A PROBLEM CONCERNING ORTHOGONAL POLYNOMIALS*

BY

G. SZEGÖ

Introduction

In his paper *Note on the orthogonality of Tchebycheff polynomials on confocal ellipses*,† Walsh has obtained a new orthogonality property of the Tchebycheff polynomials cos k arc cos z arising by orthogonalization of the set $1, z, z^2, \cdots$ over the range $-1 \leqq z \leqq 1$ with the weight function $|1-z^2|^{-1/2}$. Walsh showed that these polynomials have the same orthogonality property on all confocal ellipses with the foci at ± 1 and with the same weight function $|1-z^2|^{-1/2}$. Another example of this kind is the set of concentric circles with the weight function 1: the corresponding orthogonal polynomials $1, z, z^2, \cdots$ are the same for all curves of this set, provided the common center of the circles is at the origin.

Walsh raised the question whether there exist other pairs of curves with suitable weight functions such that the corresponding orthogonal polynomials would differ only by constant factors. A complete answer to this question seems to be rather intricate. The following theorem may furnish some indications as to the possibilities to be expected.

Theorem 1. *Let C_1 and C_2 be two analytic Jordan curves, $n_1(z)$ and $n_2(z)$ any corresponding weight functions, positive and continuous, and*

$$p_0(z),\ p_1(z),\ p_2(z),\ \cdots,\ p_k(z),\ \cdots$$

a system of polynomials, the exact degree of $p_k(z)$ being k, simultaneously orthogonal on either of the curves,

$$\int_{C_1} n_1(z)\ p_k(z)\overline{p_l(z)}\,|\,dz\,| = \int_{C_2} n_2(z)p_k(z)\overline{p_l(z)}\,|\,dz\,| = 0, \qquad k \neq l.$$

Then one of the curves, say C_1, must contain the other (C_2) and C_1 is a level curve in the conformal mapping of the region outside C_2 onto the exterior of a circle, the points at infinity corresponding to each other. Further there is an analytic function $D(z)$ regular and non-vanishing outside C_2, $z=\infty$ inclusive, such that

* Presented to the Society, December 29, 1934; received by the editors June 29, 1934.

† Bulletin of the American Mathematical Society, vol. 40 (1934), pp. 84–88.

$$|D(z)|^2 = n_1(z), \ z \text{ on } C_1; \quad \lim_{z \to z_0} |D(z)|^2 = n_2(z_0), \ z_0 \text{ on } C_2;$$

in the second formula z_0 *is an arbitrary point on* C_2 *and* z *tends to* z_0 *remaining in the region outside* C_2.

This theorem is valid also under more general assumptions. For the sake of simplicity we confine ourselves to analytic curves. The proof is a slight extension of a known line of argument used in several papers of the author.*

The result stated above suggests in a natural way the following

PROBLEM. *To determine all Jordan curves* C *and all analytic functions* $D(z)$ *regular and non-vanishing outside* C, $z = \infty$ *inclusive, possessing the following property. Let* C_r *be a level curve in the conformal mapping of the region exterior to* C *onto the region exterior to the circle* $|w| = r_0$, *the points at infinity corresponding to each other. The orthogonal polynomials*

$$p_0(z), p_1(z), \cdots, p_k(z), \cdots$$

associated with C_r *and with the weight function* $|D(z)|^2$ *are independent of* r *for* $r > r_0$. *In other words it is required that*

$$\int_{C_r} |D(z)|^2 p_k(z)\overline{p_l(z)} \, |dz| = 0, \qquad k \neq l, \qquad r > r_0.$$

This problem admits of a complete solution. The present paper is devoted to the enumeration of all the types satisfying the condition stated above. There are altogether five essentially distinct cases, two of which have been already mentioned above. In all these cases a linear transformation of the variable z and a multiplication of the weight function by a positive constant factor of course are still allowed. The orthogonal polynomials are not necessarily normalized, the normalizing factor being in general different for different curves C_r. The five types in question are as follows.

I. C_r is the set of concentric circles $|z| = r$, $r > 0$;

$$D(z) = 1, \ p_k(z) = z^k.$$

II. C_r is the set of concentric circles $|z| = r$, $r > 1$;

$$D(z) = (1 - z^{-\alpha})^{-1}, \ \alpha \text{ a positive integer,}$$
$$p_k(z) = z^k, \ 0 \leqq k < \alpha; \ p_k(z) = z^{k-\alpha}(z^\alpha - 1), \ k \geqq \alpha.$$

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* *Beiträge zur Theorie der Toeplitzschen Formen*, II, Mathematische Zeitschrift, vol. 9 (1921), pp. 167–190, especially p. 178; *Über orthogonale Polynome, die zu einer gegebenen Kurve der komplexen Ebene gehören*, Ibidem, vol. 9 (1921), pp. 218–270, especially pp. 260-262.

III. C_r is the set of confocal ellipses with foci at ± 1;

$$D(z) = \{z + (z^2 - 1)^{1/2}\}^{1/2}(z^2 - 1)^{-1/4} = \{\tfrac{1}{2}(1 - w^{-2})\}^{-1/2},$$

$$p_k(z) = w^k + w^{-k}, \ 2z = w + w^{-1}, \ |w| = r > 1.$$

IV. C_r is the same set as in III;

$$D(z) = \{z + (z^2 - 1)^{1/2}\}^{-1/2}(z^2 - 1)^{1/4} = \{\tfrac{1}{2}(1 - w^{-2})\}^{1/2},$$

$$p_k(z) = (w^{k+1} - w^{-k-1})/(w - w^{-1}),$$

where w is the same as in III.

V. C_r is the same as in III;

$$D(z) = (z - 1)^{1/4}(z + 1)^{-1/4} = (1 - w^{-1})^{1/2}(1 + w^{-1})^{-1/2},$$

$$p_k(z) = (w^{k+1/2} - w^{-k-1/2})/(w^{1/2} - w^{-1/2}).$$

It should be observed that Tchebycheff polynomials III, in addition to the property discussed here, have another analogous one, viz. that they minimize the $\max|z^k + a_1 z^{k-1} + \cdots|$ on all the ellipses defined above. This property which was pointed out by Faber,* is analogous to that obtained by Walsh. Our line of argument given in §I applies without difficulty to Tchebycheff polynomials minimizing the $\max n(z)|z^k + a_1 z^{k-1} + \cdots|$ on prescribed curves, $n(z)$ being a given weight function, positive and continuous; thus for this problem we are lead to a theorem analogous to Theorem 1.

In §I we prove Theorem 1 concerning the question raised by Walsh. §II contains a short discussion of the polynomials enumerated under I–V, particularly with respect to their orthogonality. In §III we deal with the principal problem and prove that the only possible polynomials orthogonal on all level curves of a conformal mapping are those of §II.†

I. Proof of Theorem 1

1. Let us consider an analytic Jordan curve C with a positive and continuous weight function $n(z)$. There is no difficulty in showing the existence of an analytic function $D(z)$ regular and non-vanishing outside C, $z = \infty$ inclusive, with the boundary property

$$\lim_{z \to z_0} |D(z)|^2 = n(z_0),$$

* G. Faber, *Über Tschebyscheffsche Polynome*, Journal für Mathematik, vol. 150 (1920), pp. 79–106, especially pp. 84–86.

† After having completed this paper, I communicated its main results to Professor Walsh who kindly informed me that he also obtained the first part of Theorem 1 and proposed precisely the same problem as stated above, without discussing it. These results of Walsh will appear in a monograph of the Mémorial series, Paris, Gauthier-Villars, under the title *Approximation by Polynomials in the Complex Domain*. Nevertheless, for the sake of completeness, we give here a short proof of Theorem 1.

where z_0 denotes an arbitrary point of C and z tends to z_0 from the exterior of C. The function $D(z)$ is completely determined up to a factor of the absolute value 1.

The proof of this statement can be based on the conformal mapping of the region outside C onto the region outside the unit circle $|w|=1$, the points at infinity corresponding to each other. The mapping function and its inverse,

$$z = g(w) = gw + g_0 + g_1w^{-1} + g_2w^{-2} + \cdots,$$
$$w = \gamma(z) = \gamma z + \gamma_0 + \gamma_1 z^{-1} + \gamma_2 z^{-2} + \cdots, \quad g\gamma = 1,$$

are uniquely determined under the assumption $g>0$. We write

$$D[g(w)] = \Delta(w), \quad |w| > 1; \quad n[g(w)] = \nu(w), \quad |w| = 1.$$

Then the function $\Delta(w)$ must satisfy the following condition:

$$\lim_{w\to w_0} |\Delta(w)|^2 = \nu(w_0); \quad |w_0| = 1, \quad |w| > 1,$$

from which it can be computed by means of the Poisson integral

$$2\log\Delta(w) = (2\pi)^{-1}\int_{-\pi}^{\pi} \log\nu(e^{i\phi})\,\frac{w+e^{i\phi}}{w-e^{i\phi}}\,d\phi.$$

2. Let $p_k(z)=p_kz^k+\cdots$ denote the orthogonal polynomials associated with C and with the weight function $n(z)$, the normalization being arbitrary. Then it is known* that the minimum μ_k^2 of the integral

$$(2\pi)^{-1}\int_C n(z)\,|z^k + a_1z^{k-1} + \cdots + a_k|^2\,|dz|$$

over the set of polynomials of degree k and with the highest coefficient 1 is attained for the polynomial $p_k^{-1}p_k(z)$.

We show first that

$$\lim_{k\to\infty} \mu_k g^{-k-1/2} = |D(\infty)|. \tag{1}$$

Indeed we have

$$\mu_k^2 = \min\,(2\pi)^{-1}\int_{|w|=1} |\Delta(w)|^2\,|g(w)^k + a_1g(w)^{k-1} + \cdots + a_k|^2\,|g'(w)|\,|dw|$$
$$= \min\,(2\pi)^{-1}\int_{|w|=1} |\Delta(w)\{(g(w)/w)^k + a_1w^{-1}(g(w)/w)^{k-1}$$
$$+ \cdots + a_kw^{-k}\}g'(w)^{1/2}|^2\,|dw|,$$

* See for example the second paper of the author quoted above, p. 231.

where the integrals should be interpreted as the limits of the corresponding integrals over the circle $|w|=r$ as $r\to 1+0$. The function under the absolute value sign is regular for $|w|>1$. Hence, we get a lower estimate for μ_k^2,

$$\mu_k^2 \geqq |\Delta(\infty)g^k g^{1/2}|^2 = |D(\infty)|^2 g^{2k+1}.$$

An upper estimate for μ_k^2 can be obtained for instance by using the polynomials $f_k(z)=\gamma^k z^k+\cdots$ introduced by Faber* as the principal parts of the expansions of $\gamma(z)^k$, $k=0, 1, 2, \cdots$. Faber shows by elementary methods that

$$\lim_{k\to\infty} f_k(z)\gamma(z)^{-k} = 1$$

is valid uniformly outside a level curve $|\gamma(z)|=\rho$, $\rho<1$, provided ρ is sufficiently near to 1. Now, as a consequence of the minimal property we have

$$\mu_k^2 \leqq (2\pi)^{-1}\int_C n(z)\,|q_k(z)|^2\,|dz|,$$

where $q_k(z)$ is an arbitrary polynomial of degree k with the highest coefficient 1. We put

$$q_k(z) = \sum_{h=0}^{m} \alpha_h \gamma^{h-k} f_{k-h}(z), \quad k \geqq m, \quad \alpha_0 = 1,$$

where m and the constants α_h are to be specified later. Using the asymptotic estimate above of the $f_k(z)$ we obtain

$$\begin{aligned}\overline{\lim_{k\to\infty}}\,(\mu_k g^{-k-1/2})^2 &\leqq \overline{\lim_{k\to\infty}}\,(2\pi)^{-1}\int_C n(z)\left|\sum_{h=0}^{m}\alpha_h\gamma^{h-k}g^{-k-1/2}\gamma(z)^{k-h}\right|^2|dz|\\ &\leqq (2\pi)^{-1}\int_C n(z)\left|\sum_{h=0}^{m}\alpha_h g^{-h-1/2}\gamma(z)^{-h}\right|^2|dz|\\ &\leqq (2\pi)^{-1}\int_{|w|=1}\left|\Delta(w)\sum_{h=0}^{m}\alpha_h g^{-h-1/2}w^{-h}g'(w)^{1/2}\right|^2|dw|.\end{aligned}$$

We now choose for the polynomial

$$\sum_{h=0}^{m}\alpha_h g^{-h}w^{-h} = 1+\cdots$$

the mth partial sum of the power series expansion of the analytic function

$$(g/g'(w))^{1/2}(\Delta(\infty)/\Delta(w)).$$

By taking m sufficiently large the deviation of the last integral above from

* Loc. cit., p. 83.

$|\Delta(\infty)|^2$ can be made arbitrarily small (Schwarz's inequality). Thus (1) is established.

3. The following formula is merely another expression of (1):

$$\lim_{k\to\infty}(2\pi)^{-1}g^{-2k-1}\int_C n(z)\,|\,p_k^{-1}p_k(z)\,|^2\,|\,dz\,|$$

$$= \lim_{k\to\infty}(2\pi)^{-1}g^{-2k-1}\int_{|w|=1}|\,\Delta(w)p_k^{-1}p_k[g(w)]g'(w)^{1/2}\,|^2\,|\,dw\,| = |\,\Delta(\infty)\,|^2.$$

On putting

$$g^{-k-1/2}\Delta(w)p_k^{-1}p_k[g(w)]g'(w)^{1/2} = \Delta(\infty)w^k + a_1^{(k)}w^{k-1} + a_2^{(k)}w^{k-2} + \cdots$$

we may write this as

$$\lim_{k\to\infty}(\,|\,a_1^{(k)}\,|^2 + |\,a_2^{(k)}\,|^2 + \cdots\,) = 0.$$

Hence we get, uniformly for $|w| \geqq r$, $r>1$,

$$\lim_{k\to\infty}(a_1^{(k)}w^{-1} + a_2^{(k)}w^{-2} + \cdots\,) = 0.$$

This yields the asymptotic formula

$$p_k(z) \sim p_k g^{k+1/2}D(\infty)\gamma(z)^k\gamma'(z)^{1/2}D(z)^{-1} \tag{2}$$

which is valid uniformly outside an arbitrary level curve C_r, $r>1$.

This formula shows immediately that the set $p_k(z)$ uniquely determines the mapping function $\gamma(z)$ as well as the function $D(z)$. The proof of Theorem 1 is thus complete.

II. Five types of orthogonal polynomials

1. It is well known that on an arbitrary circle $|z| = r$,

$$\int_{|z|=r} z^k\bar{z}^l\,|\,dz\,| = 0, \qquad k \neq l. \tag{3}$$

This equation is valid for arbitrary integral values of k and l.

2. The polynomials listed under II, in the special case $\alpha = 1$, were introduced by the author.* Their orthogonality may be verified in the following manner. On putting $|z| = r > 1$ we have

$$\int_{|z|=r} z^{k-1}(z-1)\bar{z}^{l-1}(\bar{z}-1)\,|\,1-z^{-1}\,|^{-2}\,|\,dz\,| = r^2\int_{|z|=r} z^{k-1}\bar{z}^{l-1}\,|\,dz\,| = 0$$

$$(k \geqq 1,\ l \geqq 1,\ k \neq l).$$

* *Über trigonometrische und harmonische Polynome,* Mathematische Annalen, vol. 79 (1919), pp. 323–339, especially p. 324.

Further for $k \geqq 1$, $z = re^{i\phi}$, $r > 1$,

$$\int_{|z|=r} z^{k-1}(z-1)\,|1-z^{-1}|^{-2}\,|dz| = r^3 \int_{-\pi}^{\pi} \frac{z^{k-1}}{\bar{z}-1}\,d\phi = r^3 \int_{-\pi}^{\pi} \frac{z^k}{r^2-z}\,d\phi = 0.$$

In the case $\alpha > 1$ the vanishing of the integral for $k \geqq \alpha$, $l \geqq \alpha$, $k \neq l$ can be shown in the same way, and is trivial for $k < \alpha$, $l < \alpha$, $k \neq l$. The only fact which still remains to be proved is that for $k \geqq \alpha$, $l < \alpha$,

$$\int_{|z|=r} z^{k-\alpha}(z^\alpha - 1)\bar{z}^l\,|1-z^{-\alpha}|^{-2}\,|dz| = r^{2\alpha+1} \int_{-\pi}^{\pi} \frac{z^{k-\alpha}\bar{z}^l}{\bar{z}^\alpha - 1}\,d\phi$$

$$= r^{2\alpha+1} \int_{-\pi}^{\pi} \frac{z^k}{r^{2\alpha} - z^\alpha}\left(\frac{r^2}{z}\right)^l d\phi = 0$$

which is easily verified.

3. Type III has been treated by Walsh. The proof can be presented in the following simple way. We have

$$|dz| = |\tfrac{1}{2}(1 - w^{-2})|\,|dw|,$$

and for $r > 1$, $k \neq l$, in view of (3),

$$\int_{|w|=r} (w^k + w^{-k})(\bar{w}^l + \bar{w}^{-l})\,|dw| = 0.$$

In case IV we have only to show that for $k \neq l$

$$\int_{|w|=r} \frac{w^{k+1} - w^{-k-1}}{w - w^{-1}}\,\frac{\bar{w}^{l+1} - \bar{w}^{-l-1}}{\bar{w} - \bar{w}^{-1}}\left|\frac{1 - w^{-2}}{2}\right|^2 |dw|$$

$$= \frac{1}{4} r^{-2} \int_{|w|=r} (w^{k+1} - w^{-k-1})(\bar{w}^{l+1} - \bar{w}^{-l-1})\,|dw| = 0,$$

which again follows from (3).

Finally, in case V, for $k \neq l$,

$$\frac{1}{2}\int_{|w|=r} \frac{w^{k+1/2} - w^{-k-1/2}}{w^{1/2} - w^{-1/2}}\,\frac{\bar{w}^{l+1/2} - \bar{w}^{-l-1/2}}{\bar{w}^{1/2} - \bar{w}^{-1/2}}\,|1 - w^{-1}|^2\,|dw|$$

$$= \frac{1}{2} r^{-1} \int_{|w|=r} (w^{k+1/2} - w^{-k-1/2})(\bar{w}^{l+1/2} - \bar{w}^{-l-1/2})\,|dw| = 0$$

if we use the equation

$$\int_{|w|=r} w^{k+1/2}\bar{w}^{l+1/2}\,|dw| = 0$$

which is valid for arbitrary integral values of k and l, provided $k \neq l$.

III. Solution of the principal problem

1. With the notation of §I our assumption can be written in the form

$$\int_{|w|=r} p_k[g(w)]\overline{p_l[g(w)]}\,|\Delta(w)g'(w)^{1/2}|^2\,|dw| = 0, \quad k \neq l, \quad r > 1.$$

As the first step of the proof we shall derive a power series expansion of the form

$$\begin{aligned}\Delta(w)g'(w)^{1/2}p_k[g(w)] &= \lambda_k w^k + \lambda_{k1}w^{-1} + \lambda_{k2}w^{-2} + \cdots \\ &= \lambda_k w^k + Q_k(w),\end{aligned} \tag{4}$$

where $\lambda_k \neq 0$. Let Aw^m be the second term of the power series expansion of the left-hand member of this formula. By hypothesis, if $0 \leqq m < k$, we must have

$$\int_{|w|=r} (\lambda_k w^k + Aw^m + \cdots)(\bar{\lambda}_m \bar{w}^m + \cdots\ )\,|dw| = 0, \quad r > 1.$$

Here the principal term for large values of r is obviously

$$A\bar{\lambda}_m \int_{|w|=r} w^m \bar{w}^m\,|dw| = 2\pi A\bar{\lambda}_m r^{2m+1}.$$

Hence $A = 0$ and the desired result follows. By (3) and the orthogonality condition we have for $k \neq l$,

$$\begin{aligned}&\int_{|w|=r} (\lambda_k w^k + Q_k(w))(\bar{\lambda}_l \bar{w}^l + \overline{Q_l(w)})\,|dw| \\ &\qquad = \int_{|w|=r} \lambda_k \bar{\lambda}_l w^k \bar{w}^l\,|dw| + \int_{|w|=r} Q_k(w)\overline{Q_l(w)}\,|dw| = 0.\end{aligned}$$

Hence

$$\int_{|w|=r} Q_k(w)\overline{Q_l(w)}\,|dw| = 0, \quad k \neq l,$$

or

$$\lambda_{k1}\bar{\lambda}_{l1} r^{-1} + \lambda_{k2}\bar{\lambda}_{l2} r^{-3} + \cdots = 0, \quad k \neq l.$$

Consequently

$$\lambda_{k1}\bar{\lambda}_{l1} = \lambda_{k2}\bar{\lambda}_{l2} = \cdots = 0, \quad k \neq l.$$

Thus we see that no column λ_{kh} $(k=0, 1, 2, \cdots)$ of the matrix (λ_{kh}) $(k=0, 1, 2, \cdots; h=1, 2, \cdots)$ can contain more than a single element $\neq 0$.

2. Let now $|w|>|t|>1$. We consider the function

$$G(w, t) = \Delta(w)g'(w)^{1/2}g'(t)^{1/2}/\{\Delta(t)(g(t) - g(w))\}. \tag{5}$$

It is regular for t fixed in $|w|>|t|$ and has a simple zero at infinity. Therefore it admits of a representation of the form

$$G(w, t) = \phi_1(t)w^{-1} + \phi_2(t)w^{-2} + \phi_3(t)w^{-3} + \cdots. \tag{6}$$

We shall show that

$$(2\pi i)^{-1}\lambda_k \int_{|t|=r} t^k G(w, t)dt = Q_k(w), \quad |w| > r > 1. \tag{7}$$

Indeed the left-hand member is, on account of (4),

$$(2\pi i)^{-1}\Delta(w)g'(w)^{1/2} \int_{|t|=r} \frac{p_k[g(t)]}{g(t) - g(w)} g'(t)dt - (2\pi i)^{-1} \int_{|t|=r} Q_k(t)G(w, t)dt.$$

On writing $\tau = g(t)$ we obtain for the first term

$$(2\pi i)^{-1}\Delta(w)g'(w)^{1/2} \int_{C_r} \frac{p_k(\tau)}{\tau - z} d\tau = 0$$

since $z = g(w)$ is outside C_r. The integral of the second term, being taken over a large circle $|t| = R$, tends to 0 as $R\to\infty$. Thus we get the residue $Q_k(w)$.

An alternative form of this result is

$$(2\pi i)^{-1}\lambda_k \int_{|t|=r} t^k\phi_h(t)dt = \lambda_{kh} \quad (k = 0, 1, 2, \cdots; h = 1, 2, 3, \cdots). \tag{8}$$

3. There is no difficulty in obtaining explicit representations for the functions $\phi_h(t)$. From (6) we have

$$\begin{aligned} \phi_1(t) &= -\Delta(\infty)g^{-1/2}g'(t)^{1/2}/\Delta(t), \\ \phi_2(t) &= -\Delta(\infty)g^{-1/2}(g'(t)^{1/2}/\Delta(t))(g(t)/g + \text{const.}). \end{aligned} \tag{9}$$

A direct expansion shows that $\phi_h(t)$ is of the form $g'(t)^{1/2}/\Delta(t)$ multiplied by a polynomial in $g(t)$ of the exact degree $(h-1)$.

In virtue of (8) and of the remark above concerning the vanishing of the λ_{kh} in a fixed column, we see at once that the Laurent series expansion of $\phi_h(t)$ cannot involve more than one negative power of t, that is, $\phi_h(t)$ must be of the form

$$bt^{-\beta} + b_0 + b_1 t + \cdots + b_{h-1}t^{h-1}, \quad b_{h-1} \neq 0, \quad \beta > 0.$$

As a consequence of this $\phi_2(t)/\phi_1(t)$, hence also $g(t)$, must be rational. This function cannot have other poles than 0 and ∞; otherwise $\phi_h(t)$ would have a further pole provided h is sufficiently large. Thus we find

$$g(t) = gt + g_0 + g_1 t^{-1} + \cdots + g_\sigma t^{-\sigma}. \tag{10}$$

4. On denoting the exact orders of $\phi_1(t)$ and of $g(t)$ at $t=0$ by ρ and σ respectively we first assume $\sigma = 0$, that is,

$$g(t) = gt + g_0, \quad \phi_1(t) = -\Delta(\infty)/\Delta(t) = bt^{-\rho} + b_0.$$

This yields types I and II given in the Introduction.

Next assume $\sigma > 0$. We now distinguish two principal cases.

(a) $Q_0(w)$ is not identically zero, that is, there is at least one coefficient $\lambda_{0h} \neq 0$. We know by (8) that $\phi_h(t)$ has a simple pole at $t=0$. Then, by (9),

$$\rho + (h-1)\sigma = 1.$$

Consequently we have to consider the following possibilities:

$$h = 1, \quad \rho = 1;$$
$$h = 2, \quad \sigma = 1, \quad \rho = 0.$$

Under the first hypothesis we have on account of (4) for $k=0$,

$$g'(t)^{1/2}/\Delta(t) = b_0 + b_1 t^{-1}, \quad g'(t)^{1/2}\Delta(t) = c_0 + c_1 t^{-1}, \quad b_0, b_1, c_0, c_1 \neq 0,$$

whence

$$g'(t) = b_0 c_0 + (b_0 c_1 + b_1 c_0)t^{-1} + b_1 c_1 t^{-2},$$

so that $b_0 c_1 + b_1 c_0 = 0$ and

$$g(t) = b_0 c_0 t + g_0 - b_1 c_1 t^{-1} = g_0 + b_0 c_0 \left(t + \frac{b_1^2}{b_0^2} t^{-1} \right),$$

while

$$\Delta(t) = \left[b_0 c_0 \left(1 - \frac{b_1^2}{b_0^2} t^{-2} \right) \right]^{1/2} / (b_0 + b_1 t^{-1}) = \left(\frac{c_0}{b_0} \right)^{1/2} \left(\frac{b_0 - b_1 t^{-1}}{b_0 + b_1 t^{-1}} \right)^{1/2}.$$

This is our type V.

The second hypothesis gives at once

$$g(t) = gt + g_0 + g_1 t^{-1} \quad (g, g_1 \neq 0); \quad g'(t)^{1/2}/\Delta(t) = \text{const.},$$

which is type IV.

(b) $Q_0(w)$ is identically zero, that is, $\Delta(t)g'(t)^{1/2}=\text{const}$. Then

$$\phi_1(t) = \text{const. } g'(t), \quad \rho = \sigma + 1,$$

and from (4) for $k=1$ we find that $\lambda_{1\sigma}\neq 0$. Consequently $\phi_\sigma(t)$ has a pole at $t=0$ of the exact order 2. Now the exact order of the pole $t=0$ of $\phi_h(t)$ is

$$\rho + (h-1)\sigma = \sigma + 1 + (h-1)\sigma = h\sigma + 1.$$

For $h=\sigma$ we have $\sigma^2+1=2$, $\sigma=1$, which corresponds to type III. Our proof is now complete.

KÖNIGSBERG, PR.