QUASI-COMMUTATIVE RINGS AND DIFFERENTIAL IDEALS*

BY

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Introduction. In the quantum mechanics, an important role is played by elements p and q, either infinite matrices or differential operators, which satisfy a commutation rule of the form

$$(1) \qquad pq - qp = c,$$

where c is a scalar and is therefore commutative with both p and q. The importance of this relation has inspired the development of a considerable number of commutation formulas for polynomials in p and q, with coefficients in the complex number field.† However, for the most part, these formulas make no use of the fact that c is a scalar, but merely that it is commutative with both p and q. And although relation (1), with c a scalar, is impossible for elements of a finite algebra, it was pointed out in a recent paper‡ that there do exist pairs of finite matrices A, B such that $AB-BA$ is not zero, but is commutative with both A and B. Thus the various commutation formulas for polynomials in p and q go over at once into corresponding ones for polynomials in A and B. This suggests the problem of characterizing all algebras, and more generally all rings, whose elements are polynomials in two given elements ξ, η with coefficients in a suitable domain, it being assumed that $\xi\eta-\eta\xi$ is commutative with both ξ and η. Such rings are of some mathematical interest in that while they are not in general commutative, they are quite closely related to commutative rings and are perhaps in certain respects the most simple non-commutative rings. It is the primary purpose of the present paper to consider rings of this type.

Let K denote a commutative ring with unit element ϵ.§ We now adjoin to K (ring adjunction) two elements ξ, η which are assumed to be commutative with elements of K, and are such that the element $\zeta=\xi\eta-\eta\xi$ is commutative with both ξ and η. This ring will be denoted by $K[\xi, \eta]$, and may be called a *quasi-commutative ring* over K. Different quasi-commutative rings may be ob-

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† See, e.g., a previous paper, *On commutation formulas in the algebra of quantum mechanics*, these Transactions, vol. 31 (1929), pp. 793–806.

‡ N. H. McCoy, *On quasi-commutative matrices*, these Transactions, vol. 36 (1934), pp. 327–340.

§ Unless otherwise stated, the notation and terminology will follow as closely as possible that of van der Waerden, *Moderne Algebra*, Berlin, 1930 and 1931.

tained by imposing additional conditions on ξ and η. But it is not obvious what new conditions are self-consistent, as well as consistent with those already imposed. Our first problem is therefore the characterization of all quasi-commutative rings over K.

In §1, we shall define a quasi-commutative ring $R=K[\alpha, \beta]$, which has the property that any quasi-commutative ring $K[\xi, \eta]$ is homeomorphic to R. It follows that $K[\xi, \eta]$ is isomorphic to the quotient ring R/M, where M is the two-sided ideal in R consisting of those elements of R which correspond to the zero element of $K[\xi, \eta]$.* The problem of characterizing the different rings $K[\xi, \eta]$ is thus reduced to that of characterizing in some simple way the two-sided ideals in R. In order to do this, we introduce in §2 a commutative polynomial ring $R'=K[x, y, z]$ whose elements can be put in a one-to-one correspondence with the elements of R. The significance of the correspondence between these rings is found to depend upon the notions of *differential ring* and *differential ideal.*† Accordingly, we discuss these concepts in some detail in §3, which is independent of the rest of the paper. In particular, we show that one of E. Noether's decomposition theorems remains valid if all ideals are required to be differential ideals.

The characterization of the two-sided ideals in R is obtained in §4. It is found that a set M of elements of R is a two-sided ideal in R, if and only if the corresponding set M' of elements of R' is a differential ideal of a certain kind in R'. If follows that there is a very close connection between the quotient rings R/M and R'/M'. Thus a number of properties of the quasi-commutative ring R/M can be determined from a knowledge of the corresponding properties of the commutative ring R'/M'.

In §5, we discuss briefly the special case in which K is a non-modular field and $K[\xi, \eta]$ is a finite algebra over K.

It may be remarked that the relations

$$\xi\eta - \eta\xi = \zeta, \qquad \xi\zeta - \zeta\xi = 0, \qquad \eta\zeta - \zeta\eta = 0$$

are precisely those which are satisfied by the infinitesimal transformations of a three-parameter continuous group with structure constants 0, 0, 1; 0, 0, 0; 0, 0, 0. The problem discussed in this paper may therefore be considered as that of determining the realizations of such three-parameter groups.

1. **The ring $R=K[\alpha, \beta]$.** Let K be a commutative ring with unit element ϵ, and $K[\xi, \eta]$ any quasi-commutative ring over K. If we set $\zeta=\xi\eta-\eta\xi$, it

* See van der Waerden, op. cit., I, pp. 56–58, for a detailed proof for the case of commutative rings. The necessary modification for the non-commutative case can be made without difficulty. The term "quotient ring" is used throughout this paper for van der Waerden's "Restklassenring."

† See H. W. Raudenbush, Jr., *Differential fields and ideals of differential forms*, Annals of Mathematics, vol. 34 (1933), pp. 509–517.

follows as a direct consequence of the fact that ζ is commutative with both ξ and η that

$$\eta^m\xi^n = \sum_{s=0} (-1)^s s! \binom{n}{s}\binom{m}{s} \xi^{n-s}\eta^{m-s}\zeta^s, \tag{2}$$

where m and n are any positive integers, the sum being extended to the smaller of n and m. We shall not give a proof of this formula as it follows readily by induction on m and n.* We remark that if h is any element of $K[\xi, \eta]$, it will be understood that $h^0 = \epsilon$.

Each coefficient on the right of (2) is a positive or negative integer. Hence if we multiply (2) by ϵ, we see that each of the resulting coefficients belongs to K. By a repeated use of formula (2), it is now clear that each element h of $K[\xi, \eta]$ can be expressed in the form

$$h = \sum c_{ijk}\xi^i\eta^j\zeta^k \qquad (i, j, k = 0, 1, \cdots), \tag{3}$$

where the coefficients c_{ijk} belong to K, and only a finite number are different from zero. In general, the expression of h in this form need not be unique.

We now pass to a consideration of the most general quasi-commutative ring over K, which may be defined in the following abstract way.† Let e_{ijk} $(i, j, k = 0, 1, \cdots)$ be undefined symbols, and denote by R the set of all finite sums of the form

$$f = \sum a_{ijk}e_{ijk},$$

where the a_{ijk} belong to K. If $g = \sum b_{ijk}e_{ijk}$, we shall write $f = g$ if and only if $a_{ijk} = b_{ijk}$ $(i, j, k = 0, 1, \cdots)$. We now define:

$$f + g = \sum (a_{ijk} + b_{ijk})e_{ijk},$$
$$af = fa = \sum (aa_{ijk})e_{ijk}, \qquad a \text{ in } K.$$

It follows from the latter of these relations that $\epsilon f = f\epsilon = f$, for all elements f of R. We now define a multiplication of the symbols e_{ijk} as follows:

$$e_{ijk}e_{lmn} = \sum_{t=0} \epsilon(-1)^t t! \binom{j}{t}\binom{l}{t} e_{i+l-t,\, j+m-t,\, k+n+t}. \tag{4}$$

This defines a multiplication of elements of R which, by a direct calculation, can be shown to be associative. Hence R is a ring with unit element $\epsilon = e_{000}$. If we set $e_{100} = \alpha$, $e_{010} = \beta$, $e_{001} = \gamma$, it follows from (4) that $\alpha\beta - \beta\alpha = \gamma$, $\alpha\gamma = \gamma\alpha$, $\beta\gamma = \gamma\beta$, $e_{ijk} = \alpha^i\beta^j\gamma^k$. Thus R is a quasi-commutative ring $K[\alpha, \beta]$ over K,

* Born and Jordan, Zeitschrift für Physik, vol. 34 (1925), p. 873.

† This will be recognized as essentially the method used by Hamilton to define an algebra over a given field. See L. E. Dickson, *Algebras and their Arithmetics*, Chicago, 1923, p. 22.

and hence relation (2) holds with ξ, η, ζ replaced by α, β, γ respectively. It is then easy to show also that relation (4) is a direct consequence of formula (2). Thus by repeated use of (4) or (2) any element f of R can be expressed *uniquely* as a finite sum

$$f = \sum a_{ijk}\alpha^i\beta^j\gamma^k, \tag{5}$$

with coefficients in K, and any such sum is an element of R.

If now $K[\xi, \eta]$ is any given quasi-commutative ring over K, we shall denote by f^* the element $\sum a_{ijk}\xi^i\eta^j\gamma^k$. Thus f^* is a uniquely defined element of $K[\xi, \eta]$ corresponding to the element f of R given by (5). It is clear that $(f\pm g)^* = f^* \pm g^*$. Let us consider $(fg)^*$. By repeated use of relation (2) (with ξ, η replaced by α, β respectively) fg may be expressed as an element $\sum c_{ijk}\alpha^i\beta^j\gamma^k$ of R, and thus $(fg)^* = \sum c_{ijk}\xi^i\eta^j\zeta^k$. But formula (2) as applied to f^*g^* in precisely the same series of operations will also yield $\sum c_{ijk}\xi^i\eta^j\zeta^k$. Hence $(fg)^* = f^*g^*$, and thus the correspondence $f\rightarrow f^*$ is a homeomorphism between R and $K[\xi, \eta]$. We have therefore shown that any quasi-commutative ring over K is homeomorphic to R, and the problem of determining the various quasi-commutative rings over K is reduced to that of finding the rings which are homeomorphic to R, and this in turn is equivalent to the determination of all two-sided ideals in R.

If f is the element (5) of R, we may define $\partial f/\partial \alpha$ to be the uniquely defined element $\sum \epsilon i a_{ijk}\alpha^{i-1}\beta^j\gamma^k$ of R. It then follows by a simple application of relation (2) that

$$\begin{aligned} f\beta - \beta f &= \gamma \frac{\partial f}{\partial \alpha}, \\ \alpha f - f\alpha &= \gamma \frac{\partial f}{\partial \beta}\cdot \end{aligned} \tag{6}$$

These are familiar formulas in the quantum mechanics.

Since the product of n consecutive integers is divisible by $n!$, we remark that no matter what the characteristic of the ring K, $(1/n!)\ \partial^n f/\partial\alpha^n$ is an element of R $(n = 1, 2, \cdots)$.

2. **The ring** $R' = K[x, y, z]$. Let x, y, z denote ordinary commutative indeterminates which are assumed to be also commutative with elements of K, and denote by R' the ring $K[x, y, z]$ consisting of all polynomials in x, y, z with coefficients in K. Corresponding to the element (5) of R, we have the element $f' = \sum a_{ijk}x^iy^jz^k$ of R'. This clearly defines a one-to-one correspondence between elements of R and those of R'. Henceforth we shall let f, f'; g, g'; $\cdots$ denote pairs of corresponding elements of R and R' respectively. The following may now be verified:

$$(f \pm g)' = f' \pm g',$$
$$(af)' = af' \qquad (a \text{ in } K),$$
$$\left(\frac{\partial^n f}{\partial \alpha^n}\right)' = \frac{\partial^n f'}{\partial x^n}, \qquad \left(\frac{\partial^n f}{\partial \beta^n}\right)' = \frac{\partial^n f'}{\partial y^n}.$$

In order to find what element of R' corresponds to a product fg of elements of R, it is necessary to express fg in the form $\sum c_{ijk}\alpha^i\beta^j\gamma^k$. We shall now prove that†

$$(7) \qquad (fg)' = \sum_{s=0} \frac{(-z)^s}{s!} \frac{\partial^s f'}{\partial y^s} \frac{\partial^s g'}{\partial x^s}.$$

Since differentiation is a linear operation, it is sufficient to establish this formula for the case in which f and g are single terms of the form $F = \alpha^i\beta^j\gamma^k$ and $G = \alpha^l\beta^m\gamma^n$, respectively. We have from formula (2)

$$FG = \alpha^i(\beta^j\alpha^l)\beta^m\gamma^{k+n} = \sum_{s=0} (-1)^s s! \binom{j}{s}\binom{l}{s} \alpha^{i+l-s}\beta^{j+m-s}\gamma^{k+n+s},$$

and hence

$$(FG)' = \sum_{s=0} \frac{(-z)^s}{s!} \frac{\partial^s F'}{\partial y^s} \frac{\partial^s G'}{\partial x^s},$$

which is the desired result. It may be noted that formula (7) is essentially a formula given by Bourlet for multiplying differential operators.‡

We shall also require a formula which exhibits the element of R which corresponds to a product $f'g'$ in R', namely

$$(8) \qquad f'g' = \left(\sum_{s=0} \frac{\gamma^s}{s!} \frac{\partial^s f}{\partial \beta^s} \frac{\partial^s g}{\partial \alpha^s}\right)'.$$

Again considering the case $f = F$, $g = G$, this formula states that

$$x^{i+l}y^{j+m}z^{k+n}$$
$$= \sum_{s=0} \binom{j}{s} \frac{l!}{(l-s)!} \sum_{t=0} (-1)^t t! \binom{j-s}{t}\binom{l-s}{t} x^{i+l-s-t}y^{j+m-s-t}z^{k+n+s+t}.$$

This is easily verified, as the term on the right given by $s = t = 0$ is precisely the left-hand side, while if $p > 0$, the coefficient of $x^{i+l-p}y^{j+m-p}z^{k+n+p}$ on the right is

† Here, as elsewhere, the existence of $s!$ in the denominator causes no difficulty, as $(1/s!)\,\partial^s f'/\partial y^s$ represents a uniquely defined element of R'.

‡ C. Bourlet, Annales de l'Ecole Normale Supérieure, (3), vol. 14 (1897).

$$\frac{j!l!}{(j-p)!(l-p)!p!}\sum_{t=0}^{p}(-1)^t\binom{p}{t}=0.$$

By means of these formulas we shall, in §4, find a characterization of all two-sided ideals in R in terms of the sets of corresponding elements of R'. But before proceeding to this, we pause to introduce some necessary concepts of a somewhat different nature. This will be done in the following section, which is independent of the rest of the paper.

3. **Rings with operators. Differential rings and ideals.** Let S be a commutative ring,* and Ω a set of operators Δ with the following properties: (1) if s is an element of S, Δs is a uniquely defined element of S; (2) if s and t are elements of S, then $\Delta(s+t)=\Delta s+\Delta t$. For convenience, we shall refer to S as an *Ω-ring*. A set I of elements of S will be said to be an *Ω-ideal* if I is an ideal in S, and in addition I is closed under the operators Δ of Ω.†

We remark that from the equation $\Delta(0+0)=\Delta 0+\Delta 0$, it follows that $\Delta 0=0$. Let now I be an Ω-ideal in S, and denote by s and $\bar{s}$ corresponding elements of S and of the quotient ring S/I respectively. We now define $\Delta\bar{s}$ to be the element $\overline{\Delta s}$ of S/I, and it follows readily that S/I is also an Ω-ring.

Let T denote another Ω-ring. The ring T will be said to be *Ω-homeomorphic* (*Ω-isomorphic*) to S, if T is homeomorphic (isomorphic) to S in the usual sense, and in addition if $s\rightarrow t$ by this homeomorphism, then $\Delta s\rightarrow\Delta t$ for each Δ in Ω. The quotient ring S/I is clearly Ω-homeomorphic to S. It is now not difficult to prove the following theorem:

THEOREM 1. *If the Ω-ring T is Ω-homeomorphic to the Ω-ring S, then T is Ω-isomorphic to the quotient ring S/I, where I is the Ω-ideal in S consisting of those elements of S which correspond to the zero element of T.*

This will be recognized as a familiar result provided the symbol Ω be omitted from the statement of the theorem.‡ We note first that if $s\rightarrow 0$ by the given Ω-homeomorphism, then $\Delta s\rightarrow\Delta 0=0$. Hence the set I of elements of S which corresponds to the zero element of T is actually an Ω-ideal. Let now s be any element of S, $\bar{s}$ the corresponding element of S/I, and t the element of T corresponding to the element s of S. Then, by the known case, it is clear that the correspondence $\bar{s}\rightarrow t$ is an isomorphism of S/I and T and we only need to show that this is also an Ω-isomorphism. But by our hypothesis, Δt

* Some of the results of this section can be extended to the case of non-commutative rings. However, we shall simplify the discussion by considering only commutative rings, as these are the ones which are important for our purpose.

† These concepts are essentially those used in the study of groups with operators. See van der Waerden, op. cit., I, p. 132.

‡ van der Waerden, op. cit., I, p. 57.

is the element of T corresponding to Δs, and by definition $\Delta\bar{s}$ is the element of S/I corresponding to Δs of S. Hence $\Delta\bar{s}\rightarrow\Delta t$, and the theorem is established.

We shall henceforth assume that the ring S has a unit element ϵ, and that every ideal in S has a finite ideal basis. We shall also assume that each operator Δ of Ω satisfies the further condition (3): if s_1, s_2 are elements of S, then $\Delta(s_1s_2)=s_1\Delta s_2+s_2\Delta s_1$. The ring S may then be called a *differential ring*, and an Ω-ideal in S a *differential ideal.**

In a differential ring we have $\Delta\epsilon=0$. For applying the condition (3) to the case in which $s_1=s_2=\epsilon$, we get $\Delta\epsilon=2\Delta\epsilon$, that is, $\Delta\epsilon=0$. We shall now prove a few theorems concerning differential ideals in S.

THEOREM 2. *An ideal $I=(a_1, a_2, \cdots, a_k)$ in S is a differential ideal, if and only if* $\Delta a_i\equiv 0\ \ (I)\ \ (i=1, 2, \cdots, k)$.

It is clearly only necessary to establish the sufficiency of these conditions. If a is any element of I, then we may write

$$a = \sum_{i=1}^{k} b_i a_i,$$

where the b_i are elements of S. Thus

$$\Delta a = \sum_{i=1}^{k} (b_i\Delta a_i + a_i\Delta b_i).$$

Hence if all $\Delta a_i\equiv 0\ \ (I)$, it follows that $\Delta a\equiv 0\ \ (I)$, and I is a differential ideal.

If I_1 and I_2 are differential ideals in S, their least common multiple $I_1\cap I_2=[I_1, I_2]$ is obviously a differential ideal in S. By Theorem 2, it is clear that their greatest common divisor (I_1, I_2) is also a differential ideal in S.

THEOREM 3. *If I is a differential ideal in S and $I=I_1\cap I_2$, where I_1 and I_2 are proper ideal divisors of I such that $(I_1, I_2)=(\epsilon)$, then I_1 and I_2 are differential ideals in S.*

Under the hypotheses of the theorem, there exist elements i_1, i_2 of S such that

$$i_1 + i_2 = \epsilon, \qquad i_1 \equiv 0 \quad (I_1), \qquad i_2 \equiv 0 \quad (I_2).$$

Let a be any element of I_1. Then $ai_2\equiv 0$ (I), and hence $\Delta(ai_2)=a\Delta i_2+i_2\Delta a\equiv 0$ (I), that is, $i_2\Delta a\equiv 0\ \ (I_1)$. But $i_2\equiv\epsilon\ \ (I_1)$, and thus $\Delta a\equiv 0\ \ (I_1)$. In like manner it can be shown that I_2 is also closed under the operations of Ω, which proves the theorem.

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* See H. W. Raudenbush, Jr., loc. cit.

An ideal (differential ideal) I may be said to be *direct indecomposable** if it cannot be expressed in the form $I_1 \cap I_2$, where I_1 and I_2 are proper ideal (differential ideal) divisors of I such that $(I_1, I_2) = (\epsilon)$. The above theorem then states that a differential ideal I is direct indecomposable if and only if it is direct indecomposable when considered as an ordinary ideal.

THEOREM 4. *Each differential ideal I in S can be expressed uniquely as the least common multiple or product of direct indecomposable differential ideals I_i:*

$$I = [I_1, I_2, \cdots, I_k] = I_1 I_2 \cdots I_k, \tag{9}$$

where the I_i are proper ideal divisors of I such that $(I_i, I_j) = (\epsilon)$, $i \neq j$.

Considered as an ordinary ideal, it is known† that I has a unique decomposition of the form stated, except that the I_i are of course not required to be differential ideals. We shall show that they are necessarily differential ideals.

We have from (9), $I = I_1 \cap [I_2, I_3, \cdots, I_k]$. Since $(I_1, I_j) = (\epsilon)$ $(j = 2, 3, \cdots, k)$, it follows‡ that $(I_1, [I_2, I_3, \cdots, I_k]) = (\epsilon)$, and hence, by Theorem 3, I_1 and $[I_2, I_3, \cdots, I_k]$ are differential ideals. A repetition of this argument proves the theorem.

If S_i $(i = 1, 2, \cdots, r)$ are differential ideals in S such that each element of S can be expressed uniquely as the sum of elements which belong respectively to the S_i, then S is said to be the *direct sum* of the differential ideals S_i, and we write

$$S = S_1 \dotplus S_2 \dotplus \cdots \dotplus S_r. \tag{10}$$

A differential ring which can be expressed as the direct sum of two or more differential ideals may be said to be *reducible*, otherwise *irreducible*.

Suppose now that relation (10) is given. From the uniqueness of the expression of any element of S as a sum of elements of the S_i, it follows that S_i and S_j have no element in common except zero, and thus S_iS_j 0, $i \neq j$. Considering now the unit element ϵ of S, we have the following relations:

$$\epsilon = \sum_{i=1}^{r} \epsilon_i, \qquad \epsilon_i \equiv 0 \ (S_i), \qquad \epsilon_i \epsilon_j = 0 \ (i \neq j), \qquad \epsilon_i^2 = \epsilon_i \neq 0. \tag{11}$$

It follows readily that S_i consists of all elements of the form $s\epsilon_i$, where s is an element of S, and thus ϵ_i is the unit element of S_i.

* This is in agreement with the terminology used by O. Ore in his paper, *Abstract ideal theory*, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 728–745.

† E. Noether, *Idealtheorie in Ringbereichen*, Mathematische Annalen, vol. 83 (1921), pp. 24–66; van der Waerden, op. cit., II, p. 46.

‡ van der Waerden, op. cit., II, p. 45.

Let us now assume that relations (11) are given, and deduce from them the decomposition (10). Let S_i be the set of all elements of S of the form $s\epsilon_i$, where s is any element of S. Then clearly S_i is an ideal in S and ϵ_i is the unit element of S_i. If s is any element of S, we have

$$s = s\epsilon = s\epsilon_1 + s\epsilon_2 + \cdots + s\epsilon_r.$$

Thus any element of S can be expressed as the sum of elements belonging respectively to S_i $(i=1, 2, \cdots, r)$. Furthermore, this expression is unique, for if $\sum a_i\epsilon_i=0$, it follows by multiplication with ϵ_j that $a_j\epsilon_j=0$ $(j=1, 2, \cdots, r)$.

We shall now show that S_i is a differential ideal. Since $\Delta\epsilon_i = \Delta\epsilon_i^2 = 2\epsilon_i\Delta\epsilon_i$, it follows that $\Delta\epsilon_i \equiv 0$ (S_i). Also from (11) we find

$$\Delta\epsilon = 0 = \Delta\epsilon_1 + \Delta\epsilon_2 + \cdots + \Delta\epsilon_r,$$

and thus $\Delta\epsilon_i=0$ $(i=1, 2, \cdots, r)$. If s_i is any element of S_i, we have therefore $\Delta s_i = \Delta(s_i\epsilon_i) = \epsilon_i\Delta s_i \equiv 0$ (S_i). Hence S_i is a differential ideal and S is the direct sum of the S_i $(i=1, 2, \cdots, r)$.

We have therefore shown by a familiar kind of calculation, that a decomposition (10) has relations (11) as a consequence, and conversely. In view of Theorem 3, it is not surprising to find that if a differential ring can be expressed as the direct sum of ordinary ideals, these ideals are of necessity differential ideals. We may remark here that if s is any element of S, the correspondence $s \rightarrow s\epsilon_i$ is an Ω-homeomorphism between S and S_i.

We conclude this section with the following theorem:

THEOREM 5. *If I is a differential ideal in S, the quotient ring S/I can be expressed as the direct sum of k differential ideals K_i, if and only if I can be expressed in the form* (9). *By a proper choice of notation we have also that K_i is Ω-isomorphic to S/I_i $(i=1, 2, \cdots, k)$.*

The theorem that can be obtained from this one by omitting the word "differential" and the symbol "Ω" is known to be true.* We shall not give a detailed proof of this extended theorem, as it follows readily from the known case by means of Theorem 3, and the fact that if S/I is reducible, the components are necessarily differential ideals.

4. **Two-sided ideals in R, and quasi-commutative rings.** We now return to a further consideration of the rings $R=K[\alpha, \beta]$ and $R'=K[x, y, z]$ introduced earlier. If M is any set of elements of R, we shall let M' denote the set of corresponding elements of R', and conversely.

We associate with the commutative ring R' the operator domain Ω con-

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* van der Waerden, op. cit., II, p. 47. See also E. Noether and W. Schmeidler, *Moduln in nichtkommutativen Bereichen*, Mathematische Zeitschrift, vol. 8 (1920), p. 11.

sisting of the two operators $z(\partial/\partial x)$ and $z(\partial/\partial y)$. Since R' consists of all polynomials in x, y, z with coefficients in K, it is clear that R' is closed under these operations. Also these operators satisfy all requirements prescribed in the preceding section, and thus R' is a differential ring with respect to these operators. Throughout the remainder of this paper, it will be understood that the terms "differential ring" and "differential ideal" refer to the particular operator domain

$$\Omega = \left(z\frac{\partial}{\partial x}, z\frac{\partial}{\partial y}\right).$$

The following theorem now gives a characterization of the two-sided ideals in R.

THEOREM 6. *A set M of elements of R is a two-sided ideal in R, if and only if the set M' of corresponding elements of R' is a differential ideal in R'.*

First let us assume that M is a two-sided ideal in R, and show that M' is a differential ideal in R'. Let f', g' be any elements of M', h' any element of R'. Then f, g are elements of M, h an element of R. Hence

$$f - g \equiv 0, \qquad hf \equiv 0, \qquad fh \equiv 0 \quad (M).$$

Also, by relations (6), it follows that

$$\gamma^i \frac{\partial^i f}{\partial \alpha^i} \equiv 0, \qquad \gamma^i \frac{\partial^i f}{\partial \beta^i} \equiv 0 \quad (M) \qquad (i = 1, 2, \cdots).$$

We therefore have at once

$$f' - g' \equiv 0, \qquad z\frac{\partial f'}{\partial x} \equiv 0, \qquad z\frac{\partial f'}{\partial y} \equiv 0 \quad (M')$$

and we have only to show that $f'h' \equiv 0 \quad (M')$. From relations (8) we find that

$$f'h' = \left(\sum_{s=0} \frac{\gamma^s}{s!} \frac{\partial^s f}{\partial \beta^s} \frac{\partial^s h}{\partial \alpha^s}\right)'.$$

But

$$\gamma^s \frac{\partial^s f}{\partial \beta^s} \equiv 0 \quad (M) \qquad (s = 0, 1, \cdots).$$

Hence the expression in parentheses belongs to M, and therefore $f'h' \equiv 0 \quad (M')$.

Now let M' be a given differential ideal in R', and M the set of corresponding elements of R. From the preceding case it is clearly sufficient to show that if f is any element of M, h any element of R, then $fh \equiv 0 \quad (M)$, $hf \equiv 0 \quad (M)$. From equation (7), we have

$$(fh)' = \sum_{s=0} \frac{(-z)^s}{s!} \frac{\partial^s f'}{\partial y^s} \frac{\partial^s g'}{\partial x^s}.$$

But since M' is a differential ideal,

$$z^s \frac{\partial^s f'}{\partial y^s} \equiv 0 \quad (M').$$

Hence $(fh)' \equiv 0 \quad (M')$, and therefore $fh \equiv 0 \quad (M)$. It follows similarly that also $hf \equiv 0 \quad (M)$, and M is a two-sided ideal in R. This completes the proof of the theorem.

Let M' be a differential ideal in R' with the ideal basis $f_1', f_2', \cdots, f_r'$. Denote by N the two-sided ideal in R with the *basis* $f_1, f_2, \cdots, f_r$; that is, N consists of all finite sums of terms of the form hf_ig, where h and g are elements of R. We shall write $M' = (f_1', f_2', \cdots, f_r')$, $N = (f_1, f_2, \cdots, f_r)$. We shall now show that $N = M$.

Let f be any element of M. Then $f' \equiv 0 \quad (M')$ and we may write $f' = \sum_{i=1}^{r} f_i' h_i'$. By relation (8) it follows that

$$f = \sum_{i=1}^{r} \left(\sum_{s=0} \frac{\gamma^s}{s!} \frac{\partial^s f_i}{\partial \beta^s} \frac{\partial^s h_i}{\partial \alpha^s} \right).$$

But from (6), it is clear that

$$\gamma^s \frac{\partial^s f_i}{\partial \beta^s} \equiv 0 \quad (N),$$

hence $f \equiv 0 \quad (N)$. Thus all elements of M are also elements of N. Since the converse is obviously true, it follows that $N = M$. This result may be stated as follows:

THEOREM 7. *If $M' = (f_1', f_2', \cdots, f_r')$ is a differential ideal in R', then the corresponding two-sided ideal in R is $M = (f_1, f_2, \cdots, f_r)$.*

We now pass to an extension of Theorem 6. Let M and M' denote respectively a two-sided ideal in R, and the corresponding differential ideal in R'. Let f be any element of R, and suppose $f \to \bar{f}$ by the homeomorphism $R \sim R/M$, and $f' \to \bar{f}'$ by the homeomorphism $R' \sim R'/M'$. The correspondence $\bar{f} \to \bar{f}'$ is then a one-to-one correspondence between elements of R/M and those of R'/M'. For if $f \equiv g \quad (M)$, it follows that $f' \equiv g' \quad (M')$, and conversely. It is also clear that if $\bar{f} \to \bar{f}'$, $\bar{g} \to \bar{g}'$, then $\bar{f} \pm \bar{g} \to \bar{f}' \pm \bar{g}'$.

If $\overline{N}$ is a set of elements of R/M, we shall let $\overline{N}'$ denote the set of corresponding elements of R'/M', and conversely. We may then extend Theorem 6 as follows:

THEOREM 8. *A set $\bar{N}$ of elements of R/M is a two-sided ideal in R/M, if and only if $\bar{N}'$ is a differential ideal in R'/M'.*

First let $\bar{N}$ be a two-sided ideal in R/M, and N the set of all elements of R which correspond to elements of $\bar{N}$ by the homeomorphism $R \sim R/M$. Then N is a two-sided ideal in R, and by Theorem 6, N' is a differential ideal in R'. Now $\bar{N}'$ is the set of all elements of R'/M' to which elements of N' correspond by the homeomorphism $R' \sim R'/M'$. We shall show that $\bar{N}'$ is a differential ideal in R'/M'. Let $\bar{n}_1'$, $\bar{n}_2'$ be any elements of $\bar{N}'$, $\bar{f}'$ any element of R'/M'. Thus there exist elements n_1', n_2' of N' and an element f' of R' such that $n_1' \to \bar{n}_1'$, $n_2' \to \bar{n}_2'$, $f' \to \bar{f}'$ by the homeomorphism $R' \sim R'/M'$. Then clearly $n_1' - n_2' \to \bar{n}_1' - \bar{n}_2'$, $f'n' \to \bar{f}'\bar{n}'$, $\Delta n_1' \to \overline{\Delta n_1'} = \Delta \bar{n}_1'$, where Δ is either of our differential operators. Now since N' is a differential ideal, $n_1' - n_2'$, $f'n_1'$, $\Delta n_1'$ are elements of N'. Thus

$$\bar{n}' - \bar{n}' \equiv 0, \qquad \bar{f}'\bar{n}' \equiv 0, \qquad \Delta \bar{n}_1' \equiv 0 \quad (\bar{N}').$$

That is, $\bar{N}'$ is a differential ideal in R'/M'.

Now let $\bar{N}'$ be a given differential ideal in R'/M', and N' the set of all elements of R' which correspond to elements of $\bar{N}'$ by the homeomorphism $R' \sim R'/M'$. It follows readily that N' is a differential ideal in R', and N is a two-sided ideal in R. Hence $\bar{N}$ is a two-sided ideal in R/M, and the theorem is established.

If T is any ring, and each element of T can be expressed uniquely as a sum of elements which belong respectively to the two-sided ideals T_i $(i = 1, 2, \cdots, k)$ in T, then T is said to be the *direct sum* of the ideals T_i, and we write

$$T = T_1 \dotplus T_2 \dotplus \cdots \dotplus T_k.$$

If T can be expressed as the direct sum of two or more two-sided ideals, we shall say that T is *reducible*, otherwise *irreducible*. These terms have been defined in §3 for commutative differential rings. From the preceding theorem we can therefore establish at once the following theorem.

THEOREM 9. *Let M and M' denote respectively a two-sided ideal in R, and the corresponding differential ideal in R'. Then*

$$R/M = \bar{R}_1 \dotplus \bar{R}_2 \dotplus \cdots \dotplus \bar{R}_k, \tag{12}$$

if and only if

$$R'/M' = \bar{R}_1' \dotplus \bar{R}_2' \dotplus \cdots \dotplus \bar{R}_k'. \tag{13}$$

This theorem shows that not only can the two-sided ideals in R be determined by working in the commutative ring R', but that reducibility of R/M corresponds also to reducibility of R'/M'.

We now assume, for the remainder of this section, that in K each ideal has a finite ideal basis, and the same is therefore true for the ring $R'=K[x, y, z]$.* Hence if M' is any differential ideal in R', there exists, by Theorem 4, a unique decomposition,

$$M' = [M_1', M_2', \cdots, M_k'], \tag{14}$$

where the M_i' are direct indecomposable differential ideals such that $(M_i', M_j')=(\epsilon)$ $(i\neq j)$. Now let M_i be the two-sided ideal in R corresponding to M_i'. Then we have $(M_i, M_j)'=(M_i', M_j')$, and $(M_i \cap M_j)'=M_i' \cap M_j'$.

In accordance with the definition given in §3, we may say that a two-sided ideal N in R is *direct indecomposable* if it cannot be expressed in the form $N_1 \cap N_2$, where N_1 and N_2 are proper two-sided ideal divisors of N, such that $(N_1, N_2)=(\epsilon)$. It follows at once that the M_i are direct indecomposable. The following theorem is then an immediate consequence of Theorem 4.

THEOREM 10.† *Corresponding to the decomposition* (14) *of M' there exists a unique decomposition of M of the form*

$$M = [M_1, M_2, \cdots, M_k], \tag{15}$$

where the M_i are direct indecomposable two-sided ideals such that $(M_i, M_j)=(\epsilon)$ $(i\neq j)$.

If M has the decomposition (15), then M' has the decomposition (14), and by Theorem 5, we have the unique decomposition (13) of R'/M', and the $\overline{R}_i'$ are irreducible. It follows from Theorem 8 that R/M has the unique decomposition (12), where the $\overline{R}_i$ are irreducible. We thus have

THEOREM 11. *The quotient ring R/M can be expressed uniquely as the direct sum of k two-sided ideals $\overline{R}_i$, if and only if M has a decomposition of the form* (15). *Furthermore, by a proper choice of notation, $\overline{R}_i \cong R/M_i$.*‡

It follows at once that each quotient ring R/M can be expressed uniquely as a direct sum of irreducible two-sided ideals.

COROLLARY. *The quotient ring R/M is irreducible if and only if M (or M') is direct indecomposable.*

We now return to a consideration of quasi-commutative rings over K. In §1, it was shown that any quasi-commutative ring over K is isomorphic to

* van der Waerden, op. cit., II, p. 23.

† Cf. W. Krull, *Zweiseitige Ideale in nichtkommutativen Bereichen*, Mathematische Zeitschrift, vol. 28 (1928), p. 499.

‡ See Noether and Schmeidler, loc. cit., p. 14.

a quotient ring R/M, where M is a two-sided ideal in R, and we have now given a characterization of the two-sided ideals in R. However, if M is a given two-sided ideal, the quotient ring R/M may clearly not be quasi-commutative over K, but over some ring homeomorphic to K.

Let L denote the ideal in K consisting of the elements of M which are also elements of K, and denote by $\overline{K}$ the ring K/L. Let $\bar{\alpha}$, $\bar{\beta}$ be the elements of R/M to which α, β respectively correspond. Then R/M is a quasi-commutative ring $\overline{K}[\bar{\alpha}, \bar{\beta}]$ over $\overline{K}$. Now $\overline{K}$ will be isomorphic to K, if and only if $L=(0)$, and if this is true, it follows that $\overline{K}[\bar{\alpha}, \bar{\beta}]$ is isomorphic to $K[\bar{\alpha}, \bar{\beta}]$.* Thus R/M is a quasi-commutative ring over K, if and only if M contains no elements of K, except the zero. By the preceding section, any such two-sided ideal M in R corresponds to a differential ideal M' in R', which contains no element of K besides the zero.

5. **Finite algebras homeomorphic to** R. We conclude with a few remarks about quasi-commutative rings over a field, which are also finite algebras over that field.

Let K now be a non-modular field with unit element 1, and A a finite algebra homeomorphic to R, and therefore isomorphic to R/M, where M is a two-sided ideal (invariant sub-algebra) in R. If M contains any non-zero element of K, then clearly $M=R$, and the homeomorphism is a trivial one. Hence we may assume that M contains no non-zero element of K, and by the homeomorphism, K corresponds to a field $\overline{K}$, simply isomorphic to K. We shall consider these fields to be identical, as we may without essential loss of generality.

By the homeomorphism $R\sim A$, suppose $\alpha\rightarrow p$, $\beta\rightarrow q$, $\gamma\rightarrow r$. Then $\sum a_{ijk}\alpha^i\beta^j\gamma^k\rightarrow\sum a_{ijk}p^iq^jr^k$. Since now A is a finite algebra over K, each element of A satisfies a unique minimum equation with coefficients in K, and leading coefficient unity.† Let $f(\lambda)=0$, $g(\lambda)=0$, $h(\lambda)=0$ be the minimum equations of p, q, and r respectively. We may now prove the following theorem:

THEOREM 12. *If $f^{(i)}(\lambda)$ denotes the ith derivative of $f(\lambda)$ with respect to λ, then we have*

$$r^i f^{(i)}(p) = 0,$$
$$r^i g^{(i)}(q) = 0 \qquad (i = 0, 1, \cdots).$$

Now M consists of those elements of R which correspond to the zero ele-

* This follows readily since the constants on the right of the multiplication formula (2) are independent of the ring K.

† See L. E. Dickson, *Algebras and their Arithmetics*, Chicago, 1923, p. 111.

ment of A, hence M contains $f(\alpha)$, $g(\beta)$, $h(\gamma)$. Thus M' contains $f(x)$, $g(y)$, $h(z)$, and since it is a differential ideal, it contains $z^i f^{(i)}(x)$, $z^i g^{(i)}(y)$. That is, M contains $\gamma^i f^{(i)}(\alpha)$, $\gamma^i g^{(i)}(\beta)$, and we have the desired result.

It follows at once from this theorem that r is nilpotent, and hence $h(\lambda)=\lambda^k$, where the index k does not exceed the degree of $f(\lambda)$ or of $g(\lambda)$. We may, however, get more information about k in the following way. Let

$$f(\lambda) = [f_1(\lambda)]^{m_1}[f_2(\lambda)]^{m_2} \cdots [f_s(\lambda)]^{m_s}$$

be the decomposition of $f(\lambda)$ into powers of distinct polynomials which are irreducible in K, and denote by m the maximum of the m_i. Then $f(\lambda)$ and $f^{(m)}(\lambda)$ have no factor in common, and their resultant D is not zero. We thus have a relation*

$$a(\lambda)f(\lambda) + b(\lambda)f^{(m)}(\lambda) \equiv D \neq 0,$$

where $a(\lambda)$ and $b(\lambda)$ are polynomials with coefficients in K. It follows that $b(p)f^{(m)}(p)=D$, and thus by the preceding theorem, $r^m D=0$. That is, $r^m=0$, and we have established

THEOREM 13. *The index k of r does not exceed the multiplicity of the factor of $f(\lambda)$ [or of $g(\lambda)$] of greatest multiplicity.*

By Theorem 9, the question of reducibility of A is equivalent to that of reducibility of R'/M', and this in turn depends upon whether M' is direct indecomposable or not. We may now prove the following theorem.

THEOREM 14. *If $f(\lambda)$ [or $g(\lambda)$] is expressible as the product of two relatively prime factors with coefficients in K, the algebra A is reducible.*

Suppose $f(\lambda)=\phi(\lambda)\psi(\lambda)$, where $\phi(\lambda)$ and $\psi(\lambda)$ are relatively prime, and have coefficients in K. There then exists a relation

$$a(\lambda)\phi(\lambda) + b(\lambda)\psi(\lambda) \equiv 1. \tag{16}$$

Let $M_1' = (M', \phi(x))$, $M_2' = (M', \psi(x))$. Since $f(x)$ is the polynomial in x of minimum degree belonging to M', it follows that M_1' and M_2' are proper divisors of M'. We have also from relation (16) that $(M_1', M_2')=(1)$. It is easy to show that $M'=M_1' \cap M_2'$. Let $c(x)$ be any element belonging to $M_1' \cap M_2'$. Then we have

$$c(x) \equiv d(x)\phi(x) \equiv e(x)\psi(x) \quad (M'),$$

and thus $d(x)\phi(x)-e(x)\psi(x)\equiv 0 \quad (M')$. If we multiply this last relation by $a(x)$, we find, by use of relation (16), that

* See, e.g., van der Waerden, op. cit., II, p. 4.

$$d(x) \equiv [d(x)b(x) + a(x)e(x)]\psi(x) \quad (M').$$

Thus $d(x)\phi(x) \equiv 0 \quad (M')$, and therefore $c(x) \equiv 0 \quad (M')$. By Theorem 5 it now follows that R'/M' is reducible, and Theorem 9 then shows that $A = R/M$ is reducible.

The converse of this theorem is not in general true even for the commutative case, as can be shown by the following example. Let K be the field of real numbers, and set $f(\lambda) = \lambda^2+1$, $g(\lambda) = \lambda^2+1$, $M' = (x^2+1, y^2+1, z)$, $m_1' = (M', 1-xy)$, $M_2' = (M', 1+xy)$. It is not difficult to show that M_1' and M_2' are proper ideal divisors of M' and that $M' = M_1' \cap M_2'$, $(M_1', M_2') = (1)$. Hence R/M is reducible.

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