INTERPOLATION IN REGULARLY DISTRIBUTED POINTS*

BY

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1. **Introduction.** Let G_n be a set of n distinct points chosen on the rectifiable Jordan curve C† in the complex z-plane, and let $\{G_n\}$ denote a sequence of such sets. This sequence may be written out in the following triangular array:

$$\begin{array}{ll} G_1: & z_1^{(1)} \\ G_2: & z_1^{(2)}, z_2^{(2)} \\ \cdot & \\ \cdot & \\ \cdot & \\ G_n: & z_1^{(n)}, z_2^{(n)}, \cdots, z_n^{(n)}. \end{array}$$

Furthermore, let $f(z)$ be a function defined‡ and integrable in the sense of Riemann on the curve C; we shall say that such a function is integrable (R) on C. By $L_n(z)$ we shall denote the unique polynomial of degree at most $n-1$ which coincides with the function $f(z)$ in the points of the set G_n; we shall call it the Lagrange polynomial interpolating to $f(z)$ in the points G_n. We shall say that the sequence $\{G_n\}$ yields effective interpolation to the function $f(z)$ if the sequence $\{L_n(z)\}$ converges to the function $f_1(z)$ at every point of B, the region interior to the curve C,§ and uniformly for z on any closed point set of B, where

$$f_1(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\,dt.$$

About half a century ago, Méray pointed out that if $f(z)=1/z$ and if the set G_n consists of the nth roots of unity, then since $L_n(z)=z^{n-1}$, the sequence $\{L_n(z)\}$ approaches zero for $|z|<1$.‖ (It will be noted that zero is the value of the function $f_1(z)$ in this case.) The following theorem, of comparatively recent origin, discloses the theory underlying this example:

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† We define a Jordan curve as a one-to-one continuous transform of a circumference.

‡ Infinity will not be admitted as a functional value in connection with the definition of functions other than the mapping function $\phi(w)$ introduced in §2.

§ By the interior of C we mean the region bounded by C which does not contain the point at infinity.

‖ Méray, Annales de l'École Normale Supérieure, (3), vol. 1 (1884), pp. 165–176.

THEOREM A.* *Let $f(z)$ be a function defined and integrable* (R) *on the unit circle, and let the set G_n be the nth roots of unity. Then the sequence $\{G_n\}$ yields effective interpolation to the function $f(z)$. But for a properly chosen function $f(z)$ a sub-sequence of the sequence $\{L_n(z)\}$ will diverge to infinity at points on the unit circle itself.*

The main results of the present paper arose from the suggestion made by Walsh that it would be of interest to extend the theorem to the consideration of curves other than the unit circle.† The extension will be derived first under the hypothesis that the function $f(z)$ is analytic on the curve C (as in Méray's example), and then under the hypothesis that the function is merely bounded in modulus and integrable (R) on C.‡ The theorems thus obtained will be supplemented by a study of the degree of convergence of the sequence $\{L_n(z)\}$. This study will result in equalities for z on the curve C as well as for z in the region B, and so will have an additional significance in that it will elucidate the statement in Theorem A concerning the possibility of divergence on C.

The paper concludes with a discussion of the results which arise from interpolation to more than one function defined on one or more Jordan curves.

2. **The choice of the points of interpolation.** An arbitrarily chosen sequence $\{G_n\}$ will not in general lead to effective interpolation, even if the function $f(z)$ is analytic in the closed region $B+C$ and if the points of the nth set G_n become everywhere dense on the curve C as n approaches infinity.§ Thus the proper choice of the set G_n is of fundamental importance in a generalization of Theorem A. We shall base our selection of the set G_n upon a notable precedent; namely, that of Fejér, who established an extension of Theorem A for functions analytic in the closed region $B+C$ by using a set G_n which he called a set of "regularly distributed" ("regelmässig verteilt") points on the curve C.|| Fejér's set G_n may be defined in the following man-

* This theorem is due to Fejér and Walsh. Fejér, in a brilliant paper entitled *Interpolation und konforme Abbildung* which appeared in the Göttinger Nachrichten, 1918, pp. 319–331, proved the theorem for the case in which the function $f(z)$ is continuous on and within the unit circle and analytic within the circle, and he also gave an example of such a function for which the corresponding Lagrange polynomials diverge at a point of the circle. Walsh showed that the theorem is true for functions more general than those considered by Fejér; Bulletin of the American Mathematical Society, vol. 38 (1932), pp. 290–291.

† Walsh, loc. cit., p. 294.

‡ The methods of approach indicated for these two cases are entirely dissimilar; see §5.

§ For a simple illustration of this statement, see Walsh, loc. cit., p. 293.

|| Fejér, loc. cit., pp. 324–327. Theorem A becomes a classical result due to Runge when the function $f(z)$ is assumed to be analytic in the closed region; see Runge, *Theorie und Praxis der Reihen*, Berlin, 1904, p. 137. In this case, and also in Fejér's extension, the convergence of the sequence $\{L_n(z)\}$ takes place in the closed region.

ner: Let the Jordan curve C lie in the z-plane and let the function $\phi(w)$ map the exterior of the unit circle in the w-plane onto the exterior of C in such a way that the points at infinity in the two planes correspond to each other.* The set G_n consists of those points on C into which the nth roots of unity are transformed by the equation $z=\phi(w)$.

Henceforth in this paper the symbol G_n, wherever it appears in connection with a curve C, will denote the nth set of Fejér's regularly distributed points on C.

3. **Restrictions on the curve.** It is assumed that the function $\phi(w)$, which we have just introduced, gives a conformal, one-to-one map of the exterior of the unit circle onto the exterior of the curve C, which means that with the exception of the point at infinity, $\phi(w)$ is analytic for $|w|>1$, univalent and continuous for $|w|\geqq 1$. The function generates a Laurent series of the following type:

$$\phi(w) \sim cw + c_0 + \frac{c_1}{w} + \frac{c_2}{w^2} + \cdots, \; c \neq 0,$$

which may be considered as a representation of the function for all $|w|\geqq 1$. In particular, we have

$$\phi'(\infty) = \lim_{w\to\infty} \frac{\phi(w)}{w} = c.$$

We shall denote the inverse of the function $\phi(w)$ by $\phi^{-1}(z)$.

We define a function $\Phi(z, w)$ by the following equations:

$$\Phi(z, w) = \log \frac{\phi(w) - z}{cw}, \quad z \text{ in } B, \; |w| \geqq 1,$$
$$\Phi(z, \infty) = 0;\dagger$$

and in the event that the function $\phi(w)$ possesses a non-vanishing first tangential derivative at every point of the circle $|w|=1$, we define also a function $\Psi(\overline{w}, w)$ as follows:

$$\left.\begin{aligned} \Psi(\overline{w}, w) &= \log \frac{\phi(\overline{w}) - \phi(w)}{c(\overline{w} - w)} \\ \Psi(\overline{w}, \overline{w}) &= \log \frac{\phi'(\overline{w})}{c} \end{aligned}\right\} |w| \geqq 1, \; |\overline{w}| \geqq 1,$$
$$\Psi(\overline{w}, \infty) = 0.\ddagger$$

* Hilbert first indicated the significance of this type of mapping function in the study of interpolation, Göttinger Nachrichten, 1897, pp. 63–70.

† This equation identifies the particular branch of the logarithmic function under consideration.

‡ See the preceding footnote.

(The variables z, w, and $\bar{w}$ are all supposed to be independent.) Both of these functions are analytic functions of w for $|w| \geqq 1$, provided that z is a point of the region B.

We shall now introduce a pair of conditions on the curve C which will be expressed in terms of these functions; the conditions play a central role in our generalization of Theorem A.

The curve C will be said to satisfy condition (a) if given an arbitrary closed point set S of the region B, there exist polynomials in $1/w$, $f_n(z, w)$, of respective degrees $n-1$, which satisfy the equation

$$\Phi(z, w) - f_n(z, w) = o\left(\frac{1}{n}\right)$$

uniformly for $|w| \geqq 1$ and for z on S.

The curve C will be said to satisfy condition (b) if the corresponding mapping function $\phi(w)$ possesses a non-vanishing first tangential derivative at every point of the circle $|w| = 1$, and if there exist polynomials in $1/w$, $F_n(\bar{w}, w)$, of respective degrees $n-1$, which satisfy the equation

$$\Psi(\bar{w}, w) - F_n(\bar{w}, w) = o\left(\frac{1}{n}\right),$$

uniformly for $|w| \geqq 1$ and $|\bar{w}| = 1$.*

A Jordan curve will satisfy condition (a) if the first tangential derivative of the corresponding mapping function $\phi(w)$ on the circle $|w| = 1$ exists and satisfies a Lipschitz condition with exponent $\alpha > 0$.† The curve C will also satisfy condition (b) if the second tangential derivative of the function $\phi(w)$ on the circle $|w| = 1$ exists and satisfies a Lipschitz condition with exponent $\alpha > 0$, and if the first tangential derivative does not vanish.

To prove that a curve of the first type satisfies condition (a), we observe that the first tangential partial derivative with respect to w of the function $\Phi(z, w)$ on the circle $|w| = 1$ satisfies a Lipschitz condition with exponent α and with a constant which is a uniformly bounded function of z for z on any closed point set of the region B. A similar assertion may also be made in connection with a curve of the second type concerning its function $\Psi(\bar{w}, w)$

* It is to be observed that no assumption is made as to the continuity of the functions $f_n(z, w)$ and $F_n(\bar{w}, w)$ in the variables z and $\bar{w}$ respectively.

† A function $f(z)$ is said to satisfy a Lipschitz condition on a curve C with exponent α and constant λ if $|f(x_1) - f(x_2)| \leqq \lambda |x_1 - x_2|^\alpha$ for all x_1 and x_2 on C.

for $|\bar{w}|=1$, $|w|=1$, although the proof is not as simple as in the first case.* The existence of the required polynomials in $1/w$ is now established by a theorem due to Sewell, which, for future reference, we shall call Theorem B.†

4. Products associated with the Lagrange polynomial. The Lagrange polynomial interpolating to the function $f(z)$ in the arbitrary set of distinct points z_k, $k=1, 2, \cdots, n$, may be written in this form:

$$L_n(z) = \sum_{k=1}^{n} \frac{f(z_k)}{z - z_k} \frac{\omega_n(z)}{\omega_n'(z_k)}, \tag{1}$$

where $\omega_n(z)$ is the following product:

$$\omega_n(z) = \prod_{k=1}^{n} (z - z_k).$$

If the set of points happens to be the nth roots of unity, $e^{2\pi ik/n}$, $k=1, 2, \cdots, n$, then

$$\omega_n(z) = z^n - 1,$$

and (1) becomes

$$L_n(z) = \sum_{k=1}^{n} f(e^{2\pi ik/n}) \frac{(z^n - 1)e^{2\pi ik/n}}{n(z - e^{2\pi ik/n})}.$$

In particular,

$$nL_n(0) = \sum_{k=1}^{n} f(e^{2\pi ik/n}); \tag{2}$$

hence if there exists an upper bound for the function $f(z)$ on the unit circle, and if this bound be denoted by f, then

$$|L_n(0)| \leqq f. \tag{3}$$

This fact leads at once to the proof of the first of three lemmas upon which our generalization of Theorem A will rest.

LEMMA I.‡ *Let the function $f(z)$ be defined for $|z| \leqq 1$, let $p_n(z)$ be a polynomial of degree $n-1$, and let ϵ_n be a positive number such that*

$$|f(z) - p_n(z)| \leqq \epsilon_n$$

for $|z|=1$ and for $z=0$.

* A proof can be given by taking the real and imaginary parts of the derivative $\Phi_w(\bar{w}, w)$ and then applying the integral form of the law of the mean.

† W. E. Sewell, Bulletin of the American Mathematical Society, vol. 41 (1935), p. 117.

‡ This lemma is a special case of Theorem IIIa.

Let the polynomial $L_n(z)$ *interpolate to the function* $f(z)$ *in the set* $e^{2\pi ik/n}$, $k=1, 2, \cdots, n$. *Then*

$$|f(0) - L_n(0)| \leqq 2\epsilon_n.$$

For let $\lambda_n(z)$ be the Lagrange polynomial interpolating to the function $p_n(z)-f(z)$ in the set $e^{2\pi ik/n}$. Then $f=\epsilon_n$, so $|\lambda_n(0)| \leqq \epsilon_n$, by (3). Therefore

$$|f(0) - p_n(0) + \lambda_n(0)| \leqq 2\epsilon_n.$$

But

$$p_n(e^{2\pi ik/n}) - \lambda_n(e^{2\pi ik/n}) = f(e^{2\pi ik/n}) \qquad (k = 1, 2, \cdots, n),$$

so $p_n(z)-\lambda_n(z)$ must be the unique polynomial of degree at most $n-1$ interpolating to the function $f(z)$ in the set $e^{2\pi ik/n}$, which is none other than $L_n(z)$ itself.

Lemma II. *Let C be a curve which satisfies condition* (a) *and let the product* $\omega_n(z)$ *be formed for the corresponding set* G_n. *Then*

$$\frac{\omega_n(z)}{-c^n} \to 1$$

uniformly for z on any closed point set S of the region B.

To prove the assertion, we first write

$$\frac{\omega_n(z)}{-c^n} = -\prod_{k=1}^{n}\left[\frac{z-\phi(e^{2\pi ik/n})}{c}\right] = \prod_{k=1}^{n}\left[\frac{\phi(e^{2\pi ik/n})-z}{ce^{2\pi ik/n}}\right].$$

We shall compute the limit by studying that branch of the function $\log [\omega_n(z)/-c^n]$ which is identified by the following equation:

$$\log \frac{\omega_n(z)}{-c^n} = \sum_{k=1}^{n} \Phi(z, e^{2\pi ik/n}) = \sum_{k=1}^{n} \Phi(z, e^{-2\pi ik/n}). \tag{4}$$

By hypothesis, there exist polynomials in the variable $W=1/w$, $f_n(z, w)$, of respective degrees $n-1$, which satisfy the equation

$$\Phi(z, w) - f_n(z, w) = o\left(\frac{1}{n}\right)$$

uniformly for $|W| \leqq 1$ and for z on S. Therefore if we denote by $\Lambda_n(z, w)$ the Lagrange polynomial in W interpolating to the function $\Phi(z,w)$ in the points of the set $W=e^{2\pi ik/n}$, $k=1, 2, \cdots, n$, we may write, by Lemma I,

$$\Phi(z, \infty) - \Lambda_n(z, \infty) = o\left(\frac{1}{n}\right),$$

or

$$n\Phi(v, \infty) - n\Lambda_n(z, \infty) = o(1)$$

uniformly for z on S. But by (2) and (4),

$$n\Lambda_n(z, \infty) = \log \frac{\omega_n(z)}{-c^n},$$

and since $\Phi(z, \infty) = 0$, the proof is complete.

LEMMA III. *Let C be a curve which satisfies condition* (b) *and let the product $\omega_n(z)$ be formed for the corresponding set G_n. Let*

$$\pi_n(\bar{w}) = \frac{\omega_n[\phi(\bar{w})]}{c^n(\bar{w}^n - 1)}.$$

Then $\pi_n(\bar{w}) \to 1$ uniformly for $|\bar{w}| = 1$.

The proof is the same as that of the preceding lemma except for obvious changes in notation.

It is worth while noticing that the existence of this limit can also be proved for $\bar{w}$ on any closed point set lying exterior to the circle $|\bar{w}| = 1$ by modifying condition (b) accordingly. The modification would have the effect of lightening the restriction on the curve C, for the function $\Psi(\bar{w}, w)$ is an analytic function of the two variables $\bar{w}$ and w for $|\bar{w}| > 1$, $|w| > 1$. Thus, in particular, the new condition would be satisfied if the first tangential derivative of the function $\phi(w)$ existed on the circle $|w| = 1$ and satisfied a Lipschitz condition with positive exponent.*

5. **The convergence of sequences of Lagrange polynomials.** We now apply the foregoing results to the theory of interpolation. Let $f(z)$ be a function known to be analytic on the curve C, but not necessarily analytic at all points of the region B. Furthermore, let the curve C satisfy condition (a) and let the product $\omega_n(z)$ be formed for the corresponding set G_n. We determine two contours C_1 and C_2 with the following properties: (1) C_1 contains C in its interior and C contains C_2 in its interior; (2) the function $f(z)$ is analytic in the closed annular region bounded by C_1 and C_2. Then we may write the following formula for the Lagrange polynomial which interpolates to the function $f(z)$ in the points G_n:

$$L_n(z) = \frac{1}{2\pi i} \int_{C_1 + C_2} \frac{f(t)}{t - z} \left[1 - \frac{\omega_n(z)}{\omega_n(t)} \right] dt.$$

* A number of writers have employed the limit, $\lim_{n\to\infty} |\pi_n(\bar{w})|^{1/n}$, $|\bar{w}| > 1$; see for example Fejér, loc. cit., pp. 322–324, and Kalmár, Mathematikai és Physikai Lapok, vol. 33 (1926), pp. 120–140.

(The integration over C_2 is taken in a sense which is positive with respect to the region exterior to C_2.) This is a slight extension of the Cauchy-Hermite form of the Lagrange polynomial; its validity may easily be checked by noting that both integrals represent polynomials in z of degree $n-1$, and that when $z=z_k^{(n)}$, we have

$$L_n(z_k^{(n)}) = \frac{1}{2\pi i}\int_{C_1+C_2} \frac{f(t)}{t - z_k^{(n)}}\, dt = f(z_k^{(n)}).$$

We may write

$$\begin{aligned} &| L_n(z) - f_1(z) | \\ &\leqq \frac{1}{2\pi}\int_{C_1}\left|\frac{f(t)}{t-z}\right|\left|\frac{\omega_n(z)}{\omega_n(t)}\right|| dt| + \frac{1}{2\pi}\int_{C_2}\left|\frac{f(t)}{t-z}\right|\left|1 - \frac{\omega_n(z)}{\omega_n(t)}\right|| dt|, \end{aligned} \tag{5}$$

where

$$f_1(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\, dt = \frac{1}{2\pi i}\int_{C_1} \frac{f(t)}{t-z}\, dt,$$

z being interior to the curve C. Fejér has shown that with the present choice of G_n and $\omega_n(z)$,

$$| \omega_n(z) |^{1/n} \to | c | \, | \phi^{-1}(z) | \tag{6}$$

for all z exterior to C and uniformly for z on any closed point set exterior to C.* On the other hand if z lies interior to C, Lemma II indicates that

$$| \omega_n(z) |^{1/n} \to | c | \tag{7}$$

and that

$$\left| 1 - \frac{\omega_n(z)}{\omega_n(t)} \right| \to 0.$$

Let z be any point on a Jordan curve C' lying between C and C_2 and containing C_2 in its interior. Then combining (6) and (7), we obtain

$$\frac{\omega_n(z)}{\omega_n(t)} \to 0$$

uniformly for z on C', t on C_1; so that inequality (5) implies that $L_n(z) \to f_1(z)$ uniformly for z on C'. The principle of the maximum then tells us that the sequence $\{L_n(z)\}$ approaches the same limit for z interior to C'. We have proved the following theorem:

* Fejér, loc. cit., pp. 322–324. See also the remark following Lemma III and the accompanying footnote.

THEOREM I. *Let C be a Jordan curve which satisfies condition* (a) *and let $f(z)$ be a function analytic on C. The sequence $\{G_n\}$ corresponding to C yields effective interpolation to the function $f(z)$.*

If we interpolate to a function $f(z)$ which is only known to be bounded in modulus and integrable (R) on the curve C, we can no longer use the convenient Cauchy-Hermite formula. To study the convergence in this case, we assume that the curve C satisfies both conditions (a) and (b) and that the tangential derivative of the corresponding mapping function on the circle $|w| = 1$ is bounded in modulus and integrable (R).

Let S be an arbitrary closed point set of the region B, let $t = \phi(e^{i\theta})$ and let $z_k^{(n)} = \phi(e^{2\pi ik/n})$. Since

$$\pi_n(e^{i\theta}) = \frac{\omega_n(t)}{c^n(e^{in\theta} - 1)},$$

we have

$$\frac{\omega_n(t) - \omega_n(z_k^{(n)})}{t - z_k^{(n)}} = \frac{\omega_n(t)}{t - z_k^{(n)}} = c^n \frac{e^{in\theta} - 1}{e^{i\theta} - e^{2\pi ik/n}} \frac{e^{i\theta} - e^{2\pi ik/n}}{\phi(e^{i\theta}) - \phi(e^{2\pi ik/n})} \pi_n(e^{i\theta}). \tag{8}$$

Therefore,

$$\frac{1}{\omega'(z_k^{(n)})} = \frac{1}{inc^n\pi_n(e^{2\pi ik/n})} \frac{d\phi(e^{2\pi ik/n})}{d\theta}. \tag{9}$$

We may now write

$$\begin{aligned} L_n(z) &= \sum_{k=1}^{n} \frac{f(z_k^{(n)})}{z - z_k^{(n)}} \frac{\omega_n(z)}{\omega_n'(z_k^{(n)})} \\ &= \frac{1}{2\pi i} \sum_{k=1}^{n} \frac{f[\phi(e^{2\pi ik/n})]}{[\phi(e^{2\pi ik/n}) - z]} \frac{\omega_n(z)}{- c^n\pi_n(e^{2\pi ik/n})} \frac{d\phi(e^{2\pi ik/n})}{d\theta} \frac{2\pi}{n}. \end{aligned} \tag{10}$$

Lemmas II and III state that

$$\frac{\omega_n(z)}{- c^n\pi_n(e^{i\alpha})} \to 1$$

uniformly for z on S and for all real α. Therefore since both $f[\phi(e^{i\theta})]$ and $d\phi(e^{i\theta})/d\theta$ are integrable (R) and of bounded modulus, we have

$$L_n(z) \to \frac{1}{2\pi i} \int_0^{2\pi} \frac{f[\phi(e^{i\theta})]}{\phi(e^{i\theta}) - z} \frac{d\phi(e^{i\theta})}{d\theta} d\theta = \frac{1}{2\pi i} \int_C \frac{f(t)}{t - z} dt$$

uniformly for z on S. We have established the following theorem:

THEOREM II. *Let C be a Jordan curve which satisfies conditions* (a) *and* (b) *and for which the mapping function possesses a tangential derivative bounded in modulus and integrable* (R) *on the circle* $|w|=1$. *Let $f(z)$ be a function bounded in modulus and integrable* (R) *on C. The sequence $\{G_n\}$ corresponding to C yields effective interpolation to the function $f(z)$.*

6. **Degree of convergence; convergence on the boundary.** We shall now study the degree of convergence of the sequence $\{L_n(z)\}$ in comparison with that of any other given sequence of approximating polynomials. At the same time, we shall be able to obtain a result which casts some light on the question of convergence on the boundary C of the Jordan region under consideration.

THEOREM IIIa. *Let C be a Jordan curve which satisfies conditions* (a) *and* (b) *and for which the mapping function possesses a first tangential derivative bounded in modulus on the circle* $|w|=1$. *Let S be any point set interior to C. Let $f(z)$ be a function defined on C and S and let the polynomials $L_n(z)$ interpolate to $f(z)$ in the set G_n corresponding to C. If there exist positive numbers ϵ_n and polynomials $p_n(z)$ of respective degrees $n-1$ such that*

$$|f(z) - p_n(z)| \leqq \epsilon_n$$

for z on C and S, then

$$|f(z) - L_n(z)| \leqq K_1\epsilon_n$$

for z on S, where K_1 depends only on C and S.

THEOREM IIIb. *Let C be a Jordan curve which satisfies condition* (b) *and for which the mapping function possesses the following property*:

$$\left| \frac{d\phi(e^{i\theta})}{d\theta} \cdot \frac{e^{i\theta} - e^{i\alpha}}{\phi(e^{i\theta}) - \phi(e^{i\alpha})} \right| \leqq M, \text{ all } \theta \text{ and all } \alpha.$$

Let $f(z)$ be a function defined on C, and let the polynomials $L_n(z)$ interpolate to $f(z)$ in the set G_n corresponding to C. If there exist positive numbers ϵ_n and polynomials $p_n(z)$ of respective degrees $n-1$ such that

$$|f(z) - p_n(z)| \leqq \epsilon_n$$

for z on C, then

$$|f(z) - L_n(z)| \leqq K_2\epsilon_n \log n, \quad n > 1,$$

for z on C, where K_2 depends only on C.

The restrictions on the curve C in both theorems are satisfied by a curve for which the mapping function possesses a non-vanishing first tangential derivative on the circle $|w|=1$, and a second tangential derivative satisfying a Lipschitz condition with a positive exponent.

For the proofs of these theorems we first consider the polynomial $\Lambda_n(z)$ which interpolates in the set G_n to a function $F(z)$ of bounded modulus on the curve C. Let F be an upper bound to the modulus of this function. If the curve C satisfies the conditions of Theorem IIIa, we may conclude at once, by referring to (10) and the reasoning which accompanies that equation, that there exists a positive number K_1 such that $|\Lambda_n(z)| \leqq (K_1-1)F$ for all n and for all z on S. The number K_1 depends only on C and S.

If the curve C satisfies the conditions of Theorem IIIb, we proceed as follows. Using (8), (9), and (10), and setting $z=\phi(e^{i\theta})$, we write

$$\Lambda_n(z) = \frac{1}{in}\sum_{k=1}^{n} F[\phi(e^{2\pi ik/n})]\frac{d\phi(e^{2\pi ik/n})}{d\theta}\,\frac{e^{i\theta}-e^{2\pi ik/n}}{\phi(e^{i\theta})-\phi(e^{2\pi ik/n})}\,\frac{\pi_n(e^{i\theta})}{\pi_n(e^{2\pi ik/n})}\,\frac{e^{in\theta}-1}{e^{i\theta}-e^{2\pi ik/n}}.$$

Lemma III establishes the existence of a positive number M_1 such that

$$\left|\frac{\pi_n(e^{i\theta})}{\pi_n(e^{i\alpha})}\right| \leqq M_1$$

for all n and for all real θ and α. Also, it can be shown that for $n>1$,

$$\sum_{k=1}^{n}\left|\frac{e^{in\theta}-1}{e^{i\theta}-e^{2\pi ik/n}}\right| = \sum_{k=1}^{n}\left|\frac{\sin\frac{1}{2}n\left(\theta-\dfrac{2\pi k}{n}\right)}{\sin\frac{1}{2}\left(\theta-\dfrac{2\pi k}{n}\right)}\right| \leqq M_2\log n,$$

where M_2 is independent of θ and n.* We may therefore write

$$|\Lambda_n(z)| \leqq FMM_1M_2\log n \leqq (K_2\log n-1)F$$

for all z on C and all $n>1$, where K_2 is independent of n.

The remaining steps in the proofs of the two theorems can now be given simultaneously. If we let $F(z)=p_n(z)-f(z)$, then $F=\epsilon_n$, and we have

$$|\Lambda_n(z)| \leqq \begin{cases}(K_1-1)\epsilon_n, & z \text{ on } S, \text{ Theorem IIIa},\\ (K_2\log n-1)\epsilon_n, & z \text{ on } C, \text{ Theorem IIIb}, n>1.\end{cases}$$

Therefore

$$|f(z)-p_n(z)+\Lambda_n(z)| \leqq \begin{cases}K_1\epsilon_n, & z \text{ on } S, \text{ Theorem IIIa},\\ K_2\epsilon_n\log n, & z \text{ on } C, \text{ Theorem IIIb}, n>1.\end{cases}$$

* For the proof of this inequality, see Jackson, *The Theory of Approximation*, New York, 1930, p. 120.

But $p_n(z)-\Lambda_n(z)\equiv L_n(z)$, so the proofs are complete.

If $\epsilon_n \log n\rightarrow 0$, then we obtain convergence of the sequence $\{L_n(z)\}$ on the curve C in Theorem IIIb. There is no implication in either Theorem IIIa or Theorem IIIb, however, that the numbers ϵ_n tend to zero; they may be any positive numbers whatsoever.

The example given by Fejér to establish the possibility of divergence on the unit circle in Theorem A employed a function $f(z)$ which was analytic throughout the interior of the unit circle and continuous in the corresponding closed region. Theorem IIIb permits us to make the general assertion in connection with this example that if the function $f(z)$ is continuous in the closed region $B+C$ and analytic in the region B (where C satisfies the condition of the theorem), then $L_n(z)=o$ ($\log n$) for z on C; for by a theorem due to Walsh there exist polynomials such that the corresponding numbers ϵ_n tend to zero.* Moreover, if the curve C is analytic, if the function $f(z)$ is continuous in the closed region $B+C$ and analytic in the region B, and if the pth tangential derivative of $f(z)$ on C satisfies a Lipschitz condition with exponent $\alpha>0$, then by Theorem IIIa,

$$|f(z)-L_n(z)|\leqq\frac{M_3\log n}{n^{p+\alpha}},\qquad z \text{ on } S,$$

and by Theorem IIIb,

$$|f(z)-L_n(z)|\leqq\frac{M_4(\log n)^2}{n^{p+\alpha}},\quad z \text{ on } C,$$

where M_3 and M_4 are both independent of n and z. This result follows directly from Theorem B. In particular, the value zero is admissible for p in these inequalities, so a sufficient condition for the convergence of the sequence $\{L_n(z)\}$ in the closed region $B+C$ is that the function $f(z)$ be continuous in the closed region, analytic in the region B, and satisfy a Lipschitz condition with positive exponent on C.

7. **Simultaneous interpolation to several functions.** We now turn to certain immediate generalizations of the results of §5. First of all, it is natural to inquire whether Theorem II admits of some form of extension when the Lagrange polynomial is defined by the requirement of coinciding, not with one function $f(z)$ at points of C, but simultaneously with several distinct functions in several distinct sets of points on C. This problem may readily be attacked by the methods which we have previously developed, and the following theorem indicates the type of result to be expected.

* Mathematische Annalen, vol. 96 (1926), pp. 430–436.

THEOREM IV. *Let C be subject to the restrictions of Theorem* II. *Let $f_1(z)$, $f_2(z), \cdots, f_m(z)$ be m functions which are bounded in modulus and integrable (R) on C. Let $L_{mn}(z)$ be the polynomial of degree at most $mn-1$ which interpolates to the function $f_\mu(z)$ in the points*

$$z_{\mu,k}^{(n)} = \phi(e^{2\pi i\mu/(mn)+2\pi ik/n}) \qquad (k = 1, 2, \cdots, n; \mu = 1, 2, \cdots, m).$$

Then

$$L_{mn}(z) \to \frac{1}{m}\sum_{\mu=1}^{m} \frac{1}{2\pi i}\int_C \frac{f_\mu(t)dt}{t-z}$$

uniformly for z on an arbitrary closed point set S interior to C.

The proof of this theorem is based on the fact that the nth polynomial under consideration may be written in the following manner:

$$L_{mn}(z) = \sum_{\mu=1}^{m}\sum_{k=1}^{n} \frac{f_\mu(z_{\mu,k}^{(n)})}{z - z_{\mu,k}^{(n)}} \frac{\Omega_n(z)}{\Omega_n'(z_{\mu,k}^{(n)})},$$

where

$$\Omega_n(z) = \prod_{\mu=1}^{m}\prod_{k=1}^{n}(z - z_{\mu,k}^{(n)}).$$

The rest of the proof follows the procedure used in that of Theorem II, with certain minor modifications.

The remainder of this section will be devoted to the discussion of two aspects of the problem of interpolation simultaneously to a finite number of functions defined respectively on the same number of Jordan curves. The first case is that in which the curves are all mutually exterior, and the second is that in which the curves lie one within another.

It is possible to generalize the theorem of Fejér mentioned in §2 to the case of a finite number of functions analytic on and within the same number of mutually exterior Jordan curves. The details have been carried through by Walsh,* who made use of the function $w = e^{G(x,y)+iH(x,y)}$, where $G(x, y)$ is the Green's function with pole at infinity for the region R exterior to the curves under consideration, and $H(x, y)$ is the harmonic conjugate of $G(x, y)$. This function maps R conformally, but not uniformly, onto the exterior of the unit circle in the w-plane so that the points at infinity in the two planes correspond.

But no similar extension of either Theorem I or Theorem II is possible with the use of this mapping function. First it should be noted that now cer-

* Unpublished.

tain of the points of the nth set G_n may coincide, because neither the function nor its inverse is single-valued if the region R is multiply connected. Thus we are no longer dealing with strictly the Lagrange type of polynomial, but rather with the Hermite type, and the existence of derivatives of the function to which we are interpolating must be postulated at the points of G_n. This fact alone precludes the possibility of generalizing Theorem II by the use of this mapping function. The Cauchy-Hermite formula used to prove Theorem I is applicable when some or all of the points of interpolation are coincident; nevertheless we shall be able to show by an example that Theorem I cannot be extended either.

The function

$$w = \frac{(z^2 - 1)^{1/2}}{\mu^{1/2}}, \qquad 0 < \mu < 1,$$

gives a map, of the type under consideration, of the region exterior to the lemniscate $|z^2-1| = \mu$ onto the exterior of the unit circle in the w-plane. This lemniscate consists of the two ovals of Cassini, and if we denote the two branches of the inverse function by

$$z = + (\mu w^2 + 1)^{1/2}$$

and

$$z = - (\mu w^2 + 1)^{1/2},$$

the right hand oval may be considered as the transform of the unit circle under the first branch, and the left hand oval, the transform under the second branch. We form the Hermite interpolation formula for the function $1/(z-1)$, using as the set G_n the following transforms of the roots of the equation $w^{2n} - 1 = 0$:

$$\left.\begin{aligned} z_{1,k}^{(n)} &= + (\mu e^{2\pi i k/n} + 1)^{1/2} \\ z_{2,k}^{(n)} &= - (\mu e^{2\pi i k/n} + 1)^{1/2} \end{aligned}\right\} \qquad (k = 1, 2, \cdots, 2n).$$

Then

$$L_{4n}(z) = (z + 1) \frac{(z^2 - 1)^{n-1}}{\mu^n} \left[2 - \left(\frac{z^2 - 1}{\mu} \right)^n \right],$$

as the reader may verify directly. When the point z lies interior to either oval, then $|z^2-1| < \mu$, so $L_n(z) \to 0$ for all points z within the ovals. But in the left oval O_1 we are seeking convergence to the value

$$\frac{1}{2\pi i} \int_{O_1} \frac{1}{t - 1} \frac{1}{t - z} dt = \frac{1}{z - 1},$$

the function $1/(z-1)$ being analytic on and within this oval. Thus Theorem I fails to generalize under this type of map.*

If the curves upon which the functions are defined lie one within another, we obtain a class of results of which the following theorems may be considered typical. For the sake of simplicity we shall state the theorems for the case of only two curves.

THEOREM V. *Let C_1 and C_2 be two Jordan curves subject to the restrictions upon C in Theorem* II, *C_2 lying interior to C_1. Let $\phi_1(w)$ denote the function which maps the exterior of the circle $|w|=1$ onto the exterior of C_1 so that the points at infinity in the z-plane and the w-plane correspond, and let $\phi_2(w)$ denote the analogous function for C_2. Let $F(n)$ be a monotonically increasing function of n such that $F(n)\to\infty$. Let $f(z)$ be a function bounded in modulus and integrable (R) on C_2. Let $\{\nu_1^{(m)}, \nu_2^{(m)}, \cdots, \nu_m^{(m}\}$, $m=[F(n)]$,† denote a sequence of sets of m numbers which is subject to the restriction that no number shall exceed a given fixed number in modulus. Form the Lagrange polynomial $L_{n+m}(z)$ of degree at most $n+m-1$ which takes on the values $\nu_h^{(m)}$ in the points $\phi_1(e^{2\pi ih/m})$, $h=1, 2, \cdots, m$, and which coincides with $f(z)$ in the points $\phi_2(e^{2\pi ik/n})$, $k=1, 2, \cdots, n$. Then*

$$L_{n+m}(z) \to \frac{1}{2\pi i}\int_{C_2}\frac{f(t)}{t-z}\,dt$$

uniformly for z on any closed point set interior to C_2.

For the proof of the theorem we employ a process similar to that used in the proof of Theorem II. The details are left to the reader.

The parallel theorem for two functions respectively analytic on and within the two curves C_1 and C_2 permits greater freedom in the choice of the curves and the function $F(n)$:

THEOREM VI. *Let C_1 and C_2 be two arbitrary Jordan curves, C_2 lying interior to C_1. Let $f_1(z)$ be a function analytic on and within C_1, and let $f_2(z)$ be a function analytic on and within C_2. Let $\phi_1(w)$ and $\phi_2(w)$ denote the mapping functions corresponding to the curves C_1 and C_2 respectively. Consider as points of interpolation to $F_2(z)$ the points $\phi_2(e^{2\pi ik/n})$ and as points of interpolation to $f_1(z)$ the points $\phi_1 e^{2\pi ih/n}$; where $m=[F(n)]$, $F(n)$ being either a positive constant or a posi-*

* This is the mapping function which has been used most frequently in the generalization to several regions of theorems concerning approximation in the complex domain. See for example Walsh and Russell, these Transactions, vol. 36 (1934), pp. 13–28. The present writer has investigated the use of other mapping functions in extensions of Theorems I and II in this direction, but so far with only negative results.

† The symbol $[x]$ means the greatest integer not greater than x.

tive monotonically increasing or decreasing function of n. Then the sequence $\{L_{n+m}(z)\}$ of corresponding Lagrange polynomials converges to $f_2(z)$ geometrically for z on and within C_2.

This theorem may be proved by writing down the appropriate extension of the Cauchy-Hermite formula and then applying (6) and Lemma III.

Divergence to infinity is possible in the annular region between C_1 and C_2 in both Theorems V and VI, as can be shown by example. The restriction to only two curves is not important, as any finite number of curves may be considered; the result will always be convergence to the value to be expected from interpolation only to the function defined on the innermost curve, for the sequence of Lagrange polynomials will ignore interpolating values assigned to outer curves. The study of combinations of the two theorems yields similar results.

It is worth pointing out that although m may remain constant with respect to n in Theorem VI, it is necessary in Theorem V that m approach infinity in some manner with n, as the following example indicates: Interpolate to the function $1/z$ in the points $e^{2\pi ik/n}$, $k=1, 2, \cdots, n$, and also in the points $Re^{2\pi ih/m}$, $h=1, 2, \cdots, m$. The corresponding Lagrange polynomial is

$$L_{n+m}(z) = \frac{1}{z}\left[1 - \left(\frac{z^n - 1}{-1}\right)\left(\frac{z^m - R^m}{-R^m}\right)\right];$$

and if m remains finite as n approaches infinity, it is apparent that the sequence $\{L_{n+m}(z)\}$ will not approach the value

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{1}{t}\,\frac{1}{t-z}\,dt = 0$$

for z interior to the circle $|z| = 1$.

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