_{\text{conv}})_{\text{cl}}$ are monotonely decreasing as $k \to \infty$, all closed and non-empty, and all subsets of the compact separable set $((\mathfrak{R}_F)_{\text{conv}})_{\text{cl}}$; thus they have a common element H. Relation (10) means that for any p and any $x, y \in \mathfrak{G}$, the relation

$$\|H(x) - f\|^{+}_{W_p} = \|H(y) - f\|^{+}_{W_p} \tag{11}$$

holds for all $f = f_n$. The f_n being dense, and the pseudo-metric being continuous, the relation (11) holds for all $f \in T$. Putting $f = H(y)$ we obtain $\|H(x) - H(y)\|^{+}_{W_p} = 0$; hence $H(x) - H(y) \in W_p$ $(p = 1, 2, \cdots)$ (cf. [4], Definition 13); hence $H(x) = H(y)$.

Corollary. *If $F \in Ap$, there is a $\phi \in L$, and a sequence of systems* $(n = 1, 2, 3, \cdots)$

$$\alpha_{n,1}, \cdots, \alpha_{n,m_n} \geqq 0, \ \alpha_{n,1} + \cdots + \alpha_{n,m_n} = 1, \ a_{n,1}, \cdots, a_{n,m_n} \in \mathfrak{G} \tag{12}$$

such that for every $U \in \mathfrak{U}$ there exists an $n_1 = n_1(U)$, for which $n \geqq n_1$ implies

$$\alpha_{n,1}F(a_{n,1}x) + \cdots + \alpha_{n,m_n}F(a_{n,m_n}x) - \phi \in U.$$

We denote the constant value of the function H of Theorem 12 by ϕ. Since $(\mathfrak{R}_F)_{\text{conv}}$ is separable, and H is an element of its closure, H is the limit of a sequence of (equal or different) elements of $(\mathfrak{R}_F)_{\text{conv}}$. Hence the corollary.

Theorem 13. *If $F \in Ap$, there is a $\psi \in L$, and a sequence of systems* (12), *such that for every $U \in \mathfrak{U}$ there exists an $n_1 = n_1(U)$, for which $n \geqq n_1$ implies*

$$\alpha_{n,1}F(xa_{n,1}y) + \cdots + \alpha_{n,m_n}F(xa_{n,m_n}y) - \psi \in U.$$

Apply the corollary to Theorem 12 to the function $F(z^{-1}y)$ of $L^{\mathfrak{G}^2}$ (cf. Definition 2), and then replace z^{-1} by x.

Definition 4. *If $F \in Ap$, every $\psi \in L$ which has the property described in Theorem* 13 *is a mean of F.*

Theorem 14. *If $F, G \in Ap$, and ψ, χ are respective means of F, G, then $\psi \pm \chi$ is a mean of $F \pm G$.*

Given $U \in \mathfrak{U}$, choose $V \in \mathfrak{U}$, with $2V - 2V \in U$. Suppose that $\alpha_\mu \geqq 0$, $\alpha_1 + \cdots + \alpha_m = 1$, $a_\mu \in \mathfrak{G}$, $\beta_\nu \geqq 0$, $\beta_1 + \cdots + \beta_n = 1$, $b_\nu \in \mathfrak{G}$, and that

$$\alpha_1 F(xa_1y) + \cdots + \alpha_m F(xa_my) - \psi \in V, \tag{13}$$

$$\beta_1 F(xb_1y) + \cdots + \beta_n F(xb_ny) - \chi \in V. \tag{14}$$

In (13) we replace y by $b_\nu y$, multiply the equation by β_ν, and sum over ν. We obtain, using [4], Theorem 12,

$$\sum_{\mu=1}^{m}\sum_{\nu=1}^{n} \alpha_\mu\beta_\nu F(xa_\mu b_\nu y) - \psi\epsilon\sum_{\nu=1}^{n}\beta_\nu V \subset 2V.$$

Similarly, if in (14) we replace x by xa_μ, multiply by α_μ and sum over μ, we get

$$\sum_{\mu=1}^{m}\sum_{\nu=1}^{n} \alpha_\mu\beta_\nu G(xa_\mu b_\nu y) - \chi\epsilon\sum_{\mu=1}^{m}\alpha_\mu V \subset 2V.$$

Hence

$$\sum_{\mu=1}^{m}\sum_{\nu=1}^{n} \alpha_\mu\beta_\nu(F(xa_\mu b_\nu y) - G(xa_\mu b_\nu y)) - (\psi - \chi)\epsilon 2V - 2V \subset U.$$

As

$$\alpha_\mu\beta_\nu \geqq 0,\ \sum_{\mu=1}^{m}\sum_{\nu=1}^{n}\alpha_\mu\beta_\nu \geqq 1,\ a_\mu b_\nu \epsilon \mathfrak{G},$$

and U was arbitrary, we conclude that $\psi-\chi$ is a mean of $F-G$. A similar argument proves that $\psi+\chi$ is a mean of $F+G$.

COROLLARY. *If $F_1, F_2, \cdots, F_N \epsilon Ap$ $(N=1, 2, 3, \cdots)$, with the means $\psi_1, \psi_2, \cdots, \psi_N$ respectively, and if $U\epsilon\mathfrak{U}$, then there exist numbers $\alpha_1, \cdots, \alpha_n \geqq 0$, $\alpha_1+\cdots+\alpha_n=1$, and elements $a_1, \cdots, a_n \epsilon \mathfrak{G}$, such that simultaneously for $\nu=1, 2, \cdots, N$,*

$$\alpha_1 F_\nu(xa_1y) + \alpha_2 F_\nu(xa_2y) + \cdots + \alpha_n F_\nu(xa_ny) - \psi_\nu \subset U.$$

The case $N=2$ was treated in the proof of Theorem 14, and the same argument can be used to extend our statement from N to $N+1$.

THEOREM 15. *Every almost periodic function has one and only one mean.*

If ψ and χ are both means of F, $\psi-\chi$ is a mean of $F-F=0$. But every mean of 0 is 0. Hence the uniqueness of the mean.

DEFINITION 5. *If $F\epsilon Ap$, its (unique) mean will be denoted by MF or $M_xF(x)$.*

THEOREM 16. *The mean has the following properties:*

1. *If $F(x)=\phi$ (constant ϵL), then $MF=\phi$.*
2. *If α is a number, $M(\alpha F)=\alpha MF$.*
3. $M(F\pm G)=MF\pm MG$.
4. $M_xF(ax)=M_xF(x)$.
5. $M_xF(xa)=M_xF(x)$.
6. $M_xF(x^{-1})=M_xF(x)$.
7. *If $U\epsilon\mathfrak{U}$,* $\|MF\|_U^+ \leqq M_x(\|F(x)\|_U^+) \leqq \|F\|_{U'}^+$.
8. *If $U\epsilon\mathfrak{U}$, and $F-G\epsilon U$, then $MF-MG\epsilon 2U$.*

1 and 2 are obvious. 3 follows from Theorem 14. 4, 5, 6 are easily deduced from Definition 5. The first half of 7 follows from the relation

$$\|\alpha_1 F(xa_1y) + \cdots + \alpha_n F(xa_ny)\|_U^+ \leqq \alpha_1\|F(xa_1y)\|_U^+ + \cdots + \alpha_n\|F(xa_ny)\|_U^+$$

and the continuity of the pseudo-metric; the second half is a proved theorem on real almost periodic functions (cf. [1], Theorem 7). With regard to 8, from $\|F-G\|_{U'}^+ \leqq 1$, it follows that $\|MF-MG\|_U^+ \leqq 1$, and thus $MF-MG \epsilon 2U$.

THEOREM 17. *The properties* 1, 2, 3, 4 (*or* 5), 8, *of Theorem* 16, *determine our mean uniquely.*

Replacing ab by ba in $\mathfrak{G}$ shows that it is sufficient to consider the properties 1, 2, 3, 4, 8. Let NF be a notion defined in Ap satisfying these properties. If $V+V \subset U \epsilon \mathfrak{U}$, $\alpha_1 + \cdots + \alpha_n = 1$, and

$$\alpha_1 F(a_1x) + \cdots + \alpha_n F(a_nx) - MF \subset V,$$

then

$$NF - MF = N(\alpha_1 F(a_1x) + \cdots + \alpha_n F(a_nx) - MF) \subset 2V \subset U.$$

U being arbitrary, $NF = MF$.

Remark. The properties 3, 7 of Theorem 16 imply the property 8. Hence we have that the properties 1, 2, 3, 4 (or 5), 7, of Theorem 16 determine our mean uniquely.

THEOREM 18. *If* $x \epsilon \mathfrak{G}$, $y \epsilon \mathfrak{G}'$, *and* $F(x, y)$ *is almost periodic in* x, y, *then* $F(x, y)$ *is almost periodic in* x *for fixed* y, *and almost periodic in* y *for fixed* x, $G(x) = M_yF(x, y)$ *is almost periodic in* x, $H(y) = M_xF(x, y)$ *is almost periodic in* y, *and*

$$\begin{aligned} MF &= M_xG(x) = M_x(M_yF(x, y)), \\ MF &= M_yH(y) = M_y(M_xF(x, y)). \end{aligned} \tag{15}$$

If the sets $F(ax, by)$, $F(xa, yb)$ are totally bounded for $a \epsilon \mathfrak{G}$, $b \epsilon \mathfrak{G}'$, then the sets $F(ax, y)$, $F(xa, y)$, which are parts of them, are totally bounded for $a \epsilon \mathfrak{G}$. Hence $F(x, y)$ is almost periodic in x for fixed y. Similarly, if we interchange x and y, $F(x, y)$ is almost periodic in y for fixed x.

Given $U \epsilon \mathfrak{U}$, choose $V \epsilon \mathfrak{U}$, with $V+V \subset U$, and elements $a_1, \cdots, a_m, b_1, \cdots, b_n$, such that

$$F(ax, y) \epsilon \mathfrak{S}(F(a_1x, y) + V', \cdots, F(a_mx, y) + V'),$$
$$F(xb, y) \epsilon \mathfrak{S}(F(xb_1, y) + V', \cdots, F(xb_n, y) + V').$$

By Theorem 16, 8,

$$G(ax) = M_yF(ax, y) \epsilon \mathfrak{S}(M_yF(a_1x, y) + U', \cdots, M_yF(a_mx, y) + U'),$$

and similarly

$$G(xb)\epsilon\mathfrak{S}(M_yF(xb_1, y) + U', \cdots, M_yF(xb_n, y) + U').$$

Hence $G(x)$ is almost periodic, and similarly $H(y)$. In order to prove the first half of (15) we have only to show that the "mean" $NF = M_x(M_yF)$ satisfies the conditions 1, 2, 3, 4, 7, of Theorem 16. But this is easily verified; for instance, in the case of the first half of condition 7, we have

$$\|NF\|_U^+ = \|M_x(M_yF(x, y))\|_U^+ \leqq M_x(\|M_yF(x, y)\|_U^+) \leqq M_xM_y(\|F\|_{U'}^+) = N(\|F\|_{U'}^+)$$

(cf. [1], Theorem 10). Similarly for the second half of (15).

Part II. Fourier expansions

Definition 6. *If $\phi(x)$ is a numerical almost periodic function, and $F\epsilon Ap$, $\phi\times F$ is the function*

$$M_y(\phi(xy^{-1})F(y)) = M_y(\phi(xy)F(y^{-1}))$$

which again belongs to Ap.

Theorem 19. *$\phi\times F$ is linear in ϕ and in F, and associative: $(\phi\times\psi)\times F = \phi\times(\psi\times F)$.*

The first follows from Theorem 16, 2, 3. The second follows from the following formal calculations, each step of which is justified by one of the foregoing theorems:

$$\begin{aligned}(\phi\times\psi)\times F &= M_y(M_z(\phi(xy^{-1}z^{-1})\psi(z))F(y)) = M_yM_z(\phi(xy^{-1}z^{-1})\psi(z)F(y)) \\ &= M_{(y,z)}(\phi(xy^{-1}z^{-1})\psi(z)F(y)). \\ \phi\times(\psi\times F) &= M_y(\phi(xy^{-1})M_z(\psi(yz^{-1})F(z))) = M_yM_z(\phi(xy^{-1})\psi(yz^{-1})F(z)) \\ &= M_zM_y(\phi(xy^{-1})\psi(yz^{-1})F(z)) = M_z(M_y(\phi(xy^{-1})\psi(yz^{-1}))F(z)),\end{aligned}$$

and substituting yz for y,

$$\begin{aligned}&= M_z(M_y(\phi(xz^{-1}y^{-1})\psi(y))F(z)) = M_zM_y(\phi(xz^{-1}y^{-1})\psi(y)F(z)) \\ &= M_{(y,z)}(\phi(xy^{-1}z^{-1})\psi(z)F(y)).\end{aligned}$$

Theorem 20. *If $F(x, y)$ is an almost periodic function of $L^{\mathfrak{G}\times\mathfrak{G}'}$, and $U\epsilon\mathfrak{U}$, there exist numbers $\alpha_1, \cdots, \alpha_n \geqq 0$, $\alpha_1 + \cdots + \alpha_n = 1$, and elements $a_\nu\epsilon\mathfrak{G}$, such that, for all $x\epsilon\mathfrak{G}$, $y\epsilon\mathfrak{G}'$,*

$$\sum_{\nu=1}^{n}\alpha_\nu F(a_\nu x, y) - M_xF(x, y)\epsilon U. \tag{16}$$

For fixed y, $F(x, y)$ is almost periodic in x. Hence, by the corollary to Theorem 14, if any finite number of fixed values is assigned to the variable y, it is possible to choose quantities α_ν, a_ν which satisfy (16) for all x.

Now, choose $V \in \mathfrak{U}$, with $2V+V-2V \subset U$, and then elements $b_1, \cdots, b_m$, such that $F(x, yb) \in \mathfrak{S}(F(x, yb_1)+V', \cdots, F(x, yb_m)+V')$. Putting $y=1$, we obtain

$$F(x, b) \in \mathfrak{S}(F(x, b_1) + V', \cdots, F(x, b_m) + V').$$

We determine quantities α_ν, a_ν such that

$$\sum_{\nu=1}^{n} \alpha_\nu F(a_\nu x, b_\mu) - M_x F(x, b_\mu) \in V \tag{17}$$

for $\mu=1, \cdots, m$. To each b there corresponds a b_μ such that

$$F(x, b) - F(x, b_\mu) \in V. \tag{18}$$

Hence by Theorem 16, 8,

$$M_x F(x, b) - M_x F(x, b_\mu) \in 2V. \tag{19}$$

On the other hand, it follows from (18) that

$$\sum_{\nu=1}^{n} \alpha_\nu F(a_\nu x, b) - \sum_{\nu=1}^{n} \alpha_\nu F(a_\nu x, b_\mu) \in 2V. \tag{20}$$

Combining (17), (19), (20), we obtain, for any $b \in \mathfrak{G}'$,

$$\sum_{\nu=1}^{n} \alpha_\nu F(a_\nu x, b) - M_x F(x, b) \in 2V + V - 2V \subset U,$$

and this proves the theorem.

DEFINITION 7. *A weight function is a real almost periodic function $\phi(x)$ with the properties: $\phi(x) \geqq 0$, $M\phi=1$. A special weight function is a weight function which is a finite linear aggregate of representation coefficients.* (Cf. [1], Chapter III.)

THEOREM 21. *If $F \in Ap$, and $\phi(x)$ is a weight function, then $\phi \times F \dashv \mathfrak{Z} F$.*

If $U \in \mathfrak{U}$, choose $V \in \mathfrak{U}$, with $V+V \subset U$. Then $\phi \times F = M_y(\phi(y)F(y^{-1}x))$. By Theorem 20 there are numbers $\alpha_1, \cdots, \alpha_n \geqq 0$, $\alpha_1 + \cdots + \alpha_n = 1$, and elements $a_1, \cdots, a_n \in \mathfrak{G}$, such that

$$\phi \times F - \sum_{\nu=1}^{n} \alpha_\nu \phi(a_\nu y) F(y^{-1} a_\nu^{-1} x) \in V. \tag{21}$$

The function

$$\psi(y) = \sum_{\nu=1}^{n} \alpha_\nu \phi(a_\nu y)$$

is a weight function. Hence there exist elements y_1, y_2, such that $\psi(y_1) \geqq 1$, $\psi(y_2) \leqq 1$. Therefore we may find a γ, $0 \leqq \gamma \leqq 1$, with $\gamma\psi(y_1)+(1-\gamma)\psi(y_2)=1$. From (20) we easily deduce that

$$\phi \times F \subset \gamma \sum_{\nu=1}^{n} \alpha_\nu \phi(a_\nu y_1) F(y_1^{-1} a_\nu^{-1} x) + (1-\gamma) \sum_{\nu=1}^{n} \alpha_\nu \phi(a_\nu y_2) F(y_2^{-1} a_\nu^{-1} x) + U'.$$

It is easily seen that the function in x on the right side $\epsilon(\mathfrak{R}_F)_{\mathrm{conv}}$. Hence $\phi \times F \epsilon (\mathfrak{R}_F)_{\mathrm{conv}} + U'$, for any $U' \epsilon \mathfrak{U}'$. Thus ([4], Theorem 5) $\phi \times F \subset ((\mathfrak{R}_F)_{\mathrm{conv}})_{\mathrm{cl}}$.

LEMMA 1. *If $\phi(x, y)$ is a real almost periodic function in x, y, then*

$$\psi(x) = \underset{y \epsilon \mathfrak{G}'}{\text{l.u.b.}}\ \phi(x, y)$$

is almost periodic in x.

If $|\phi(ax, by) - \phi(a_\nu x, b_\nu y)| \leqq \epsilon$, then also

$$\left| \underset{y \epsilon \mathfrak{G}'}{\text{l.u.b.}}\ \phi(ax, by) - \underset{y \epsilon \mathfrak{G}'}{\text{l.u.b.}}\ \phi(a_\nu x, b_\nu y) \right| \leqq \epsilon,$$

that is,

$$\left| \underset{y \epsilon \mathfrak{G}'}{\text{l.u.b.}}\ \phi(ax, y) - \underset{y \epsilon \mathfrak{G}'}{\text{l.u.b.}}\ \phi(a_\nu x, y) \right| \leqq \epsilon.$$

Hence, if $\mathfrak{R}_\phi$ is totally bounded, then so is $\mathfrak{R}_\psi$. Similarly for $\mathfrak{L}_\psi$.

LEMMA 2. *If $\psi(x)$ is a weight function, and $\epsilon > 0$, there is a special weight function $\chi(x)$ such that*

$$|\psi(x) - \chi(x)| \leqq \epsilon, \qquad x \epsilon \mathfrak{G}.$$

By the approximation theorem ([1], Theorem 30), for each $\delta > 0$, there is a finite linear aggregate of representation coefficients $\chi_1(x)$ such that

$$|\psi(x) - \chi_1(x)| \leqq \delta. \tag{22}$$

We may assume $\chi_1(x)$ real, otherwise we replace it by $(\chi_1(x)+\overline{\chi_1(x)})/2$. After that, we may assume it $\geqq 0$, otherwise $\chi_1(x)+\delta$ satisfies (22) with 2δ instead of δ. Since $M\psi = 1$, (22) implies $1-\delta \leqq M\chi_1 \leqq 1+\delta$; hence, for $\chi = \chi_1/(M\chi_1)$, we obtain

$$\begin{aligned} |\psi(x) - \chi(x)| &\leqq |\psi(x) - \chi_1(x)| + \left| \left(1 - \frac{1}{M\chi_1}\right)\chi_1 \right| \\ &\leqq \delta + \frac{\delta}{1-\delta}\left[\underset{x \epsilon \mathfrak{G}}{\text{l.u.b.}}\ \psi(x) + \delta \right], \end{aligned}$$

and this is $\leqq \epsilon$, if δ is small enough.

THEOREM 22. *If $F \epsilon Ap$, and $U \epsilon \mathfrak{U}$, there exists a special weight function χ such that $\chi \times F - F \epsilon U'$.*

$F(x^{-1}y) - F(y)$ is almost periodic in x, y, hence $\|F(x^{-1}y) - F(y)\|_U^+$ is also almost periodic in x, y (Theorem 5). By Lemma 1,

$$t(x) = \underset{y \epsilon \mathfrak{G}}{\text{l.u.b.}} \|F(x^{-1}y) - F(y)\|_U^+$$

is almost periodic in x.

Given $\epsilon > 0$, we form with the function

$$\omega_\epsilon(u) = \begin{cases} 1 - \dfrac{|u|}{\epsilon} & \text{for } |u| \leqq \epsilon, \\ 0 & \text{for } |u| \geqq \epsilon \end{cases}$$

the almost periodic (Theorem 5) function $\psi_\epsilon(x) = \omega_\epsilon(t(x))$. We have (1) $\psi_\epsilon(x) \geqq 0$, (2) $\psi_\epsilon(1) = 1$, since $t(0) = 0$, (3) if $\psi_\epsilon(x) > 0$, then $\|F(x^{-1}y) - F(y)\|_U^+ \leqq \epsilon$ for all $y \epsilon \mathfrak{G}$, (4) $m_\epsilon = M_x \psi_\epsilon(x) > 0$ by [1], Theorem 7, 4. Therefore

$$\|\psi_\epsilon(z)(F(z^{-1}x) - F(x))\|_U^+ \leqq \psi_\epsilon(z)\|F(z^{-1}x) - F(x)\|_U^+ \leqq \psi_\epsilon(z)t(z),$$

and this is equal to 0 if $\psi_\epsilon(z) = 0$, and $\leqq \epsilon\psi_\epsilon(z)$ if $\psi_\epsilon(z) \neq 0$. Thus

$$\|\psi_\epsilon \times F - m_\epsilon F\|_U^+ \leqq \epsilon m_\epsilon,$$

and for the weight function $\psi(x) = \psi_\epsilon(x)/m_\epsilon$ we have

$$\|\psi \times F - F\|_U^+ \leqq \epsilon.$$

For the special weight function $\chi(x)$ from Lemma 2 we obtain

$$\|\chi \times F - F\|_U^+ \leqq \epsilon + \|(\chi - \psi) \times F\|_U^+ \leqq \epsilon + M_x\|(\psi(xy^{-1}) - \chi(xy^{-1}))F(y)\|_U^+$$

$$\leqq \epsilon + \epsilon M_x(\max\| \pm F(y)\|_U^+) \leqq \epsilon + \epsilon C,$$

where C is a constant independent of ϵ. Choosing $\epsilon < (1+C)^{-1}$, we obtain the theorem.

THEOREM 23. *If $F \epsilon Ap$, there exists a sequence of special weight functions $\phi_n(x)$, $n = 1, 2, 3, \cdots$, such that F is the limit of the sequence $\phi_n \times F$, $n = 1, 2, 3, \cdots$.*

Consider the functions $\phi \times F$, for all special weight functions ϕ, and denote their set by S. By Theorem 22, $F \subset S_{\text{cl}}$; but S_{cl}, being $\subset ((\mathfrak{R}_F)_{\text{conv}})_{\text{cl}}$ (Theorem

21), is separable; hence, F is the limit of a sequence of (equal or different) elements ϵS.

DEFINITION 8. *If* $D(x)=\{D_{\rho\sigma}(x)\}_{\rho,\sigma=1,\cdots,s}$ *is an irreducible normal representation of* $\mathfrak{G}$ (cf. [1], Chapter III, Definitions 9, 10) *we form, for* $F\epsilon Ap$,

$$f_{\rho\sigma}(D) = M_x(D_{\sigma\rho}(x^{-1})F(x)) = M(\overline{D_{\rho\sigma}}F),$$

and the matrix

$$f^D = \{f_{\rho\sigma}(D)\}_{\rho,\sigma=1,\cdots,s}.$$

$f_{\rho\sigma}(D)$ *is the* (D, ρ, σ)*-expansion-coefficient of* F, f^D *its D-expansion-matrix.*

THEOREM 24. *The expansion coefficients and matrices have the following properties:*

1. $M(\overline{D_{\rho\sigma}}(\alpha F))=\alpha M(\overline{D_{\rho\sigma}}F)$, $\alpha\epsilon L$.
2. $M(\overline{D_{\rho\sigma}}(F+G))=M(\overline{D_{\rho\sigma}}F)+M(\overline{D_{\rho\sigma}}G)$.
3. *If* F *is the limit of a sequence* F_N, $N=1, 2, \cdots$, *then, for each* (D, ρ, σ), $M(\overline{D_{\rho\sigma}}F)$ *is the limit of the sequence* $M(\overline{D_{\rho\sigma}}F_N)$, $N=1, 2, \cdots$.
4. *If* $G=\phi\times F$, *and if* $g_{\rho\sigma}(D)$, $\phi_{\rho\sigma}(D)$, $f_{\rho\sigma}(D)$ *are expansion coefficients of* G, ϕ, F *respectively, then*

$$g^D = \phi^D f^D,$$

that is to say,

$$g_{\rho\sigma}(D) = \sum_{\tau=1}^{s} \phi_{\rho\tau}(D) f_{\tau\sigma}(D).$$

In particular, the matrix g^D *vanishes if* f^D *or* ϕ^D *vanishes (or both).*

5. *If* D^N, $N=1, 2, 3, \cdots$, *is a (finite or countable) sequence of irreducible inequivalent normal representations, if the* s^N *are their respective degrees, if* $h_{\rho\sigma}$ *are elements of* L, *and if*

$$F = \sum_{N=1}^{\infty}\left(s^N \sum_{\rho,\sigma=1}^{s^N} \overset{N}{h}_{\rho\sigma} \overset{N}{D}_{\rho\sigma}\right) \tag{23}$$

[*meaning that* $F\epsilon Ap$ *is the limit,* $m\to\infty$, *of the sequence of elements*

$$F_m = \sum_{N=1}^{m}\left(s^N \sum_{\rho,\sigma=1}^{s^N} \overset{N}{h}_{\rho\sigma} \overset{N}{D}_{\rho\sigma}\right)$$

from Ap], *then*

$$M(\overline{D_{\rho\sigma}}F) = \begin{cases} 0 \text{ if } D \neq D^N \\ \overset{N}{h}_{\rho\sigma} \text{ if } D = D^N \end{cases} \qquad (N = 1, 2, 3, \cdots).$$

1 and 2 are obvious; 3 is easily deducible from the fact that $G(x, y)\epsilon U$ implies $M_y G(x, y)\epsilon 2U$ (Theorem 16, 8); and 4 follows from the relation

$$\begin{aligned} \overset{D}{g}_{\rho\sigma} &= M_x(D_{\sigma\rho}(x^{-1})M_y(\phi(xy^{-1})F(y))) = M_y(F(y)M_x(D_{\sigma\rho}(x^{-1})\phi(xy^{-1}))) \\ &= M_y(F(y)M_z(D_{\sigma\rho}(y^{-1}z^{-1})\phi(z))) \\ &= \sum_{\tau=1}^{n} M_y(F(y)D_{\sigma\tau}(y^{-1}))\cdot M_z(D_{\tau\rho}(z^{-1})\phi(z)). \end{aligned}$$

As regards 5, if the number of the representations D^N is finite, it follows by direct computation of $M_x(\overline{D_{\rho\sigma}}(x)F(x))$, applying 1 and 2 and [1], Theorem 21, and in the general case by using this and 3.

THEOREM 25. *If $F\epsilon Ap$, it has only countably many expansion matrices $\neq 0$.*

By Theorem 24, 4 and 3, this is true if F has the form $\phi\times G$, with ϕ a special weight function, or if F is the limit of such functions. Now apply Theorem 23.

DEFINITION 9. *If $F\epsilon Ap$; if $D^N, N=1, 2, \cdots$, is the sequence of irreducible normal representations (in any fixed order) for which its expansion matrices $\neq 0$; if the s^N are their respective degrees; and if $f_{N,\rho\sigma}=M_x(D^N_{\rho\sigma}(x^{-1})F(x))$; then we call the formal series*

$$\sum_{N=1}^{\infty} s^N \sum_{\rho,\sigma=1}^{s^N} f_{N,\rho\sigma} D^N_{\rho\sigma} \tag{24}$$

the Fourier expansion of F.

We call a sequence F_m, $m=1, 2, \cdots$, formally convergent (to F), if the sequence of the Fourier expansions of the functions F_m is formally convergent (to the Fourier expansion of F), i.e., if for any (D, ρ, σ), the sequence $M(\overline{D_{\rho\sigma}}F_m)$ has a limit (namely $M(\overline{D_{\rho\sigma}}F)$).

Remark. Theorem 24 states properties of the Fourier expansion. 1 and 2 state additivity. 3 states that the Fourier series of a limit is the formal limit of the Fourier series. 4 gives an important rule for the computation of the Fourier series of a convolution of an almost periodic function with a numerical almost periodic function. Finally, 5 states that the sum of a uniformly convergent series of the form (23) is its own Fourier expansion.

THEOREM 26. (Uniqueness Theorem.) *Almost periodic functions which have the same Fourier expansion are equal.*

If G and H have the same expansion, then the expansion of $F=G-H$ vanishes identically. From

$$(25)\qquad D_{\rho\sigma} \times F = M_y(D_{\rho\sigma}(xy^{-1})F(y)) = \sum_{\tau=1}^{s} D_{\rho\tau}(x) M_y(D_{\tau\sigma}(y^{-1})F(y))$$

it follows that $D_{\rho\sigma}\times F$ vanishes for any (D, ρ, σ). Hence $\phi\times F=0$ for any special weight function. But F is the limit of such functions (Theorem 23). Therefore $F=0$.

Part III. Theorems on approximation and summation

Theorem 27. (Approximation Theorem.) *If F is almost periodic, it is the limit of a sequence of finite linear aggregates of the form $\sum fD_{\rho\sigma}$ with $f\epsilon L$. More precisely, if D^N, $N=1, 2, 3, \cdots$, are the representations occurring in the Fourier expansion of F, if the s^N are their respective degrees, there exist elements $f^m_{N,\rho\sigma}$ of L $(m=1, 2, 3, \cdots)$ such that for each m only a finite number of them is $\neq 0$, and that F is the limit, for $m\to\infty$, of the finite aggregates*

$$F_m = \sum_{N=1}^{\infty}\left(s^N \sum_{\rho,\sigma=1}^{s^N} f^m_{N,\rho\sigma} D^N_{\rho\sigma}\right).$$

F is the limit of a sequence $F_m=\phi_m\times F$, each ϕ_m being a special weight function. Using (25) we find that this sequence has the property stated in the theorem.

Theorem 28. *Let the sequence F_m, $m=1, 2, 3, \cdots$, be part of a totally bounded or compact set of Ap. In order that the sequence have a limit it is sufficient that it be formally convergent.*

As the closure of a totally bounded set is compact (cf. [4], Theorem 11 and Definition 10), it is sufficient to consider the second case. Owing to the compactness, the sequence has at least one condensation point, and, using the compactness again, we have to show that it has no more than one condensation point. Otherwise there would exist two subsequences F_{p_m}, F_{q_m} of F_m, having two different limits, G and H respectively. The Fourier expansion of G is the formal limit of the Fourier expansions of the sequence F_{p_m}, and therefore, the sequence F_m being formally convergent, of the sequence F_m. Similarly, the Fourier expansion of H is the formal limit of the expansions of the sequence F_m. Thus G and H have the same Fourier expansion, and by Theorem 26, $G=H$, which contradicts our hypothesis.

Definition 10. *The function $F\epsilon Ap$ will be called a class function if for all x and y, $F(yxy^{-1})=F(x)$, or, which is equivalent, if for all x and y, $F(yx)=F(xy)$.*

A formal series (24) *will be called a class series if*

$$(26)\qquad f_{N,\rho\sigma} = f_N\delta_{\rho\sigma}.$$

THEOREM 29. *If $F(x)\epsilon Ap$, then $F_0(x)=M_yF(yxy^{-1})$ is a class function, and if (24) is the Fourier expansion of $F(x)$, then the Fourier expansion of $F_0(x)$ is*

$$\sum_{N=1}^{\infty} s^N f_N \sum_{\rho=1}^{s^N} D_{\rho\rho}^N,$$

where

$$f_N = \frac{1}{s^N}\sum_{\rho=1}^{s^N} f_{N,\rho\rho}.$$

In order that a function F of Ap be a class function, it is necessary and sufficient that its Fourier expansion be a class series.

We have

$$F_0(zxz^{-1}) = M_y F(yzxz^{-1}y^{-1}) = M_y F(yxy^{-1}) = F_0(x);$$

hence $F_0(x)$ is a class function. For a fixed representation $\{D_{\rho\sigma}\}$ we have

$$\begin{aligned} M_x(D_{\rho\sigma}(x^{-1})F_0(x)) &= M_xM_y(D_{\rho\sigma}(x^{-1})F(yxy^{-1})) = M_xM_y(D_{\rho\sigma}(y^{-1}x^{-1}y)F(x)) \\ &= M_y\sum_{p,q=1}^{s} D_{\rho p}(y^{-1})D_{q\sigma}(y)\cdot M_x(D_{pq}(x^{-1})F(x)) \\ &= \frac{\delta_{\rho\sigma}}{s}\sum_{p=1}^{s} M_x(D_{pp}(x^{-1})F(x)) \end{aligned}$$

and this proves the statement about the Fourier series of $F_0(x)$.

If the Fourier expansion of F is a class series, the functions F and F_0 have the same Fourier expansion. By Theorem 26, $F=F_0$, and hence F is a class function, because F_0 is one.

Conversely if $F(x)=F(yxy^{-1})$, then $F(x)=M_yF(yxy^{-1})=F_0(x)$, and by the first part of our theorem, the Fourier expansion of $F_0(x)$ is a class series.

THEOREM 30. (Summation Theorem.) *Let D^N, $N=1, 2, 3, \cdots$, denote a sequence of irreducible normal representations, and the s^N their respective degrees. There exists a sequence of special weight functions ϕ_m, $m=1, 2, 3, \cdots$, with the following properties:*

1. *Each ϕ_m is a class function.*
2. *All Fourier coefficients of ϕ_m are $\geqq 0$, $\leqq 1$.*
3. *Any almost periodic function F whose Fourier expansion contains no other representations than the given ones, is the limit of the sequence $\phi_m \times F$, $m=1, 2, \cdots$.*

In particular, there exists a square array of numbers r_N^m, m, $N=1, 2, \cdots$, with the following properties:

α. *For each m only a finite number of them is* $\neq 0$.
β. $0 \leqq r_N^m \leqq 1$.
γ. *If an almost periodic function F has a Fourier expansion*

$$\sum_{N=1}^{\infty} s^N \sum_{\rho,\sigma=1}^{s^N} f_{N,\rho\sigma} D_{\rho\sigma}^{N}$$

[*any number of the coefficients $f_{N,\rho\sigma}$ may vanish*], *then F is the limit, $m\to\infty$, of the finite aggregates*

$$F_m = \sum_{N=1}^{\infty} r_N^m s^N \sum_{\rho,\sigma=1}^{s^N} f_{N,\rho\sigma} D_{\rho\sigma}^{N}.$$

We determine numbers ϵ_N, all $\neq 0$, such that the series

(27) $$\sum_{N=1}^{\infty} \epsilon_N s^N \left(\sum_{\tau=1}^{s^N} D_{\tau\tau}^{N}(x) \right)$$

is uniformly convergent, thus representing a numerical almost periodic function $f(x)$. There exist special weight functions $\chi_m(x)$, $m=1, 2, \cdots$, such that the sequence of functions $f_m(x)=\chi_m \times f(x)$ is uniformly convergent to $f(x)$. By [1], Theorem 21,

$$f \times D_{\rho\sigma}^{N} = \epsilon_N D_{\rho\sigma}^{N}.$$

But $f_m \times D_{\rho\sigma}^N = \chi_m \times (f \times D_{\rho\sigma}^N)$. Considering $\epsilon_N \neq 0$, we conclude that

(28) $$\lim_{m\to\infty} \chi_m \times D_{\rho\sigma}^{N} = D_{\rho\sigma}^{N}.$$

In particular,

$$\lim_{m\to\infty} M_x(D_{\rho\sigma}^{N}(x^{-1})\chi_m(x)) = \delta_{\rho\sigma}.$$

We now consider the class function

(29) $$\psi_m(x) = M_y \chi_m(yxy^{-1});$$

$\psi_m(x)$ is obviously a weight function, and Theorem 29 shows that it has the form

(30) $$\psi_m(x) = \sum_D s_D\, a_D^m \left(\sum_{\tau=1}^{s_D} D_{\tau\tau} \right)$$

with a finite number of terms, so that it is a special weight function. From (28) it follows that

$$\lim_{m\to\infty} \overset{m}{a_D} = 1 \qquad \text{for } D = D^N.$$

The function

$$\psi_m' = \bar{\psi}_m(x^{-1}) = \sum_D s_D \overline{\overset{m}{a_D}}\left(\sum_{\tau=1}^{s_D} D_{\tau\tau}\right)$$

has all the properties of ψ_m. Finally we introduce the class function $\phi_m = \psi_m \times \psi_m'$, for which we find

$$\phi_m = \sum_D s_D \overset{m}{r_D}\left(\sum_{\tau=1}^{s_D} D_{\tau\tau}\right),$$

where

$$\overset{m}{r_D} = \left| \overset{m}{a_D} \right|^2,$$

with

(31) $$\lim_{m\to\infty} \overset{m}{r_D} = 1 \qquad \text{for } D = D^N.$$

It is easy to find that ϕ_m is again a weight function. Its coefficients $\overset{m}{r_D}$ are $\geqq 0$, and as $\overset{m}{r_D}$, $\phi_m(x)$ are real and $\geqq 0$,

$$\overset{m}{r_D} = M_x(D_{11}(x^{-1})\phi_m(x)) \leqq M_x(\left| D_{11}(x^{-1}) \right| \cdot \left| \phi_m(x) \right|) \leqq M_x(\phi_m(x)) = 1.$$

Hence the properties 1 and 2 of our theorem are fulfilled. Using (31) we conclude from Theorem 24, 4, that for any F whose Fourier expansion contains no other representations than the given ones, the sequence $\phi_m \times F$ converges formally to F. By Theorem 21, this sequence is part of the compact set $((\mathfrak{R}_F)_{\text{conv}})_{\text{cl}}$. Thus we may apply Theorem 28, and this proves property 3.

The second half of Theorem 30 is an immediate consequence of the first half.

DEFINITION 11. *A system of irreducible normal representations will be called a module $\mathfrak{M}$ if it contains with every representation its complex conjugate, and the Fourier series of the product of any two occurring representation coefficients contains no other representations than the given ones.* A module will be called countable if it contains only a finite or enumerable number of representations.*

---

* If D^M, D^N are any two normal representations, then, in the sense in which these symbols are used in the theory of group-representations, the direct product $D^M D^N$ is a finite sum

$$\sum_P c_P^{MN} D^P$$

(the c_P^{MN} are the "composition coefficients"). Hence the product of any two representation coefficients has a *finite* Fourier expansion.

LEMMA 3. *There is always a smallest module containing a given set of representations; if the given set is countable, then so is also the smallest module.*

Consider all finite subsystems of the given set of representation coefficients and their complex conjugates and form all possible power-products with them. The totality of the representations occurring in their Fourier series is the desired module.

DEFINITION 12. *Given any module $\mathfrak{M}$, we shall denote by $\{\mathfrak{M}\}$ the set of Ap consisting of those functions whose Fourier expansion contains no other representations than those occurring in $\mathfrak{M}$.*

THEOREM 31. *Given $\mathfrak{M}$, the set $\{\mathfrak{M}\}$ has the following properties:*

1. *If $F(x)$ is in $\{\mathfrak{M}\}$, every $F(xa)$ is in $\{\mathfrak{M}\}$.*
2. *If $F(x)$ is in $\{\mathfrak{M}\}$, every $F(ax)$ is in $\{\mathfrak{M}\}$.*
3. *If $F(x)$ is in $\{\mathfrak{M}\}$, every $\alpha F(x)$ is in $\{\mathfrak{M}\}$.*
4. *If $F(x)$ and $G(x)$ are in $\{\mathfrak{M}\}$, $F(x)\pm G(x)$ is in $\{\mathfrak{M}\}$.*
5. *If $F_1, F_2, \cdots$ are in $\{\mathfrak{M}\}$, and if F is the limit of F_n, then F is in $\{\mathfrak{M}\}$.*
6. *If $f_1, \cdots, f_n$ are numerical functions of $\{\mathfrak{M}\}$, and $F(u_1, \cdots, u_n)$ is a numerical function which is defined and uniformly continuous for the range of $f_1, \cdots, f_n$, then $F(f_1(x), \cdots, f_n(x))$ is also contained in $\{\mathfrak{M}\}$.*
7. *If at least one of the two functions $\alpha(x), F(x)$ is in $\{\mathfrak{M}\}$, then $\alpha\times F(x)$ is also. ($\alpha(x)$ is numerical.)*

1–5 follow easily from the formal properties of Fourier expansion. 7 is an immediate consequence of Theorem 24, 4. As regards 6, if f and g are numerical finite aggregates of representations from $\{\mathfrak{M}\}$, then $\bar{f}$ as well as fg is also $\epsilon\{\mathfrak{M}\}$ by definition of $\{\mathfrak{M}\}$. Applying the approximation theorem and 5, the same holds for $\bar{f}$ and fg for any numerical functions f, g from $\{\mathfrak{M}\}$. Hence 6 is true if $F(u_1, \cdots, u_n)$ is a polynomial in $u_1, \cdots, u_n$ and their complex conjugates. $F(u_1, \cdots, u_n)$ being uniformly continuous on the range of $f_1, \cdots, f_n$, which is a bounded set since $f_1, \cdots, f_n$ are a.p., $F(f_1(x), \cdots, f_n(x))$ is a uniform limit of polynomials in $f_1(x), \cdots, f_n(x)$ and their complex conjugates. Thus 5 completes the proof of 6.

THEOREM 32. *If $\mathfrak{M}=(D^1, D^2, \cdots)$ is a countable module, there exists in $\{\mathfrak{M}\}$ a sequence of special weight functions $\phi_1, \phi_2, \cdots$ such that*

(i) $$\phi_m = \sum_{N=1}^{\infty} s^N \overset{m}{r}_N \left(\sum_{\tau=1}^{s^N} D^N_{\tau\tau} \right);$$

(ii) $$0 \leqq \overset{m}{r}_N \leqq 1, \qquad \lim_{m\to\infty} \overset{m}{r}_N = 1.$$

As in the proof of Theorem 30, we choose the $\epsilon_N \neq 0$ such that the series (27) is uniformly convergent, and construct the special weight functions $\phi_1, \phi_2, \cdots$ mentioned therein.

By Theorem 30 these $\phi_1, \phi_2, \cdots$ have the properties (i), (ii) of our theorem, so we need prove only that they belong to $\{\mathfrak{M}\}$.

As $f(x)$ belongs to $\{\mathfrak{M}\}$, for every finite subset $a_1, \cdots, a_n$ of $\mathfrak{G}$, all $f(xa_1), \cdots, f(xa_n)$ also do, and with them their continuous function

$$\max\left(|f(xa_1) - f(a_1)|, \cdots, |f(xa_n) - f(a_n)|\right) = \underset{y=a_1,\cdots,a_n}{\text{l.u.b.}} \left(|f(xy) - f(y)|\right).$$

Since $f(z)$ is almost periodic, we can select a sequence of such sets $a_1, \cdots, a_n$, so that this converges uniformly to the translation function

$$t(x) = \underset{y\in\mathfrak{G}}{\text{l.u.b.}} \left(|f(xy) - f(y)|\right).$$

Thus $t(x)$, as well as the $\psi_\epsilon(x) = \omega_\epsilon(t(x))$ of Theorem 22, which is a continuous function of $t(x)$, belongs to $\{\mathfrak{M}\}$, and with it

$$\psi(x) = \frac{\psi_\epsilon(x)}{M_x \psi_\epsilon(x)}.$$

The $\chi_1(x)$ of Lemma 2 contains only such representations as occur in the Fourier series of $\psi(x)$; therefore it also belongs to $\{\mathfrak{M}\}$, and with it $\chi(x)$. The same is true of the χ of Theorem 22 and also of the $\chi_1, \chi_2, \cdots$ in the proof of Theorem 30 as well as of the $\psi_1, \psi_2, \cdots$, since these contain only such representations as occur in the Fourier series of the corresponding functions $\chi_1, \chi_2, \cdots$. It follows finally that $\phi_1, \phi_2, \cdots \subset \{\mathfrak{M}\}$, and the proof is complete.

THEOREM 33. (Isolation Theorem.) *Let F be an almost periodic function, with a Fourier expansion*

$$\sum_D \left(s_D \sum_{\rho,\sigma=1}^{s_D} f_{D_{\rho\sigma}} D_{\rho\sigma} \right), \tag{32}$$

and let $\mathfrak{M}$ be any module. There exists an almost periodic function, we shall denote it by $F_{\mathfrak{M}}$, whose Fourier expansion consists of exactly those terms

$$s_D \sum_{\rho,\sigma=1}^{s_D} f_{D_{\rho\sigma}} D_{\rho\sigma} \tag{33}$$

of (32), *for which D is contained in $\mathfrak{M}$. And we have $F_{\mathfrak{M}} \dashv F$.*

The given module $\mathfrak{M}$ need not be enumerable from the outset, but we may replace it by the enumerable module generated by those representations of $\mathfrak{M}$ which occur in (32). Hence $\mathfrak{M}$ may be assumed to be enumerable. Let it contain the representations $D^1, D^2, \cdots$. With the special weight functions $\phi_1, \phi_2, \cdots$ from Theorem 32 we construct the functions $\phi_1 \times F, \phi_2 \times F, \cdots$. By Theorem 21 the functions $\phi_m \times F$ are all $\dashv F$, and by the properties of ϕ_m, the functions $\phi_m \times F$ are formally convergent. By Theorem 28 the sequence $\phi_m \times F$ has a limit function $f_{\mathfrak{M}}$, and by Theorems 24, 4 and 32 the Fourier series of $f_{\mathfrak{M}}$ has the stated form.

THEOREM 34. *Let F be an almost periodic function and $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ a sequence of monotonically increasing modules which in their sum contain all representations occurring in F.*

The sequence of functions

$$F_{\mathfrak{M}_n} \qquad (n = 1, 2, \cdots) \tag{34}$$

converges to F.

The functions (34) converge formally to F, and are all $\dashv F$. By Theorem 28, F is the limit of the sequence.

THEOREM 35. (Decomposition Theorem.) *Let F be an almost periodic function and let it be possible to divide the representations occurring in F in a sequence of systems $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \cdots$, in such a way that for each $k(=1, 2, 3, \cdots)$, the least module containing $\mathfrak{S}(\mathfrak{s}_1, \mathfrak{s}_2, \cdots, \mathfrak{s}_k)$ has no element in common with $\mathfrak{S}(\mathfrak{s}_{k+1}, \mathfrak{s}_{k+2}, \cdots)$. There exists, for each k, an almost periodic function, we shall denote it by $F_{\mathfrak{s}_k}$, whose Fourier expansion consists of exactly those terms* (33) *of* (32), *for which D is contained in $\mathfrak{s}_k$; and the series*

$$F_{\mathfrak{s}_1} + F_{\mathfrak{s}_2} + \cdots$$

converges to F.

If we denote the smallest module containing $\mathfrak{s}_1, \cdots, \mathfrak{s}_k$ by $\mathfrak{M}_k$, the desired function $F_{\mathfrak{s}_k}$ is $F_{\mathfrak{M}_k}$ for $k=1$, and $F_{\mathfrak{M}_k} - F_{\mathfrak{M}_{k-1}}$ for $k \geqq 2$, and our theorem reduces to Theorem 34.

PART IV. APPLICATIONS TO HILBERT SPACE

In this section we shall assume L to be either a Hilbert space $\mathfrak{H}$, or the space $\mathfrak{B}$ of all bounded transformations in $\mathfrak{H}$. We shall consider these spaces with the help of the various topologies which have been discussed in [4], Chapter IV, as well as in the places quoted there. According to whether we consider $\mathfrak{H}$ in the topology based on the neighborhoods $\mathfrak{U}_1$ or $\mathfrak{U}_2$, we shall denote $\mathfrak{H}$ by $\mathfrak{H}_1$ or $\mathfrak{H}_2$ respectively. As for $\mathfrak{B}$, corresponding to the neighborhoods $\mathfrak{U}_3, \mathfrak{U}_4, \mathfrak{U}_5$ we shall denote it by $\mathfrak{B}_3, \mathfrak{B}_4, \mathfrak{B}_5$, respectively.

THEOREM 36. (α) *If* $F(x)\epsilon\,\mathfrak{H}^{\mathfrak{G}}$ *is almost periodic in* $\mathfrak{H}_1^{\mathfrak{G}}$ *it is also almost periodic in* $\mathfrak{H}_2^{\mathfrak{G}}$. ($\beta$) *If* $F(x)\epsilon\,\mathfrak{B}^{\mathfrak{G}}$ *is almost periodic in* $\mathfrak{B}_3^{\mathfrak{G}}$ *it is also almost periodic in* $\mathfrak{B}_4^{\mathfrak{G}}$, *and* ($\gamma$) *if it is almost periodic in* $\mathfrak{B}_4^{\mathfrak{G}}$ *it is also almost periodic in* $\mathfrak{B}_5^{\mathfrak{G}}$.

(δ) $F(x)$ *is almost periodic in* $\mathfrak{H}_2^{\mathfrak{G}}$ *if and only if the numerical function* $(F(x), g)$, *which is the inner product of* $F(x)$ *with the constant* $g\epsilon\mathfrak{H}$, *is almost periodic for every* $g\epsilon\mathfrak{H}$.

(ϵ) $F(x)$ *is almost periodic in* $\mathfrak{B}_4^{\mathfrak{G}}$ *if and only if* $F(x)f$, *that is, the value of* $F(x)$ *operated upon the constant* $f\epsilon\mathfrak{H}$, *is almost periodic in* $\mathfrak{H}_1^{\mathfrak{G}}$ *for every* $f\epsilon\mathfrak{H}$.

(ζ) $F(x)$ *is almost periodic in* $\mathfrak{B}_5^{\mathfrak{G}}$ *if and only if* $F(x)f$ *is almost periodic in* $\mathfrak{H}_2^{\mathfrak{G}}$ *for every* $f\epsilon\mathfrak{H}$, *that is, if and only if the numerical function* $(F(x)f, g)$ *is almost periodic for every pair* $f, g, \epsilon\mathfrak{H}$.

Ad (α). The topology of $\mathfrak{H}_2$ is weaker than the topology of $\mathfrak{H}_1$. More precisely: to any $U_2\epsilon\,\mathfrak{U}_2$ there corresponds a $U_1\epsilon\,\mathfrak{U}_1$ such that $U_1\subset U_2$. Hence, if a set $S\subset\mathfrak{H}$ is totally bounded in $\mathfrak{H}_1$ it is also totally bounded in $\mathfrak{H}_2$. And this proves the proposition, if we remember Definition 1.

Ad (β) and (γ). A similar argument as ad(α).

Ad (δ). Let S be any set of $\mathfrak{H}_b^{\mathfrak{G}}$. If $g\epsilon\mathfrak{H}$ we denote by S_g the set of numerical functions $(F(x), g)$, $F\epsilon S$, and for any finite number of elements $g_1, \cdots, g_n\epsilon\mathfrak{H}$ we denote by $S_{g_1,\ldots,g_n}$ the set of n-dimensional vector functions with components $(F(x), g_1), \cdots, (F(x), g_n)$, $F\epsilon S$. It is easy to see that S is totally bounded in $\mathfrak{H}_2^{\mathfrak{G}}$ if and only if all sets $S_{g_1,\cdots,g_n}$ are totally bounded. On the other hand, we have to prove that S is totally bounded in $\mathfrak{H}_2$ if and only if all S_g are totally bounded. Thus it is sufficient to prove that, for a fixed S, the total boundedness of all sets S_g implies the total boundedness of all sets $S_{g_1,\cdots,g_n}$. This follows from [4], Theorem 9, if we observe that a vector $\{\phi_1, \cdots, \phi_n\}$ may be considered as a continuous function of its components $\phi_1, \cdots, \phi_n$ in the obvious topology of vector spaces.

Ad (ϵ) and (ζ). A similar argument as ad (δ).

THEOREM 37. *Let* $F(x)$ *be an almost periodic function, its Fourier expansion being (throughout this part, we will write the Fourier coefficients* $\alpha_{\rho\sigma}(D)$ *without the factors* s_D, *the degree of* D)

$$F(x)\sim\sum_{D,\rho,\sigma}\alpha_{\rho\sigma}(D)D_{\rho\sigma}(x). \tag{35}$$

If $F(x)$ *is in* $\mathfrak{H}^{\mathfrak{G}}$, *we have* $\alpha_{\rho\sigma}(D)\epsilon\mathfrak{H}$, *and* $(F(x), g)$, *for any* $g\epsilon\mathfrak{H}$, *has the Fourier expansion*

$$(F(x), g) \sim \sum_{D,\rho,\sigma} (\alpha_{\rho\sigma}(D), g) D_{\rho\sigma}(x). \tag{36}$$

If $F(x)$ is in $\mathfrak{B}^{\mathfrak{G}}$, we have $\alpha_{\rho\sigma}(D)\epsilon\mathfrak{B}$; for any $f\epsilon\mathfrak{H}$, $F(x)f$ has the Fourier expansion

$$F(x)f \sim \sum_{D,\rho,\sigma} (\alpha_{\rho\sigma}(D)f)\cdot D_{\rho\sigma}(x), \tag{37}$$

and for any $f, g\epsilon\mathfrak{H}$, $(F(x)f, g)$ has the Fourier expansion

$$(F(x)f, g) \sim \sum_{D,\rho,\sigma} (\alpha_{\rho\sigma}(D)f, g)\cdot D_{\rho\sigma}(x). \tag{38}$$

Remembering the definition of $\alpha_{\rho\sigma}(D)$, it is sufficient to prove

$$M_x(F(x), g) = (M_xF(x), g) \tag{39}$$

in the case of (36), and

$$M_x(F(x)f) = (M_xF(x))f \tag{40}$$

in the case (37); (38) follows from their combination.

The mean with respect to the strong topology ($\mathfrak{H}_1$ or $\mathfrak{B}_4$) has, a fortiori, all the properties of the mean in the weak topology ($\mathfrak{H}_2$ or $\mathfrak{B}_5$). Hence, by the uniqueness property of the mean (Theorem 17), it is sufficient to consider the cases of weak topology. In these cases the relations (39), (40) follow by a new application of Theorem 17, since $(M_xF(x), g)$ and $(M_xF(x))f$ have the properties required in this theorem.

THEOREM 38. *If $F(x)$ is an almost periodic function of the class $\mathfrak{B}_\nu^{\mathfrak{G}}(\nu=3, 4, 5)$, and $\alpha\epsilon\mathfrak{B}$, then $\alpha F(x)$ and $F(x)\alpha$ [αF and $F\alpha$ are the operational products of F and α] are again almost periodic functions of the same class; and the Fourier expansions of αF and $F\alpha$ may be obtained from the Fourier expansion of $F(x)$ by term-by-term multiplication with α on the right or on the left.*

The first half of the theorem follows from the fact that, for a fixed α, αF and $F\alpha$ are continuous functions of F in each of the topologies $\mathfrak{B}_3$, $\mathfrak{B}_4$, $\mathfrak{B}_5$. The second half (in the case of αF) follows from Theorem 37, Formula (38), if, in the formula of our statement connecting the expansions of αF and F, we apply (38) and then replace F, f, g by $\alpha F, f, g$ in the left-hand member, and by F, f, α^*g(α^* is the adjoint of α) in the right-hand member and compare the results; in the case of $F\alpha$ we replace F, f, g by $F\alpha, f, g$ and by $F, \alpha f, g$ respectively.

THEOREM 39. *Let $F(x)$ be an almost periodic function of $\mathfrak{B}_5^{\mathfrak{G}}$; let* (35) *be its Fourier expansion. If $F(x)$ is a representation of the group by means of unitary transformations, that is, if $F(xy)=F(x)F(y)$, $F(1)=1$, $F(x^{-1})=F(x)^*$, then we have the following*:

(i) *$F(x)$ is almost periodic as a function of $\mathfrak{B}_4^{\mathfrak{G}}$; moreover, F is the limit, in the $\mathfrak{B}_4$-topology, of the sequence*

$$F_m = \sum_{N=1}^{m}\left(\sum_{\rho,\sigma=1}^{s_{D^N}} \alpha_{\rho\sigma}(D^N)D^N_{\rho\sigma}(x)\right). \tag{41}$$

(ii) *The Fourier coefficients have the properties*

$$\alpha_{\rho\sigma}(D)\alpha_{\tau\upsilon}(E) = 0 \qquad \text{if } D \neq E, \tag{42}$$

$$\alpha_{\rho\sigma}(D)\alpha_{\tau\upsilon}(D) = \delta_{\sigma\tau}\alpha_{\rho\upsilon}(D) \qquad [\rho, \sigma, \tau, \upsilon = 1, \cdots, s_D], \tag{43}$$

$$\alpha_{\rho\sigma}(D)^* = \alpha_{\sigma\rho}(D) \qquad [\rho, \sigma = 1, \cdots, s_D]. \tag{44}$$

(iii) *The system of operators $\alpha_{\rho\rho}(D)$ is a resolution of the identity, that is to say, each of them is a projection operator, any two of them are orthogonal, and their sum is the identity.*

Conversely, if a set of elements $\alpha_{\rho\sigma}(D)$ of $\mathfrak{B}$ fulfills the conditions (ii) *and* (iii), *then the series* (35) *is the Fourier expansion of an almost periodic function $F(x)$ with the assigned properties.*

Remark concerning the theorem. Before proving the theorem we want to point out the algebraic aspect of the proposition.

Since $\alpha_{\rho\rho}(D)$ is a projection it corresponds to a closed linear manifold, say $\mathfrak{M}_\rho^D$. Select an orthonormal system determining $\mathfrak{M}_1^D$: ψ_{11}^D, ψ_{12}^D, $\cdots$ (the sequence is empty, finite, or countable). Now, (43) and (44) imply

$$\alpha_{\rho 1}(D)^*\alpha_{\rho 1}(D) = \alpha_{11}(D), \quad \alpha_{\rho 1}(D)\alpha_{\rho 1}(D)^* = \alpha_{\rho\rho}(D),$$

and this means that $\alpha_{\rho 1}(D)$ maps $\mathfrak{M}_1^D$ in a one-to-one and unitary way on $\mathfrak{M}_\rho^D$, while $\alpha_{\rho 1}(D)f\equiv 0$ if f is orthogonal to $\mathfrak{M}_1^D$. Thus, if we define $\psi_{\rho\nu}(D) =\alpha_{\rho 1}(D)\psi_{1\nu}(D)$, the $\psi_{\rho 1}(D)$, $\psi_{\rho 2}(D)$, $\cdots$ determine $\mathfrak{M}_\rho^D$. By (43) we have $\alpha_{\rho\nu}(D)\alpha_{\nu 1}(D)=\alpha_{\rho 1}(D)$, and this implies $\alpha_{\rho\sigma}(D)\psi_{\sigma\nu}(D)=\psi_{\rho\nu}(D)$.

Since, by (iii), $\sum_{D,\rho}\alpha_{\rho\rho}(D)=1$, the $\mathfrak{M}_\rho^D$ together determine $\mathfrak{H}$, and, since, by (42), (43), (44), $\alpha_{\sigma\rho}(D)^*\alpha_{\tau\upsilon}(E)=0$ if $D\neq E$ or $\sigma\neq\tau$, the $\mathfrak{M}_\rho^D$ are mutually orthogonal. Thus the $\psi_{\rho\nu}(D)$ form a complete orthonormal system in $\mathfrak{H}$. For this system we found

$$\alpha_{\tau\upsilon}(E)\psi_{\rho\nu}(D) = \begin{cases} 0, \text{ if } D \neq E \text{ or } \upsilon \neq \rho, \\ \psi_{\tau\nu}(D) \text{ if } D = E,\ \upsilon = \rho. \end{cases}$$

Remembering that the $\alpha_{\tau\upsilon}(D)$ form the Fourier expansion of $F(x)$, we obtain the formulas

$$F(x)\psi_{\rho\nu}(D) = \sum_{\tau=1}^{s_D} D_{\tau\rho}(x)\psi_{\tau\nu}(D).$$

Thus our theorem proves that the representation $F(x)$ may be reduced to split up according to the finite irreducible representations $D(x)$. The subspaces which correspond to this reduction are determined by the

$$\psi_{1\nu}(D),\ \psi_{2\nu}(D),\ \cdots,\ \psi_{s_D\nu}(D) \text{ for all } D,\ \nu.\text{—} \tag{45}$$

We pass on to the proof. Let $F(x)$ have the assumed properties. $F(xy)$ is an almost periodic function in (x, y) and for each y an almost periodic function in x. From the approximating properties of the Fourier expansion it follows that the Fourier expansion of $F_y(x) = F(xy)$ as x-function may be derived from the series (35) in a formal way, namely

$$F_y(x) \sim \sum_{D,\rho,\sigma} \alpha_{\rho\sigma}(D;\ y)D_{\rho\sigma}(x), \tag{46}$$

where

$$\alpha_{\rho\sigma}(D;\ y) = \sum_{\tau=1}^{s_D} \alpha_{\rho\tau}(D)D_{\sigma\tau}(y). \tag{47}$$

On the other hand, $F_y(x) = F(x)F(y)$, and by Theorem 38,

$$F_y(x) \sim \sum \alpha_{\rho\sigma}(D)\,F(y)D_{\rho\sigma}(x). \tag{48}$$

A comparison of (46) and (48) yields, for any D, ρ, σ,

$$\alpha_{\rho\sigma}(D)F(y) = \sum_{\tau=1}^{s} \alpha_{\rho\tau}(D)D_{\sigma\tau}(y). \tag{49}$$

Now we take y variable, and we again apply Theorem 38. This gives the relations (42) and (43). In order to prove (44) we need only compare the relations

$$F(x^{-1}) \sim \sum \alpha_{\rho\sigma}(D)D_{\rho\sigma}(x^{-1}) = \sum \alpha_{\rho\sigma}(D)\overline{D_{\sigma\rho}(x)}, \tag{50}$$

$$F(x)^* \sim \sum \alpha_{\rho\sigma}(D)^*\overline{D_{\rho\sigma}(x)} \tag{51}$$

(which follow from the approximating properties of the Fourier series), and observe that the representations $\{\overline{D_{\rho\sigma}}\}$ also form a set of irreducible normal representations of $\mathfrak{G}$.

If we apply (44) to $\rho=\sigma$ we find that $\alpha_{\rho\rho}(D)$ is self-adjoint, and (43) gives $(\alpha_{\rho\rho}(D))^2 = \alpha_{\rho\rho}(D)$. Hence, each $\alpha_{\rho\rho}(D)$ is a projection. It follows from (42) and (43) that any two of them are orthogonal. Hence any finite number of them, and also their (infinite) sum, is a projection again. We still have to prove that $\sum_{D,\rho}\alpha_{\rho\rho}(D)$ is the unity and that (i) holds.

Let f be any element of $\mathfrak{H}$. We want to evaluate the difference

$$F_p(x)f - F_q(x)f, \qquad p > q,$$

$F_p(x)$ being given by (41). Let D be any representation occurring in $F_p(x)f - F_q(x)f$. Writing

$$\beta^D(x) = \sum_{\rho,\sigma=1}^{s_D} \alpha_{\rho\sigma}(D)D_{\rho\sigma}(x),$$

we have

$$\|\beta(x)f\|^2 = (\beta(x)f, \beta(x)f) = \sum_{\rho,\sigma,\tau,\upsilon=1}^{s_D} D_{\rho\sigma}(x)\overline{D_{\tau\upsilon}(x)}(\alpha_{\rho\sigma}(D)f, \alpha_{\tau\upsilon}(D)f);$$

but

$$(\alpha_{\rho\sigma}(D)f, \alpha_{\tau\upsilon}(D)f) = (\alpha_{\tau\upsilon}(D)^*\alpha_{\rho\sigma}(D)f, f) = (\alpha_{\upsilon\tau}(D)\alpha_{\rho\sigma}(D)f, f) = \delta_{\tau\rho}(\alpha_{\upsilon\sigma}(D)f, f);$$

hence

$$\begin{aligned}\|\beta(x)f\|^2 &= \sum_{\upsilon,\sigma=1}^{s_D}\left(\sum_{\tau=1}^{s_D} D_{\tau\sigma}(x)\overline{D_{\tau\upsilon}(x)}\right)(\alpha_{\upsilon\sigma}(D)f, f)\\ &= \sum_{\upsilon,\sigma=1}^{s_D} \delta_{\upsilon\sigma}(\alpha_{\upsilon\sigma}(D)f, f)\\ &= \sum_{\rho=1}^{s_D}(\alpha_{\rho\rho}(D)f, f) = \sum_{\rho=1}^{s_D}\|\alpha_{\rho\rho}(D)f\|^2.\end{aligned}$$

Thus

$$\|F_p(x)f - F_q(x)f\|^2 = \sum_{N=q+1}^{p}\sum_{\rho=1}^{s_{D^N}}\|\alpha_{\rho\rho}(D^N)f\|^2. \tag{52}$$

Since the $\alpha_{\rho\rho}(D)$ are mutually orthogonal projections and the right side of (52) is independent of x, it follows easily that the sequence of functions $F_m(x)$ is uniformly convergent in the topology $\mathfrak{B}_4$. Thus $F(x)$ is the limit of the sequence (41) and is almost periodic also in the class $\mathfrak{B}_4^{\mathfrak{G}}$. Now

$$F(1) = \lim_{m\to\infty} F_m(1) = \lim_{m\to\infty}\sum_{N=1}^{m}\sum_{\rho=1}^{s_{D^N}}\alpha_{\rho\rho}(D^N) = \sum_{D,\rho}\alpha_{\rho\rho}(D).$$

But $F(1)=1$, by assumption, and this proves the last missing part of (iii).

Conversely, let a set of elements $\alpha_{\rho\sigma}(D)$ of $\mathfrak{B}$ satisfy (ii) and (iii). As before, we deduce the relation (52). Hence the sequence $F_m(x)$, defined by

(41), is convergent, in the topology $\mathfrak{B}_4$, to an almost periodic function $F(x)$ whose Fourier expansion is the uniformly convergent series (35). The group properties of $F(x)$ are easily verified from its series (35) on the basis of the known properties (ii), (iii).

THEOREM 40. *Let the group $\mathfrak{G}$ be a metric, locally compact, separable space in which the product xy is a continuous function of (x, y). We consider in $\mathfrak{G}$ a right invariant Haar measure and the Hilbert space $\mathfrak{H}$ consisting of all Lebesgue measurable functions of integrable square in $\mathfrak{G}$, and let $\mathfrak{T}(x)$ denote, for each x, the unitary operator which transforms every element $f(t)\in\mathfrak{H}$ into $f(tx)$.*

In order that $\mathfrak{G}$ be compact it is necessary and sufficient that $\mathfrak{T}(x)$ be an almost periodic function of $\mathfrak{B}_5^{\mathfrak{G}}$.

Obviously $\mathfrak{T}(x)$ has the properties

$$\mathfrak{T}(x)^{-1} = \mathfrak{T}(x)^*, \quad \mathfrak{T}(xy) = \mathfrak{T}(x)\,\mathfrak{T}(y), \quad \mathfrak{T}(1) = 1.$$

It follows easily that $\mathfrak{T}(x)$ is continuous in the topology $\mathfrak{B}_5$.

If $\mathfrak{G}$ is compact, every continuous function is almost periodic and this proves the necessity of our condition. Conversely, let $\mathfrak{T}(x)$ be almost periodic. We consider its Fourier expansion and the complete orthonormal system $\psi_{\rho\nu}(D)$ constructed in the remark to Theorem 39. Consider one of its non-empty subsystems (45) and denote it by $\psi_1, \cdots, \psi_s$. By the last relation of the remark we have

$$F(x)\psi_\rho(t) = \psi_\rho(xt) = \sum_\tau D_{\tau\rho}(x)\psi_\tau(t), \tag{53}$$

except perhaps for a t-set of measure 0, depending on x. By the theorem of Fubini there is a value $t = t_0$ for which (53) holds for all x except a set of measure 0. As $D(x)$ is unitary we obtain

$$\sum_\rho |\psi_\rho(xt_0)|^2 = \sum_\rho |\psi_\rho(t_0)|^2 = C.$$

Upon integrating over $\mathfrak{G}$, the left side gives s_D whereas the right side is C times the total measure of $\mathfrak{G}$. Thus $s_D > 0$ implies that $C \neq 0$ and that the total measure of $\mathfrak{G}$ is finite. If the total measure of $\mathfrak{G}$ is finite there cannot exist an $\epsilon_0 > 0$ and an infinite number of points on $\mathfrak{G}$ any two of which have a distance $> \epsilon_0$. This implies that $\mathfrak{G}$ is totally bounded. Being locally compact, $\mathfrak{G}$ is also complete. Hence the compactness of $\mathfrak{G}$.

BIBLIOGRAPHY

[1] J. von Neumann, *Almost periodic functions in a group*, I, these Transactions, vol. 36 (1934), pp. 445–492.

[2] S. Bochner, *Abstrakte fastperiodische Funktionen*, Acta Mathematica, vol. 61 (1933), pp. 149–184.

[3] S. Bochner, *Fastperiodische Lösungen der Wellengleichung*, Acta Mathematica, vol. 62 (1934), pp. 227–237.

[4] J. von Neumann, *On complete topological spaces*, these Transactions, vol. 37 (1935), pp. 1–20.

[5] S. Bochner, *Beiträge zur Theorie der fastperiodischen Funktionen*, I, Mathematische Annalen, vol. 96 (1926), pp. 119–147.

PRINCETON UNIVERSITY,
PRINCETON, N. J.
INSTITUTE FOR ADVANCED STUDY,
PRINCETON, N. J.