ON SOME FUNCTIONALS, II*

BY
S. SAKS

1. This article is primarily intended to correct the mistakes in §§4 and 5 of the former paper by the author *On some functionals*.† On the whole these errors do not affect the theorems themselves with the exception of the obviously false remarks that Theorem 4 and a part of Theorem 3 of S. F. hold for the space R of characteristic functions. In the present note we complete the gaps in the proofs of those theorems‡ and slightly strengthen Theorem 3 of S. F. in the part concerning the equal continuity of the operations considered (cf. below Theorem 3, (ii)). We also prove two theorems which were not stated in S. F., namely, Theorems 1 and 4 so as to obtain a more symmetric set of results.

2. We shall recall briefly the notation. Let $\{\xi_n(t)\}$ be a sequence of measurable functions on a measurable set U . Then

(i) $\{\xi_n(t)\}$ is *bounded in measure* on U if to every $\eta > 0$ there corresponds a number $M = M(\eta)$ such that

$$\text{meas } E_t[t\epsilon U; |\xi_n(t)| > M] < \eta \quad (n = 1, 2, \dots);$$

(ii) $\{\xi_n(t)\}$ *converges in measure* on U if to every η there corresponds a $k = k(\eta)$ such that

$$\text{meas } E_t[t\epsilon U; |\xi_n(t) - \xi_m(t)| > \eta] < \eta$$

whenever $n > k, m > k$;

(iii) $\{\xi_n(t)\}$ *has property $B(\epsilon)$* on U , if there exists a number M independent of n , such that

$$\text{meas } E_t[t\epsilon U; |\xi_n(t)| > M] < \epsilon \quad (n = 1, 2, \dots);$$

(iv) $\{\xi_n(t)\}$ *has property $C(\epsilon)$* on U ,§ if to every $\eta > 0$ there corresponds a $k = k(\eta)$ such that

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† These Transactions, vol. 35 (1932), pp. 549–556. This paper will be referred to as S. F.

‡ For the proof of Parts (i) and (ii) of Theorem 3 of S. F. see also the recent book by Kaczmarz and Steinhaus, *Theorie der Orthogonalreihen*, Monografie Matematyczne, Warszawa-Lwow, 1935, pp. 24–26. The author's attention was called to these mistakes by Kaczmarz and by Gowurin. (Cf. Gowurin, *On sequences of indefinite integrals*, Bulletin of the American Mathematical Society, vol. 42 (1936), pp. 930–936.)

§ We shall write $\{\xi_n(t)\} \varepsilon B(\epsilon)$ or $\varepsilon C(\epsilon)$ to indicate that the sequence $\{\xi_n(t)\}$ has property $B(\epsilon)$ or $C(\epsilon)$ respectively.

$$\text{meas } E[t \in U; |\xi_n(t) - \xi_m(t)| > \eta] < \epsilon, \quad n > k, m > k.$$

In what follows we denote by I a measurable set of finite measure (e.g., an interval) and by E a Banach space. We shall consider sequences of functions $\{\xi_n(x, t)\}$ depending on $x \in E$ and $t \in I$. For each x fixed in E , $\xi_n(x, t)$ are finite and measurable functions of t in I . On the other hand, as functions of x , they are supposed to be linear operations on E ; i.e., additive and continuous. This means that $\xi_n(x_1 + x_2, t) = \xi_n(x_1, t) + \xi_n(x_2, t)$ almost everywhere on I , whenever $x_1 \in E, x_2 \in E$ and that $\xi_n(x, t)$ tends in measure to 0 on I when $|x| \rightarrow 0$ (for n fixed). In considering $\xi_n(x, t)$ as operations on E we shall often write $\xi_n(x, t) \equiv F_n(x)$.

The operations $F_n(x) = \xi_n(x, t)$ will be said to be *equally continuous* if to every $\eta > 0$ there corresponds an $r > 0$ such that

$$\text{meas } E[t \in I; |\xi_n(x, t)| > \eta] < \eta \quad (n = 1, 2, \dots),$$

whenever $|x| < r$. They will be said to be *equally continuous with respect to a set* $U \subset I$ if the condition above is satisfied with I replaced by U .

Finally $\{T_k\}$ ($k = 1, 2, \dots$) will denote a sequence of measurable sets in I such that the characteristic functions of the sets T_k form an everywhere dense set in the space of all characteristic functions defined on I . Thus for each set Q and every $\eta > 0$ there is a set T_k , $k = k(Q, \eta)$, such that $\text{meas}(Q - QT_k) < \eta$ and $\text{meas}(T_k - QT_k) < \eta$.

3. We have

LEMMA 1. Let $\sum_1^\infty \epsilon_k$ be a converging series of positive numbers, $\{P_q\}$ a sequence of measurable sets in I and

$$(3.1) \quad P = \limsup_k P_k = \prod_{k=1}^\infty \sum_{q=k}^\infty P_q.$$

Then, if a sequence $\{\xi_n(t)\} \in B(\epsilon_k)$ on P_k ($k = 1, 2, \dots$), it is bounded in measure on P ; and if a sequence $\{\xi_n(t)\} \in C(\epsilon_k)$ on P_k ($k = 1, 2, \dots$), it converges in measure on P .

Let

$$Q_k = \sum_{q=k}^\infty P_q, \quad Q_{k,s} = \sum_{q=k}^s P_q,$$

and let η be an arbitrary positive number. Choose k_0 and next s_0 , so that

$$(3.2) \quad \sum_{i=k_0}^\infty \epsilon_i < \frac{\eta}{2}, \quad \text{meas}(Q_{k_0} - Q_{k_0, s_0}) < \frac{\eta}{2}.$$

Suppose now that the sequence under consideration $\in B(\epsilon_k)$ on P_k . Then there is a number M such that

$$\text{meas } E[t\epsilon Q_{k_0, s_0}; |\xi_n(t)| > M] < \sum_{q=k_0}^{s_0} \epsilon_q < \frac{\eta}{2} \quad (n = 1, 2, \dots);$$

thus, by (3.1) and (3.2)

$$\text{meas } E[t\epsilon P; |\xi_n(t)| > M] \leq \text{meas } E[t\epsilon Q_{k_0}; |\xi_n(t)| > M] < \eta \quad (n = 1, 2, \dots);$$

i.e., the sequence $\{\xi_n(t)\}$ is bounded in measure on P .

Next suppose that the sequence $\{\xi_n(t)\} \in C(\epsilon_k)$ on P_k . Then there is a positive integer n_0 such that

$$\text{meas } E[t\epsilon Q_{k_0, s_0}; |\xi_n(t) - \xi_m(t)| > \eta] < \sum_{i=k_0}^{s_0} \epsilon_i < \frac{\eta}{2}, \quad n \geq n_0, \quad m \geq n_0,$$

and so, by (3.1) and (3.2)

$$\text{meas } E[t\epsilon P; |\xi_n(t) - \xi_m(t)| > \eta] \leq \text{meas } E[t\epsilon Q_{k_0}; |\xi_n(t) - \xi_m(t)| > \eta] < \eta, \quad n \geq n_0, \quad m \geq n_0.$$

LEMMA 2. If a sequence $\{\xi_n(x, t)\} \in B(\epsilon)$ on a measurable set $P \subset I$ for every x belonging to a set H of the second category in E , then, to every $\eta > 0$, there corresponds an $r > 0$ such that for an arbitrary $x \in E$ with $|x| < r$,

$$(3.3) \quad \text{meas } E[t\epsilon P; |\xi_n(x, t)| > \eta] < 3\epsilon \quad (n = 1, 2, \dots).$$

Let H_m denote the set of all $x \in H$ such that

$$\text{meas } E[t\epsilon P; |\xi_n(x, t)| > m] < \epsilon \quad (n = 1, 2, \dots).$$

We have $H = \sum_1^\infty H_m$, and so there is a positive integer m_0 such that H_{m_0} is of the second category. Since $\xi_n(x, t)$ is continuous in x on E , the inequalities

$$\text{meas } E[t\epsilon P; |\xi_n(x, t)| > m_0] \leq \epsilon \quad (n = 1, 2, \dots)$$

hold for all $x \in \overline{H_{m_0}}$, and consequently for all x in a sphere $K_0 \subset \overline{H_{m_0}}$. Let r_0 be the radius of K_0 . By the linearity of the operations $\xi_n(x, t)$, we have

$$\text{meas } E[t\epsilon P; |\xi_n(x, t)| > 2m_0] \leq 2\epsilon \quad (n = 1, 2, \dots),$$

whenever $|x| < r_0$, and therefore condition (3.3) is satisfied with $r = \eta r_0 / (2m_0)$.

4. We now prove the following theorem:

THEOREM 1. For any given sequence $\{\xi_n(x, t)\}$ there exists a set A in I such that

- (i) the sequence $\{\xi_n(x, t)\}$ is bounded in measure on A for all $x \in E$;

- (ii) the operations $F_n(x) = \xi_n(x, t)$ are equally continuous with respect to A ;
 (iii) for every x in E , except perhaps for a set of the first category in E , the sequence $\{\xi_n(x, t)\}$ is bounded in measure on no set in $I - A$ of positive measure.

Denote by α_0 the least upper bound of all numbers α with the property that there exists a set $H(\alpha)$ of the second category in E such that for each x in $H(\alpha)$ the sequence $\{\xi_n(x, t)\}$ is bounded in measure on a set of measure $> \alpha$. Let q be a positive integer. We shall prove first that there exists a set H_q of the second category in E and a set P_q in I of measure $> \alpha_0 - 1/q^2$ such that

$$(4.1) \quad \{\xi_n(x, t)\} \epsilon B(1/q^2) \text{ on } P_q, \quad x \in H_q.$$

Indeed to every $x \in H(\alpha_0 - 1/q^2)$ we can attach a set T_k , $k = k(x)$ (see §2), such that

$$(4.2) \quad \{\xi_n(x, t)\} \epsilon B(1/q^2) \text{ on } T_k, \quad \text{meas } T_k > \alpha_0 - \frac{1}{q^2}.$$

Let $H_{q,k}$ denote the set of all $x \in H(\alpha_0 - 1/q^2)$ for which (4.2) is satisfied for a fixed k . Since $H(\alpha_0 - 1/q^2) = \sum_{k=1}^{\infty} H_{q,k}$ is of the second category, there exists a $k = k_q$ such that H_{q,k_q} is of the second category. On putting $H_q = H_{q,k_q}$ and $P_q = T_{k_q}$, we see at once that H_q and P_q satisfy condition (4.1).

We shall prove next that for every x in E

$$(4.3) \quad \{\xi_n(x, t)\} \epsilon B(8/q^2) \text{ on } P_q.$$

To show this consider an arbitrary point $x_0 \in \overline{H}_q$. By (4.1) and Lemma 2

$$(4.4) \quad \text{meas } E[t \epsilon P_q; |\xi_n(x_0 - x_1, t)| > 1] < \frac{3}{q^2} \quad (n = 1, 2, \dots),$$

if $|x_0 - x_1|$ is sufficiently small. Hence we can choose $x_1 \in H_q$ so as to satisfy (4.4). Since $\{\xi_n(x_1, t)\} \epsilon B(1/q^2)$ on P_q this implies that $\{\xi_n(x_0, t)\} \epsilon B(4/q^2)$ on P_q . But x_0 is an arbitrary element of the set $\overline{H}_q$ which certainly contains interior points (a sphere), whence it follows that $\{\xi_n(x, t)\} \epsilon B(8/q^2)$ on P_q for all x in E .

Now set $A = \prod_{k=1}^{\infty} \sum_{q=k}^{\infty} P_q$. It results from (4.3) and Lemma 1 that the sequence $\{\xi_n(x, t)\}$ is bounded in measure on A , i.e., that A satisfies condition (i) of the theorem.

Next let η be an arbitrary positive number, and let, as in the proof of Lemma 1,

$$(4.5) \quad Q_k = \sum_{q=k}^{\infty} P_q, \quad Q_{k,s} = \sum_{q=k}^s P_q.$$

Let k_0 and s_0 be positive integers such that

$$(4.6) \quad 24/(k_0 - 1) < \frac{\eta}{2}, \quad \text{meas } (Q_{k_0} - Q_{k_0, s_0}) < \frac{\eta}{2}.$$

By (4.3), Lemma 2 and (4.6), there exists an $r = r(\eta)$ such that, for $|x| < r$,

$$\text{meas } E[t \varepsilon Q_{k_0, s_0}; |\xi_n(x, t)| > \eta] < 24 \sum_{q=k_0}^{s_0} q^{-2} < \frac{\eta}{2} \quad (n = 1, 2, \dots),$$

and thus, again by (4.6)

$$\text{meas } E[t \varepsilon A; |\xi_n(x, t)| > \eta] \leq \text{meas } E[t \varepsilon Q_{k_0}; |\xi_n(x, t)| > \eta] < \eta$$

which is condition (ii).

Finally, in order to prove condition (iii) suppose that there is a set H of the second category in E such that for every $x \in H$ the sequence $\{\xi_n(x, t)\}$ is bounded in measure on a set $Q(x) \subset I - A$ of positive measure. Then, for every $x \in H$ the sequence $\{\xi_n(x, t)\}$ would be bounded in measure on $A + Q(x)$. Now since

$$\text{meas } [A + Q(x)] > \text{meas } A \geq \limsup_q \text{meas } P_q \geq \lim_q (\alpha_0 - 1/q^2) = \alpha_0,$$

we have

$$H = \sum_{n=1}^{\infty} H_n, \quad H_n = E_x[x \varepsilon H; \text{meas } \{A + Q(x)\} \geq \alpha_0 + 1/n].$$

Since at least one of the sets H_n is of the second category, this contradicts the definition of α_0 .

5. Next we prove

THEOREM 2. *There exists a set B in I such that*

- (i) *for all x in E , the sequence $\{\xi_n(x, t)\}$ converges in measure on B ;*
- (ii) *for every x in E , except perhaps for a set of the first category in E , the sequence $\{\xi_n(x, t)\}$ converges in measure on no set in $I - B$ of positive measure.*

First observe that for every $x \in E$, except perhaps for a set of the first category, the sequence $\{\xi_n(x, t)\}$ does not converge in measure on any set of positive measure contained in $I - A$ (where A is the set defined in Theorem 1). Hence to prove Theorem 2 we may assume that $I = A$, or, which amounts to the same (see Theorem 1, (ii)) that the operations $F_n(x) = \xi_n(x, t)$ are equally continuous with respect to the whole set I .

From now on we shall follow the line of the proof of Theorem 1. Denote by β_0 the least upper bound of all numbers β with the property that there exists a set $H(\beta)$ of the second category in E such that for every $x \in H(\beta)$ the sequence $\{\xi_n(x, t)\}$ converges in measure on a set of measure $> \beta$. Let q be an arbitrary

positive integer, and let $H_{q,k}$ denote the set of all $x \in H(\beta_0 - 1/q^2)$ such that $\{\xi_n(x, t)\} \in C(1/q^2)$ on a set T_k (see §2) of measure $> \beta_0 - 1/q^2$. Clearly $H(\beta_0 - 1/q^2) = \sum_{k=1}^{\infty} H_{q,k}$, and thus there is a $k = k_q$ such that H_{q,k_q} is of the second category. Hence upon putting $H_q = H_{q,k_q}$ and $P_q = T_{k_q}$ we see that there exists a set H_q of the second category in E and a set P_q in I of measure $> \beta_0 - 1/q^2$ such that

$$(5.1) \quad \{\xi_n(x, t)\} \in C(1/q^2) \text{ on } P_q, \quad x \in H_q.$$

Now as in the proof of Theorem 1 we shall show that for every x in E

$$(5.2) \quad \{\xi_n(x, t)\} \in C(6/q^2) \text{ on } P_q.$$

Indeed, let $x_0 \in \bar{H}_q$ and η be an arbitrary positive number. The equal continuity of the operations $F_n(x)$ implies the existence of an $x_1 \in H_q$ sufficiently near to x_0 such that

$$(5.3) \quad \text{meas } E_t \left[t \in I; |\xi_n(x_1 - x_0, t)| > \frac{\eta}{6} \right] < \frac{1}{q^2} \quad (n = 1, 2, \dots).$$

Again, in view of (5.1) there exists a positive integer n_0 such that, for $n \geq n_0$, $m \geq n_0$,

$$\text{meas } E_t \left[t \in P_q; |\xi_m(x_1, t) - \xi_n(x_1, t)| > \frac{\eta}{6} \right] < \frac{1}{q^2}.$$

Thus, by (5.3),

$$(5.4) \quad \text{meas } E_t \left[t \in P_q; |\xi_m(x_0, t) - \xi_n(x_0, t)| > \frac{\eta}{2} \right] < \frac{3}{q^2}, \quad m \geq n_0, n \geq n_0.$$

Let K_0 be an arbitrary sphere contained in $\bar{H}_q$ and r_0 the radius of K_0 . Since (5.4) holds for all elements $x_0 \in K_0$, we have, by the linearity of the transformations $F_n(x)$,

$$\text{meas } E_t \left[t \in P_q; |\xi_n(x, t) - \xi_m(x, t)| > \eta \right] < \frac{6}{q^2}, \quad m \geq n_0, n \geq n_0, |x| < r_0.$$

This means however that $\{\xi_n(x, t)\} \in C(6/q^2)$ on P_q whenever $|x| < r_0$ and consequently for every x in E .

Now let

$$B = \prod_{k=1}^{\infty} \sum_{q=k}^{\infty} P_q.$$

By Lemma 1 the sequence $\{\xi_n(x, t)\}$ converges in measure on B for every x . Hence, the set B satisfies condition (i). To establish condition (ii) suppose

that there is a set H of the second category such that for every $x \in H$ the sequence $\{\xi_n(x, t)\}$ converges in measure on a set $Q(x) \subset I - B$ of positive measure. Then, for every $x \in H$ the sequence would converge in measure on $B + Q(x)$. But

$$\text{meas } [B + Q(x)] > \text{meas } B \geq \limsup_q \text{meas } P_q \geq \beta_0,$$

which contradicts the definition of the number β_0 (cf. the proof of Theorem 1). Condition (ii) is thus established.

6. We now have

THEOREM 3. *There exists a set C in I such that*

- (i) *for all x in E , $\sup_n |\xi_n(x, t)| < \infty$ almost everywhere in C ;*
- (ii) *to every $\eta > 0$ there corresponds an $r > 0$ such that*

$$\text{meas } E[t_\epsilon C; \sup_n |\xi_n(x, t)| > \eta] < \eta, \quad |x| < r;$$

- (iii) *for every x in E , except perhaps for a set of the first category in E ,*

$$\sup_n |\xi_n(x, t)| = \infty$$

almost everywhere in $I - C$.

Let γ_0 be the least upper bound of all numbers γ such that the set $H(\gamma)$ of x for which

$$\text{meas } E[t_\epsilon \sup_n |\xi_n(x, t)| < \infty] > \gamma$$

is of the second category. Let q be an arbitrary positive integer and let $H_{q,p,k}$ denote the set of all $x \in H(\gamma_0 - 1/q^2)$ such that

$$\text{meas } E[t_\epsilon T_k; \sup_n |\xi_n(x, t)| > p] < \frac{1}{q^2}, \quad \text{meas } T_k > \gamma_0 - \frac{1}{q^2}.$$

It is clear that $H(\gamma_0 - 1/q^2) = \sum_{p,k=1}^{\infty} H_{q,p,k}$ and so there exist integers $k = k_q$ and $p = p_q$ such that H_{q,p_q,k_q} is of the second category. Put $H_q = H_{q,p_q,k_q}$ and $P_q = T_{k_q}$. Thus for every $x \in H_q$

$$\text{meas } E[t_\epsilon P_q; \sup_n |\xi_n(x, t)| > p_q] < \frac{1}{q^2}$$

while $\text{meas } P_q > \gamma_0 - 1/q^2$. Hence, by continuity of $\xi_n(x, t)$, for every $x \in H_q$ and all s ,

$$\text{meas } E[t_\epsilon P_q; \sup_{n \leq s} |\xi_n(x, t)| > p_q] \leq \frac{1}{q^2},$$

and therefore, for all $x \in \overline{H}_q$,

$$(6.1) \quad \text{meas } E_t[t\epsilon P_q; \sup_n |\xi_n(x, t)| > p_q] \leq \frac{1}{q^2}.$$

Let K_0 be a sphere contained in $\overline{H}_q$. If r_0 is the radius of K_0 , (6.1) implies in view of the linearity of $\xi_n(x, t)$,

$$\text{meas } E_t[t\epsilon P_q; \sup_n |\xi_n(x, t)| > 2p_q] \leq \frac{2}{q^2},$$

no matter what $x \in E$, provided $|x| < r_0$, and consequently, for any $\sigma > 0$,

$$(6.2) \quad \text{meas } E_t[t\epsilon P_q; \sup_n |\xi_n(x, t)| > \sigma] \leq \frac{2}{q^2}, \quad |x| < \frac{r_0 \sigma}{2p_q}.$$

Now, again by linearity of $\xi_n(x, t)$ it results from (6.2) that for every $x \in E$, $\sup_n |\xi_n(x, t)| < \infty$ for all $t \in P_q$, with the exception of at most a subset of measure $\leq 2/q^2$. Hence, upon putting $C = \prod_{k=1}^{\infty} \sum_{q=k}^{\infty} P_q$ we see at once that, for every $x \in E$, $\sup_n |\xi_n(x, t)| < \infty$ almost everywhere in C . Thus C satisfies condition (i) of the theorem.

Next, as in the proof of Lemma 1, let

$$Q_k = \sum_{q=k}^{\infty} P_q, \quad Q_{k,s} = \sum_{q=k}^s P_q.$$

Let η be an arbitrary positive number, and let k_0 and s_0 be positive integers such that

$$(6.3) \quad \frac{2}{k_0 - 1} < \frac{\eta}{2}, \quad \text{meas } (Q_{k_0} - Q_{k_0, s_0}) < \frac{\eta}{2}.$$

In virtue of (6.2) there is an $r > 0$ such that whenever $|x| < r$,

$$\text{meas } E_t[t\epsilon Q_{k_0, s_0}; \sup_n |\xi_n(x, t)| > \eta] \leq \sum_{q=k_0}^{s_0} 2q^{-2} < \frac{\eta}{2}$$

and therefore, by (6.3),

$$\begin{aligned} \text{meas } E_t[t\epsilon C; \sup_n |\xi_n(x, t)| > \eta] \\ \leq \text{meas } E_t[t\epsilon Q_{k_0}; \sup_n |\xi_n(x, t)| > \eta] < \eta, \quad |x| < r. \end{aligned}$$

Thus condition (ii) for C is established. Finally condition (iii) follows at once from the definition of γ_0 , since

$$\text{meas } C \geq \limsup_q \text{meas } P_q \geq \gamma_0.$$

7. Next we have

THEOREM 4. *There exists a set D in I such that*

- (i) *for all x in E the sequence $\{\xi_n(x, t)\}$ converges almost everywhere on D ;*
- (ii) *for every x in E except perhaps for a set of the first category, the sequence $\{\xi_n(x, t)\}$ diverges almost everywhere on $I-D$.*

First observe that for every x , except perhaps for a set of the first category, the sequence $\{\xi_n(x, t)\}$ diverges almost everywhere on $I-C$, where C is the set defined in Theorem 3. Thus, without loss of generality, we may assume that $I=C$.

For a fixed $x \in E$ let $\Gamma(x)$ denote the subset of I on which $\{\xi_n(x, t)\}$ converges. Further let $H(\delta)$ denote the set of all $x \in E$ such that $\text{meas } \Gamma(x) > \delta$, $\delta > 0$, and let δ_0 be the upper bound of the numbers δ for which $H(\delta)$ is of the second category.

Now, let q be an arbitrary positive integer and $H_{q,k}$ the set of all $x \in H(\delta_0 - 1/q^2)$ such that

$$\text{meas } T_k > \delta_0 - \frac{1}{q^2}, \quad \text{meas } (T_k - T_k \Gamma(x)) < \frac{1}{q^2}.$$

We thus have $H(\delta_0 - 1/q^2) = \sum_{k=1}^{\infty} H_{q,k}$, and so there exists a $k = k_q$ such that $H_{q,k}$ is of the second category. Put $H_q = H_{q,k_q}$ and $P_q = T_{k_q}$. Then

$$(7.1) \quad \text{meas } (P_q - P_q \Gamma(x)) \leq \frac{1}{q^2}, \quad x \in H_q,$$

but we shall show that the latter inequality holds for every $x \in \overline{H_q}$. Indeed, let $x_0 \in \overline{H_q}$ and η be an arbitrary positive number. In virtue of Theorem 3 (condition (ii) with $C=I$) we can find an $x_1 \in H_q$ such that

$$\text{meas } E_t [t \in I; \sup_n |\xi_n(x_0 - x_1, t)| > \eta] < \eta.$$

Again, by (7.1) there is a positive integer $n_0 = n_0(\eta)$ such that

$$\text{meas } E_t [t \in P_q; \sup_{n \geq n_0} |\xi_n(x_1, t) - \xi_{n_0}(x_1, t)| > \eta] \leq \frac{1}{q^2} + \eta,$$

whence

$$\text{meas } E_t [t \in P_q; \sup_{n \geq n} |\xi_n(x_0, t) - \xi_{n_0}(x_0, t)| > 3\eta] \leq \frac{1}{q^2} + 3\eta.$$

Since $\eta > 0$ is arbitrary this implies the convergence of the sequence $\{\xi_n(x, t)\}$ for every $x \in \overline{H_q}$ and for all $t \in P_q$, with the exception of at most a subset of

measure $\leq 1/q^2$. Since $\overline{H}_q$ certainly contains a sphere the same holds for every $x \in E$ by the linearity of $\xi_n(x, t)$. Therefore, for each $x \in E$ the sequence converges almost everywhere on the set $D = \prod_{k=1}^{\infty} \sum_{q=k}^{\infty} P_q$. Thus the set D satisfies condition (i) of the theorem and condition (ii) follows at once since

$$\text{meas } D \geq \limsup_q \text{meas } P_q \geq \delta_0.$$

8. In view of Theorems 1, 2, 3, and 4 to every sequence $\{\xi_n(x, t)\}$ of the type considered there are attached four sets A, B, C , and D . We obviously have $B \subset A$ and $D \subset C$ (except for sets of measure zero). However, on account of condition (ii) of Theorem 1 (or 3) we have $B = A$ (or $D = C$) whenever E contains an everywhere dense set E_1 such that the sequence $\{\xi_n(x, t)\}$ converges in measure (or converges almost everywhere) in I for every x belonging* to E_1 . It follows that if E is a separable space then from the sequence $\{\xi_n(x, t)\}$ a subsequence $\{\xi'_n(x, t)\}$ may be extracted which converges almost everywhere in BC for every $x \in E$; in fact it is sufficient to select the required subsequence so that it converges almost everywhere in B for every x belonging to an everywhere dense, denumerable set in E .

9. The theorems of the preceding sections do not hold† in general, if the linear space E is replaced by the space R of characteristic functions. However, for the latter we have the following theorem:

THEOREM 5. (i) *If for all x belonging to a set H of the second category in R the sequence $\{F_n(x) = \xi_n(x, t)\}$, $x \in R$, $t \in I$, converges in measure on I , then the operations $F_n(x)$ are equally continuous in I , i.e., for every $\eta > 0$ there exists an $r > 0$ such that*

$$\text{meas } E_t[t \in I; |\xi_n(x, t)| > \eta] \leq \eta \quad (n = 1, 2, \dots),$$

whenever $|x| < r$.

(ii) *If for all x belonging to a set H of the second category in R , the sequence $\{\xi_n(x, t)\}$ converges almost everywhere on I , then for every $\eta > 0$ there exists an $r > 0$ such that*

$$\text{meas } E_t[t \in I; \sup_n |\xi_n(x, t)| > \eta] < \eta,$$

whenever $|x| < r$.

We shall confine ourselves to the proof of statement (ii) of the theorem. The proof of statement (i) is the same as that of the lemma of S. F., p. 555.

* Cf. Banach, *Sur la convergence presque partout de fonctionnelles linéaires*, Bulletin des Sciences Mathématiques, vol. 50 (1926), pp. 27–32, 36–43; Saks, *Sur les fonctionnelles de M. Banach et leur application aux développements des fonctions*, Fundamenta Mathematicae, vol. 10 (1927), pp. 186–196; Mazur and Orlicz, *Über Folgen linearer Operationen*, Studia Mathematica, vol. 4 (1933), pp. 152–157; esp. p. 157; Kaczmarz and Steinhaus, op. cit., pp. 177–178.

† See Gowurin, loc. cit.

Let η be any positive number and k a positive integer; let H_k denote the set of all $x \in H$ such that

$$\text{meas}_t E_t \left[t \in I; \sup_{n \geq k} |\xi_n(x, t) - \xi_k(x, t)| > \frac{\eta}{4} \right] \leq \frac{\eta}{4}.$$

We have $H = \sum_1^\infty H_k$ and thus there exists a $k = k_0$ such that H_{k_0} is of the second category. Hence

$$\text{meas}_t E_t \left[t \in I; \sup_{n \geq k_0} |\xi_n(x, t) - \xi_{k_0}(x, t)| > \frac{\eta}{4} \right] \leq \frac{\eta}{4},$$

for each $x \in H_{k_0}$, and therefore, by continuity (cf. the proof of Theorem 3), for all $x \in \overline{H}_{k_0}$. Let K_0 be a sphere which is contained in $\overline{H}_{k_0}$ and let r_0 be its radius. For every element $x \in R$, $|x| < r_0$, there are two elements x_1 and x_2 in R such* that $x_1 \in K_0$, $x_2 \in K_0$ and $x_1 = x_2 + x$. It is readily seen that, no matter what x is, $|x| < r_0$,

$$(9.1) \quad \text{meas}_t E_t \left[t \in I; \sup_{n \geq k_0} |\xi_n(x, t) - \xi_{k_0}(x, t)| > \frac{\eta}{2} \right] \leq \frac{\eta}{2}.$$

Let now $r < r_0$ be a positive number such that

$$\text{meas}_t E_t \left[t \in I; |\xi_n(x, t)| > \frac{\eta}{2} \right] \leq \frac{\eta}{2k_0} \quad (n = 1, 2, \dots, k_0),$$

whenever $|x| < r$. Then in view of (9.1), for every x , $|x| < r$,

$$\text{meas}_t E_t \left[t \in I; \sup_n |\xi_n(x, t)| > \eta \right] \leq \eta.$$

* We put $x_1 = x + x_0(1-x)$, $x_2 = x_0(1-x)$ where x_0 is the center of the sphere K_0 .