LINEAR DIVISIBILITY SEQUENCES*

BY

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I. INTRODUCTION

1. A sequence of rational integers

(u) : $u_0, u_1, \dots, u_n, \dots$

is called a *divisibility sequence* if u_n divides u_m whenever n divides m . (u) is *linear*† if it satisfies a linear difference equation with integral coefficients and *normal* if $u_0=0, u_1=1$. Marshall Hall has shown that a linear divisibility sequence is usually normal [2]. If

$$(1.1) \quad f(x) = x^k - c_1 x^{k-1} - \dots - c_k, \quad c_1, \dots, c_k \text{ integers,}$$

is the polynomial associated with the difference equation of lowest order which (u) satisfies, (u) is said to be of *order* k and to *belong* to its *characteristic polynomial* $f(x)$.

An integer dividing every term of (u) beyond a certain point is called a *null divisor* of (u) [3]. If (u) has no null divisors save ± 1 , it is said to be *primary*.

If u_s is any fixed non-vanishing term of (u) , the sequence

$$u_0/u_s, u_1/u_s, u_2/u_s, \dots, u_n/u_s, \dots$$

is called a *subsequence* of (u) . The various subsequences of (u) are themselves normal linear divisibility sequences of order $\leq k$.

2. The object of this paper is to prove the following results:

Let the characteristic polynomial of the linear divisibility sequence (u) have no repeated roots, and let its coefficients be relatively prime. Then:

I. *If (u) is primary and if q is any large prime number,*

$$(2.1) \quad u_q^\sigma \equiv 1 \pmod{q},$$

where σ is the least common multiple of $1, 2, 3, \dots, k$.

II. *If (u) is not primary it always contains an infinity of subsequences which are primary. Furthermore the characteristic polynomials of such subsequences satisfy the hypotheses imposed above upon the polynomial (1.1).*

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† T. A. Pierce appears to have been the first to discuss sequences of order greater than two [1]. (Numbers in square brackets refer to the bibliography at the end of the paper.)

III. There exists a rational number

$$B = B(u) = B(u_0, u_1, \dots, u_{k-1}; c_1, \dots, c_k) = \frac{P}{Q}, \quad (P, Q) = 1$$

such that

(i) if p is a prime number dividing neither the numerator P nor the denominator Q of B , then the rank of apparition* of p in the sequence (u) is the restricted period* of (u) modulo p ;

(ii) the prime factors of the denominator of B all divide the discriminant of the polynomial to which (u) belongs;

(iii) the numerator of B can never vanish if the galois group of $f(x)$ is alternating or symmetric.†

II. PROOF OF FIRST RESULT

3. Given any modulus m , the least period of (u) modulo m is called its characteristic number and the number of non-periodic terms in (u) modulo m its numeric. The reader will be assumed to be familiar with my previous paper in these Transactions [4] (referred to hereafter as T) devoted to the determination of these numbers.

Henceforth let (u) be a normal linear divisibility sequence of order k , and let D denote the discriminant of its characteristic polynomial. We assume:

$$(3.1) \quad D \neq 0.$$

LEMMA 3.1 [4]. If $\dagger (q, D) = 1$, q a prime, and if σ is the least common multiple of $2, 3, \dots, k$, then (u) admits the period $q^\sigma - 1$ modulo q .

THEOREM 3.1. If (u) is a linear divisibility sequence of order k and q a prime such that $u_q \equiv 0 \pmod{q}$, then either q divides D or q divides c_k .

Assume that $\dagger q \nmid u_q$, q a prime. The assumption $(q, c_k) = (q, D) = 1$ then yields a contradiction. For if $(q, c_k) = 1$, (u) is purely periodic modulo q [5]. And if $(q, D) = 1$, (u) admits the period $q^\sigma - 1$ modulo q . Determine positive integers x and y such that $xq = y(q^\sigma - 1) + 1$. Then $u_{xq} \equiv u_1 \equiv 1 \pmod{q}$. But $q \mid u_q$ and $u_q \mid u_{xq}$.

The following lemma is a direct consequence of Theorem 3.1.

* The rank of apparition of p is the index ρ of the first term of (u) excluding u_0 which divides: $u_r \equiv 0 \pmod{p}$; $u_n \not\equiv 0 \pmod{p}$, $0 < n < \rho$. The restricted period [5] of (u) modulo is the least positive integer τ such that $u_{n+\tau} \equiv cu_n \pmod{p}$, $n = 0, 1, 2, \dots$, c an integer. ρ always divides τ [2].

† It is unknown whether divisibility sequences exist whose characteristic polynomial is restricted as in (iii). No such sequences exist when $k = 3$ [2].

‡ If $a, b, c, \dots$ are rational integers, we write as usual $(a, b, c, \dots)$ for the greatest common divisor of $a, b, c, \dots$, and $a \mid b$ for a divides b .

LEMMA 3.2. *There exists a rational integer q_0 such that*

$$(3.2) \quad u_q \not\equiv 0 \pmod{q}, \quad q \text{ a prime } \geq q_0.$$

LEMMA 3.3 [4]. *For any prime p , $p^k(p^\sigma - 1)$ is a period of (u) modulo p .*

LEMMA 3.4 [4]. *For any prime p , the numeric of (u) modulo p is less than or equal to k .*

THEOREM 3.2. *If p is a prime dividing a term u_q of the divisibility sequence (u) with a sufficiently large prime index q , then either*

$$(3.3) \quad p^\sigma \equiv 1 \pmod{q}$$

or else (u) is a null sequence modulo p .

Choose a prime $q > k$ and q_0 of (3.2), and assume that $u_q \equiv 0 \pmod{p}$, p a prime. By (3.2), $p \neq q$. Hence if $(p^\sigma - 1, q) = 1$, for each positive integer r there exist positive integers x, y, z such that

$$(3.4) \quad xq + yp^k(p^\sigma - 1) = r + zp^k(p^\sigma - 1).$$

By Lemma 3.3, $p^k(p^\sigma - 1)$ is a period of (u) modulo p . Therefore if $r > k$, (3.4) and Lemma 3.4 give $u_{xq} \equiv u_r \pmod{p}$. Since $p \mid u_q$ and $u_q \mid u_{xq}$, $u_r \equiv 0 \pmod{p}$ so that (u) is a null sequence modulo p .

THEOREM 3.3. *If the linear divisibility sequence (u) is primary, and if k is its order and σ the least common multiple of the numbers $2, 3, \dots, k$, then for all sufficiently large prime indices q we have*

$$(2.1) \quad u_q^\sigma \equiv 1 \pmod{q}.$$

Choose the prime $q > k$ and q_0 of (3.2), and let the factorization of u_q be $u_q = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$. Since (u) is assumed primary none of the primes p_i are null divisors. Therefore Theorem 3.2, $p_i^\sigma \equiv 1 \pmod{q}$, so that

$$p_i^{\sigma e_i} \equiv 1 \pmod{q}, \quad (i = 1, 2, \dots, t).$$

On multiplying these t congruences together, we obtain (2.1), and our first result is proved.

III. PROOF OF SECOND RESULT

4. We assume that (u) is a normal linear divisibility sequence for which

$$(4.1) \quad (c_1, c_2, \dots, c_k) = 1.$$

A *proper* null divisor of a linear sequence is one which divides neither its initial terms nor the coefficients of its recursion. Any other null divisor is called *trivial*. (u) obviously has no trivial null divisors.

THEOREM 4.1. *No subsequence of (u) has trivial null divisors.*

LEMMA 4.1 (Schatanovskis Principle) [6, 7, 8]. *If $\Phi(x_1, x_2, \dots, x_k)$ is an integral symmetric function of the arguments $x_1, \dots, x_k$ with integral coefficients, and if for a natural number m*

$$f(x) \equiv (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_k) \equiv (x - \gamma_1)(x - \gamma_2) \cdots (x - \gamma_k) \pmod{m},$$

where $f(x)$ is a polynomial with integral coefficients, then

$$\Phi(\alpha_1, \alpha_2, \dots, \alpha_k) \equiv \Phi(\gamma_1, \gamma_2, \dots, \gamma_k) \pmod{m}.$$

LEMMA 4.2. *Let*

$$f^{(s)}(x) = x^k - d_1 x^{k-1} - \dots - d_k$$

be the polynomial whose roots are the s th powers of the roots of $f(x)$, and p a prime number. Then if t is any positive integer $\leq k$, (A) $p \mid (c_k, c_{k-1}, \dots, c_{k-t+1})$ when and only when (B) $p \mid (d_k, d_{k-1}, \dots, d_{k-t+1})$.

Assume that (A) holds. Then

$$f(x) \equiv g(x) = x^{k-t}(x^t - c_1 x^{t-1} - \dots - c_{k-t}) \pmod{p}.$$

Let the k roots of $g(x)=0$ be $\gamma_1, \gamma_2, \dots, \gamma_t; \gamma_{t+1}=\gamma_{t+2}=\dots=\gamma_k=0$. If the roots of $f(x)=0$ are $\alpha_1, \alpha_2, \dots, \alpha_k$, then $d_i = \Phi(\alpha_1, \alpha_2, \dots, \alpha_k)$, where Φ is a symmetric polynomial in its arguments with rational integral coefficients. Hence by the preceding lemma

$$d_i \equiv \Phi(\gamma_1, \gamma_2, \dots, \gamma_k) \pmod{p}.$$

But if $g^{(s)}(x) = x^k - e_1 x^{k-1} - \dots - e_k$ is the equation whose roots are the s th powers of the roots of $g(x)=0$, then

$$e_i = \Phi(\gamma_1, \gamma_2, \dots, \gamma_k) = \Sigma \gamma_1^s \gamma_2^s \cdots \gamma_i^s = 0 \text{ if } i > k - t.$$

Hence $d_i \equiv 0 \pmod{p}$ if $i > k - t$, so that (B) follows.

To prove the converse, it suffices to show that (A) and $c_{k-t} \not\equiv 0 \pmod{p}$ imply that $d_{k-t} \not\equiv 0 \pmod{p}$. But by what precedes,

$$d_{k-t} \equiv \Sigma (\gamma_1 \gamma_2 \cdots \gamma_t)^s \equiv (\gamma_1 \gamma_2 \cdots \gamma_t)^s \equiv c_{k-t}^s \pmod{p}.$$

Proof of Theorem 4.1. With the notation of Lemma 4.2, any subsequence $(v): v_n = u_{ns}/u_s$ of (u) is normal, so that the only possible trivial null divisors of (v) are common divisors of $d_1, d_2, \dots, d_k$. On taking $t=k$ in Lemma 4.2, we see that if $(c_1, c_2, \dots, c_k) = 1$ then $(d_1, d_2, \dots, d_k) = 1$.

5. We begin our discussion of the proper null divisors of (u) by restating some properties of linear sequences used in T. Let

$$f_0(x) = 0, \quad f_r(x) = x^r - c_1 x^{r-1} - \dots - c_r, \quad (r = 1, 2, \dots, k).$$

The polynomial

$$(5.1) \quad u(x) = u_0 f_{k-1}(x) + u_1 f_{k-2}(x) + \cdots + u_{k-1} f_0(x)$$

is called the *generator* of the sequence (u) .^{*} If furthermore

$$(5.2) \quad \Delta(u) = \begin{vmatrix} u_0, & u_1, & \cdots, & u_{k-1} \\ u_1, & u_2, & \cdots, & u_k \\ \vdots & \vdots & & \vdots \\ u_{k-1}, & u_k, & \cdots, & u_{2k-2} \end{vmatrix},$$

then

$$(5.3) \quad \Delta(u) = (-1)^{k(k-1)/2} \text{Res} \{u(x), f(x)\} = \beta_1 \beta_2 \cdots \beta_k D,$$

where $u_n = \beta_1 \alpha_1^n + \cdots + \beta_k \alpha_k^n$ and $\alpha_1, \cdots, \alpha_k$ are the roots of $f(x)$. Since (u) is of order k and $D \neq 0$, $\Delta(u) \neq 0$.

Consider next the $k+1$ greatest common divisors

$$\begin{aligned} e_0 &= (u_0, u_1, u_2, \cdots, u_{k-1}) \\ e_1 &= (c_k, u_1, u_2, \cdots, u_{k-1}) \\ e_2 &= (c_k, c_{k-1}, u_2, \cdots, u_{k-1}) \\ &\vdots \\ e_{k-1} &= (c_k, c_{k-1}, c_{k-2}, \cdots, u_{k-1}) \\ e_k &= (c_k, c_{k-1}, c_{k-2}, \cdots, c_1). \end{aligned}$$

Then

$$e_0 = e_1 = e_k = 1.$$

The following lemma easily follows from formula (5.1) and the results of part IV of T.

LEMMA 5.1. *Necessary and sufficient conditions that a linear sequence of order k be primary are that the $k+1$ greatest common divisors e_i be all equal to unity.*

THEOREM 5.1. *If the prime p is a null divisor of the normal linear divisibility sequence (u) , then p divides both $\Delta(u)$ and the discriminant D of the characteristic polynomial $f(x)$ of (u) .*

It is easily shown that every such p must divide one or the other of the numbers e_i . Since $e_k = 1$, $p \mid u_{k-1}$. Hence $p \mid u_k$, $p \mid u_{k+1}$, $\cdots$ by Lemma 3.4.

^{*} We have the identity $u(x)/f(x) = \sum_0^\infty u_n/x^{n+1}$ for $|x|$ large. See T, p. 606, and [3].

Hence $p \mid \Delta(u)$ by formula (5.2). Since $e_0 = e_1 = 1$, $p \mid c_k$ and $p \mid c_{k-1}$. Hence $x=0$ is a multiple root of the congruence $f(x) \equiv 0 \pmod{p}$ and $p \mid D$.

As a corollary, we have

LEMMA 5.2. *A sufficient condition that the divisibility sequence (u) be primary is that D and $\Delta(u)$ be co-prime.*

If p is a prime proper null divisor of (u) , the exponent of the highest power of p which is a null divisor of (u) is called the *index* of p in (u) [3].

LEMMA 5.3 [3]. *Let (u) be a linear sequence for which (4.1) holds. Then the index of any prime null divisor p is $\leq r$, where p^r is the highest power of p dividing $\Delta(u)$.*

THEOREM 5.2. *A subsequence of a normal linear divisibility sequence can have no prime null divisor which is not a possible null divisor of (u) itself.*

Every prime null divisor of (u) must divide c_k in (1.1) [5]. Let (v) be any subsequence of (u) . By Theorem 4.1, (v) can have only proper null divisors. Hence any prime null divisor of (v) must divide the constant term d_k of the polynomial to which (v) belongs. But obviously d_k divides some power of c_k .

6. Let $f^{(s)}(x) = (x - \alpha_1^s) \cdots (x - \alpha_k^s)$ be the polynomial whose roots are the s th powers of the roots of $f(x)$, and let $D^{(s)}$ be its discriminant. $D^{(s)}/D$ is clearly an integer.

THEOREM 6.1. *The integer s may be chosen in an infinite number of ways so that $D^{(s)}/D$ is prime to D .*

Let p be any prime factor of D , $\mathfrak{F}$ the Galois field of $f(x)$, and $\mathfrak{p}$ a prime ideal factor of p in $\mathfrak{F}$. Then since $D^{1/2} = \prod_{i < j} (\alpha_i - \alpha_j)$, $p \mid D$ only when $\alpha_i - \alpha_j \equiv 0 \pmod{\mathfrak{p}}$ for some values of the subscripts i and j .

Now

$$\left(\frac{D^{(s)}}{D}\right)^{1/2} = \prod_{i < j} \frac{\alpha_i^s - \alpha_j^s}{\alpha_i - \alpha_j} \quad \text{and} \quad \frac{\alpha_i^s - \alpha_j^s}{\alpha_i - \alpha_j} \equiv s \pmod{[\alpha_i - \alpha_j]}.*$$

Hence if $\alpha_i - \alpha_j \equiv 0 \pmod{\mathfrak{p}}$, then $\alpha_i^s - \alpha_j^s / (\alpha_i - \alpha_j) \equiv 0 \pmod{\mathfrak{p}}$ if and only if $s \equiv 0 \pmod{\mathfrak{p}}$; that is, if and only if $s \equiv 0 \pmod{p}$. Choose s prime to D . Then if $D^{(s)}/D$ and D have a common factor, and hence a common prime factor p , we must have for some k and l

$$(6.1) \quad \alpha_k^s \equiv \alpha_l^s \pmod{\mathfrak{p}}, \quad (6.11) \quad \alpha_k \not\equiv \alpha_l \pmod{\mathfrak{p}},$$

where $\mathfrak{p} \mid p$. If both (6.1) and (6.11) hold, then

* The square bracket denotes a principal ideal.

$$(6.2) \quad (\alpha_k, \mathfrak{p}) = (\alpha_l, \mathfrak{p}) = (\alpha_k - \alpha_l, \mathfrak{p}) = \mathfrak{o},$$

where $\mathfrak{o}$ as usual is the unit ideal of $\mathfrak{F}$.

Now for each pair of distinct roots α_i, α_j of $f(x)$ for which $(\alpha_i, \mathfrak{p}) = (\alpha_j, \mathfrak{p}) = (\alpha_i - \alpha_j, \mathfrak{p}) = \mathfrak{o}$, let s_{ij} be the least positive integer y such that

$$(6.3) \quad \alpha_i^y \equiv \alpha_j^y \pmod{\mathfrak{p}}.$$

Then s_{ij} divides every other such y , and in particular the number $N(\mathfrak{p}) - 1 = p^t - 1$. Here $t \leq k!$, the maximum possible degree of $\mathfrak{F}$.

Let m_p be the least common multiple of the numbers $p-1, p^2-1, \dots, p^{k!}-1$ and if D has in all k distinct prime factors $p_1, p_2, \dots, p_k$ let $\mathfrak{m}$ be the least common multiple of $m_{p_1}, m_{p_2}, \dots, m_{p_k}$. Then if s is chosen prime to both $\mathfrak{m}$ and D (and this choice can be made in an infinity of ways), $D^{(s)}/D$ is prime to D .

For if $(s, D) = 1$ and $(D^{(s)}/D, D) \neq 1$, (6.1) holds. Then $s_{kl} | s$. Since $(s, \mathfrak{m}) = 1$ and $s_{kl} | \mathfrak{m}$, $s_{kl} = 1$ contradicting (6.11).

7. As in §6, let $p_1, \dots, p_k$ be the distinct prime factors of D . By Theorems 4.5, 5.1 and Lemma 5.4, these primes are the only possible prime null divisors of (u) and its subsequences. Write

$$\Delta(u) = p_1^{r_1} \cdots p_k^{r_k} q, \quad (q, D) = 1, \quad r_i \geq 0,$$

and let θ_i be the index of p_i in (u) , where if p_i is not a null divisor, $\theta_i = 0$. By Lemma 5.3, $0 \leq \theta_i \leq r_i$, ($i = 1, 2, \dots, k$).

Now if R is the largest of $r_1, r_2, \dots, r_k$, the numeric of p^{θ_i} is always less than kR . Choose $s > kR$ as in Theorem 6.1, and let (v) be the subsequence of (u) with general term $v_n = u_{ns}/u_s$ belonging to the polynomial $f^{(s)}(x)$. As in Theorem 6.1, let the discriminant of $f^{(s)}(x)$ be $D^{(s)}$. Then since $u_{ns} = \beta_1 \alpha_1^{ns} + \dots + \beta_k \alpha_k^{ns}$, we have by formula (5.3),

$$(7.1) \quad \Delta(v) = \frac{\Delta(u)}{u_s^k} \frac{D^{(s)}}{D}.$$

Now $u_s \equiv 0 \pmod{p_i^{\theta_i}}$ and $(p_i, D^{(s)}/D) = 1$. Hence since $\Delta(v)$ is an integer, $\Delta(u) \equiv 0 \pmod{p_i^{k\theta_i}}$. Therefore $r_i \geq k\theta_i$. If $\Delta(v) = p_1^{r'_1} \cdots p_k^{r'_k} q'$, $(q', D) = 1$, then $r'_i = r_i - k\theta_i$. Therefore

$$(7.2) \quad r'_i < r_i \text{ if } \theta_i > 0; \quad r'_i = r_i \text{ if } \theta_i = 0.$$

8. We now prove our second result indirectly. Suppose that the result is false. Then in any infinite set of normal divisibility sequences

$$\mathfrak{S}: \quad (u^{(1)}) = (u), (u^{(2)}), (u^{(3)}), \dots, (u^{(m)}), (u^{(m+1)}), \dots,$$

such that each sequence is a subsequence of its immediate predecessor, there must occur an infinity of non-primary sequences. Therefore there must exist a prime p dividing D which is a null divisor of an infinite number of the sequences $(u^{(m)})$. The general term of $(u^{(m+1)})$ is of the form $u_n^{(m+1)} = u_{ns_m}^{(m)} / u_{s_m}^{(m)}$, where the integer s_m specifies the particular subsequence of $(u^{(m)})$ selected. Consider now a set $\mathfrak{S}$ in which each $u^{(m)}$ satisfies the conditions imposed upon s in §6.

The considerations of the preceding section carry over to the relationship between $(u^{(m)})$ and $(u^{(m+1)})$. With an obvious extension of notation, let $\theta^{(m)}$ denote the index of p in $(u^{(m)})$ and p^{r_m} and $p^{r_{m+1}}$ the highest powers of p dividing $\Delta(u^{(m)})$ and $\Delta(u^{(m+1)})$. Then as in (7.2)

$$(8.1) \quad r_{m+1} < r_m \text{ if } \theta^{(m)} > 0; \quad r_{m+1} = r_m \text{ if } \theta^{(m)} = 0.$$

By our hypothesis, an infinite number of the $\theta^{(m)}$ are positive. But then (8.1) leads to an absurdity; for obviously $r = r_1 \geq r_2 \geq r_3 \geq \dots \geq 0$.

IV. PROOF OF THIRD RESULT

9. We assume as in the previous proofs that $D \neq 0$. In the Galois field $\mathfrak{F}$ of $f(x)$, a rational prime p which does not divide D remains unramified [9]. Accordingly the decomposition of p into prime ideal factors in $\mathfrak{F}$ is of the form

$$p = \mathfrak{p}_1 \mathfrak{p}_2 \cdots \mathfrak{p}_l,$$

where the $\mathfrak{p}$ are all distinct.

Let σ_i be the least positive integer n such that

$$(9.1) \quad \alpha_1^n \equiv \alpha_2^n \equiv \cdots \equiv \alpha_k^n \pmod{\mathfrak{p}_i} \quad (i = 1, \dots, l).$$

The restricted period τ of (u) modulo p is defined as the least value of n such that

$$u_{n+m} \equiv au_n \pmod{p} \quad (m = 0, 1, 2, \dots),$$

where a is some rational integer [5]. If p is prime to $\Delta(u)$, τ may be equally defined as the least positive integer n such that we have in $\mathfrak{F}$

$$\alpha_1^n \equiv \alpha_2^n \equiv \cdots \equiv \alpha_k^n \pmod{p}.$$

The following lemma therefore follows.

LEMMA 9.1. *If p is a prime dividing neither $\Delta(u)$ nor D , then the restricted period τ of (u) modulo p is the least common multiple of the numbers $\sigma_1, \sigma_2, \dots, \sigma_l$ associated with the congruence (9.1) above.*

10. Since $u_n = \beta_1 \alpha_1^n + \cdots + \beta_k \alpha_k^n$ and the α_i are distinct,

$$(10.1) \quad \beta_i = u(\alpha_i) / f'(\alpha_i) \neq 0, \quad (i = 1, \dots, k).$$

where the β' occur in the sets (10.2) of sums of β 's. The determinant of the first m of these congruences as the difference product of the ζ is prime to p . Thus $\beta'_1 \equiv \beta'_2 \equiv \dots \equiv \beta'_m \equiv 0 \pmod{p}$, so that $p \mid B$, contrary to hypothesis.

From (10.4) and the definition of the numbers σ in §9, we see that $\sigma \mid \rho$. Since this argument applies to all of the prime ideal factors of p , the least common multiple of $\sigma_1, \dots, \sigma_l$ divides ρ . That is, by Lemma 9.1, $\tau \mid \rho$. But ρ always divides $\tau[2]$. Hence $\rho = \tau$.

LEMMA 10.1. *If the number $B = B(u)$ is not zero, the rank of apparition of all save a finite number of primes in (u) is their restricted period.*

11. We now prove

THEOREM 11.1. *A sufficient condition that the number B be not zero is that the group of the characteristic polynomial of (u) be either alternating or symmetric.*

If B vanishes, one of the numbers of the set (10.2) vanishes. With a proper choice of notation we may assume that*

$$(11.1) \quad \beta_1 + \beta_2 + \dots + \beta_i = 0, \quad (k/2 \leq i \leq k).$$

We may also assume that $k > 4$, as the cases $k = 2, 3, 4$ may be easily discussed directly (see next theorem). Hence $i \geq 3$.

If we represent the Galois group $\mathfrak{G}$ of $f(x)$ as a permutation group upon the k roots $\alpha_1, \dots, \alpha_k$, then formula (10.1) shows that any permutation of the α induces the corresponding permutation upon the β . If $\mathfrak{G}$ is alternating or symmetric, it contains the permutation $S = (\alpha_1 \alpha_{i+1})(\alpha_2 \alpha_3)$. On applying S to (11.1), we obtain $\beta_{i+1} + \beta_2 + \beta_3 + \dots + \beta_i = 0$. Hence $\beta_1 = \beta_{i+1}$. Similarly, $\beta_2 = \beta_{i+1}, \dots, \beta_i = \beta_{i+1}$. Hence $\beta_{i+1} = 0$ contrary to (10.1).

The following result is proved by similar reasoning.

THEOREM 11.2. *For low orders of (u) , sufficient conditions that $B(u) \neq 0$ are as follows:*

Order of (u)	Condition of Galois group or characteristic polynomial
2, 3	none
4	order of group divisible by 3
5	$f(x)$ irreducible, or product of an irreducible quartic and linear factor
6, 7	group transitive and primitive.

* It will be recalled that $\beta_1 + \beta_2 + \dots + \beta_k = u_0 = 0$.

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