A NEW CLASS OF TRANSCENDENTAL NUMBERS*

BY

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1. **Introduction.** Every algebraic number except zero and unity (e^0), when raised to a power which is an irrational algebraic number gives a transcendental number. This was conjectured by Hilbert, and proposed as a problem in 1900.† It was only recently that Gelfond,‡ and independently at about the same time, Schneider,§ succeeded in proving this theorem. G. Ricci|| has shown that the transcendentality persists if the algebraic numbers are replaced by products of algebraic numbers and a restricted type of Liouville number. The question we consider here is the more general one where the algebraic numbers are replaced by numbers capable of approximation by sequences of algebraic numbers in an appropriate manner.

These form a restricted type of certain generalized Liouville transcendental numbers very closely related to those treated by Ore.¶ They bear a relation to the numbers of an algebraic field similar to that of the ordinary Liouville numbers to the rational numbers.

Our principal result is that if X and Y are each suitably approximable by sequences of algebraic numbers, then X^Y and $\log X/\log Y$ are transcendental. Some numerical examples are given in §7.

Though the transcendental numbers found by us have the power of the continuum, they form a set of zero measure as we show.

My interest in this subject was stimulated by a report on the Hilbert-Gelfond-Schneider theorem which I gave to the seminar of Professor H. Weyl at Princeton, and in particular by a conjecture of Professor J. von Neumann that natural generalizations of these results would involve approximations by algebraic numbers, rather than by rational numbers. I also wish to acknowledge my appreciation of the opportunity to work without distraction on this field, made possible by the courtesy of the Massachusetts Institute of Technology and the Institute for Advanced Study.

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† For reference to the earlier work on this and related questions concerning transcendental numbers, the reader may consult J. F. Koksma, *Diophantische Approximationen*, Ergebnisse der Mathematik, vol. 4, No. 4, Berlin, 1936, pp. 58–65.

‡ A. Gelfond, Bulletin de l'Académie des Sciences de l'U.R.S.S., vol. 7 (1934), p. 623.

§ T. Schneider, Crelle's Journal, vol. 172 (1934), p. 65.

|| G. Ricci, Annali delle R. Scuola Normale Superiore di Pisa, vol. 4 (1935), p. 341.

¶ O. Ore, Avhandlinger, Norske Videnskaps-Akademie, Oslo, Matematisk-naturvidenskapelig Klasse, vol. 1 (1925), p. 11.

2. **The lemmas.** We shall adopt the following notation for our lemmas. K_ν will mean an algebraic field of degree ν, fixed throughout. For a concise presentation of the concepts and theory on algebraic fields used here, see, e.g., Landau, *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale*, Leipzig, 1927. The symbols α, β, and η will denote algebraic numbers belonging to this field. We will attach subscripts to indicate that we have an infinite sequence of such numbers. However, the conjugates of these numbers in K_ν will all be uniformly bounded. Since, except for repetitions, a number has the same conjugates in any field containing it, the specification of K_ν here is not essential. We indicate this by writing

$$\|\alpha\| < V, \quad \|\beta\| < V, \quad \|\eta\| < V. \tag{2.1}$$

Moreover, $|\log \beta|$ and its reciprocal will be uniformly bounded by V.

The letters c, q, s, t, S, T will denote positive integers. In particular, c is such that $c\alpha$, $c\beta$, and $c\eta$ are all algebraic integers. Usually s and t will take on the values $0, 1, 2, \cdots, S-1$ and $0, 1, 2, \cdots, T-1$, respectively, unless otherwise indicated.

We use γ or γ_i as a generic symbol for any positive quantity which depends only on K_ν and V.

Finally we define

$$g(x) = \sum_{k=0}^{q-1} \sum_{l=0}^{q-1} C_{kl} \alpha^{kx} \beta^{lx}, \tag{2.2}$$

where the C_{kl} are algebraic integers belonging to K_ν, more precisely specified later.

We also define

$$g_s(x) = (\log \beta)^s \sum_{k=0}^{q-1} \sum_{l=0}^{q-1} C_{kl}(k\eta + l)^s \alpha^{kx} \beta^{lx}. \tag{2.3}$$

It may be noticed that, if η were equal to $\log \alpha / \log \beta$, this would be the sth derivative of $g(x)$.

LEMMA I.* *If $q^2 \geqq 2ST$, there exists a $g(x)$ such that $g_s(t) = 0$, with coefficients C_{kl} not all zero satisfying the inequality*

$$\log \|C_{kl}\| < S[\log (\gamma c) + \log q] + 2qT \log (\gamma c). \tag{2.4}$$

To prove this, we note that the ST equations $g_s(t) = 0$ are linear in the q^2 variables C_{kl}, so that, if we denote the coefficients by a_{klst}, we may write them in the form

* Cf. Schneider, loc. cit., and C. Siegel, Abhandlungen der Preussischen Akademie der Wissenschaften, No. 1, 1929.

$$\sum a_{klst}C_{kl} = 0. \tag{2.5}$$

If we multiply the equations by a suitable power of c before doing this the coefficients a_{klst} will be algebraic integers in K_ν, and we shall assume this done. Furthermore, we may deduce bounds for the conjugates of these coefficients from those for α, β, and η. Thus:

$$\|a_{klst}\| < (\gamma_1 c)^{S+2q(T-1)}q^{S-1} \equiv \mu. \tag{2.6}$$

Now let ρ_i, $i=1, 2, \cdots, \nu$ denote an integral basis for K_ν. Suppose that $\|\rho_i\| < R$. Put

$$y_{st} = \sum a_{klst}x_{kl}, \tag{2.7}$$

and

$$x_{kl} = \sum_{i=1}^{\nu} B_{kli}\rho_i. \tag{2.8}$$

If the B_{kli} take on the values $0, \pm 1, \pm 2, \cdots, \pm h$, it follows that

$$\|x_{kl}\| < \nu Rh \equiv \gamma_2 h. \tag{2.9}$$

Hence

$$\|y_{st}\| < q^2\mu\gamma_2 h. \tag{2.10}$$

Next we put

$$y_{st} = \sum_{i=1}^{\nu} B_{sti}\rho_i. \tag{2.11}$$

We may write similar equations, with the same B_{sti} but with y_{st} and ρ_i replaced by their conjugates in K_ν. These may be solved, since the ρ_i formed a basis, and the determinant of them and their conjugates depends essentially only on K_ν. Thus we will have

$$|B_{sti}| < q^2\mu\gamma_3 h \equiv \delta. \tag{2.12}$$

We now observe that there are $(2h+1)^{\nu q^2}$ choices of B_{kli} giving rise to sets of y_{st}, and there are only $(2\delta+1)^{\nu ST}$ choices of B_{sti} giving distinct sets of y_{st}. Hence two sets of y_{st} arising from two distinct sets of x_{kl} must agree if

$$(2h + 1)^{\nu q^2} > (2\delta + 1)^{\nu ST}, \tag{2.13}$$

or, since $\nu q^2 \geqq 2\nu ST$, if

$$(2h + 1)^2 > 4h^2 + 1 > 2\delta + 1. \tag{2.14}$$

That is, if

$$4h^2 > 2h\gamma_3(\gamma_1 c)^{S+2q(T-1)}q^{S+1}, \tag{2.15}$$

or

$$2h > \gamma_3(\gamma_1 c)^{S+2q(T-1)}q^{S+1}. \tag{2.16}$$

But, on taking the difference of these two sets of x_{kl}, we will have a set of C_{kl} of the kind required by the lemma, with

$$\|C_{kl}\| < 2\|x_{kl}\| < 2\gamma_2 h. \tag{2.17}$$

That is, not all zero solutions exist with

$$\|C_{kl}\| < (\gamma_4 c)^{S+2qT}q^S. \tag{2.18}$$

This proves Lemma I.

The bound for the coefficients readily leads to a bound for the function, expressed in

LEMMA II.* *The* $g_s(x)$ *of Lemma* I *satisfies the inequality*

$$\log |g_s(x)| < \log \|C_{kl}\| + \gamma q|x| + (s+2)\log q + \gamma s. \tag{2.19}$$

This is proved directly from the definition of $g_s(x)$, which gives

$$|g_s(x)| < \|C_{kl}\|\gamma_1^{s+2qx}q^{s+2}|\log\beta|^s. \tag{2.20}$$

A lower bound for the functions $g_s(t)$ is given by

LEMMA III.* *Except when* $g_s(t)=0$, *it satisfies the inequality*

$$\log |g_s(t)| \geqq -\gamma\log\|C_{kl}\| - \gamma(s+2)\log q - \gamma qt - \gamma s\log c - \gamma qt\log c. \tag{2.21}$$

This lemma is proved by consideration of the algebraic integer

$$c^{s+2qt}g_s(t)(\log\beta)^{-s} = \sum_{k=0}^{q-1}\sum_{l=0}^{q-1} C_{kl}c^{s+2qt}(k\eta+l)^s\alpha^{kt}\beta^{lt}. \tag{2.22}$$

As bounds for the conjugates of this number we have

$$\|c^{s+2qt}g_s(t)(\log\beta)^{-s}\| < q^2c^{s+2qt}\|C_{kl}\|\gamma^{s+2qt}q^s. \tag{2.23}$$

Since the norm of an algebraic integer not zero must be at least unity in absolute value, it follows from this that, unless $g_s(t)=0$,

$$|g_s(t)c^{s+2qt}(\log\beta)^{-s}|\cdot\|c^{s+2qt}g_s(t)(\log\beta)^{-s}\|^{\nu-1} \geqq 1. \tag{2.24}$$

Hence we have

* Cf. Gelfond, loc. cit.

$$(2.25) \qquad |g_s(t)| \geqq \|C_{kl}\|^{-\nu+1}[c^{s+2qt}]^{-\nu}[\gamma^{s+2qt}]^{-\nu+1}[q^{s+2}]^{-\nu+1}|\log\beta|^{s\nu},$$

which establishes Lemma III.

Our next lemma concerns the polynomial of interpolation determined by its value and that of its derivatives at certain places. Specifically it reads:

LEMMA IV.* *If $S>T$, and $|P^s(t)|<M$, the polynomial of degree at most $ST-1$ determined by the ST values $P^s(t)$, when $|x|\geqq T$, satisfies the inequality*

$$(2.26) \qquad \begin{aligned} |P(x)| < M \exp\,&[-ST\log T + 2ST + 2S\log S] \\ &\cdot |x(x-1)\cdots(x-T+1)|^S. \end{aligned}$$

Let us first derive some formulas concerning the polynomial $P(x)$ of degree at most n, determined by the values of it and its first s_i-1 derivatives at the points a_i, where $\sum_{i=1}^{\tau} s_i = n+1$. We write

$$(2.27) \qquad \phi(x) = (x-a_1)^{s_1}(x-a_2)^{s_2}\cdots(x-a_\tau)^{s_\tau} = (x-a_i)^{s_i}/\psi_i(x).$$

Then

$$(2.28) \qquad P(x) = \sum_{i=1}^{\tau}\left[\frac{\sum_{k=0}^{s_i-1} A_k(x-a_i)^k}{(x-a_i)^{s_i}}\right]\phi(x)$$

is a polynomial of degree at most n. Also, for $x=a_i$, $k=0, 1, \cdots, s_i-1$, we find that

$$(2.29) \qquad \frac{d^k}{dx^k}\left[\frac{P(x)}{\phi(x)}\cdot(x-a_i)^{s_i}\right]_{a_i} = \frac{d^k}{dx^k}\left[\sum_{k=0}^{s_i-1}A_k(x-a_i)^k\right]_{a_i} = k!A_k.$$

Hence we have

$$(2.30) \quad A_k k! = \frac{d^k}{dx^k}\,[P(x)\cdot\psi_i(x)]_{a_i} = \sum_{a=0}^{k}\frac{k!}{a!(k-a)!}P^a(a_i)\psi_i^{k-a}(a_i),$$

and

$$(2.31) \quad P(x) = \left[\sum_{i=1}^{\tau}\sum_{k=0}^{s_i-1}\sum_{a=0}^{k}\frac{P^a(a_i)}{a!}\frac{\psi_i^{k-a}(a_i)}{(k-a)!}(x-a_i)^{k-s_i}\right]\phi(x).$$

But, if the prime indicates that $j\neq i$, $\psi_i(x)=\prod'(x-a_j)^{-s_j}$, and

$$(2.32) \qquad \begin{aligned} \frac{d^h}{dx^h}\psi_i(x) &= h!\sum_{\Sigma l_j=h}\prod{}'\frac{1}{l_j!}\frac{d^{l_j}}{dx^{l_j}}(x-a_j)^{-s_j} \\ &= h!(-1)^h\sum_{\Sigma l_j=h}\prod{}'\frac{(s_j+l_j-1)!}{l_j!(s_j-1)!}(x-a_j)^{-s_j-l_j}. \end{aligned}$$

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* Cf. Ricci, loc. cit.; Hermite, Crelle's Journal, 1878, pp. 70–79.

Hence, putting $b=s_i-1-k$, a, b and the l_j all run from 0 to s_i-1, subject only to the condition that $a+b+\sum l_j=s_i-1$, and we have finally,

$$\frac{P(x)}{\phi(x)}=\sum_{i=1}^{r}\sum_{a+b+\Sigma l_j=s_i-1}(x-a_i)^{-b-1}\frac{P^a(a_i)}{a!} \cdot\prod{}'\frac{(s_j+l_j-1)!}{l_j!(s_j-1)!}(a_i-a_j)^{-s_j-l_j}(-1)^{\Sigma l_j}. \tag{2.33}$$

We now specialize the values by putting $a_1, a_2, \cdots, a_r=0, 1, 2, \cdots, T-1$ and $s_1=s_2=\cdots=s_r=S$. We also require $|x|\geqq T$. Then we have

$$|x-a_i|\geqq 1 \quad\text{and}\quad |(x-a_i)^{-b-1}|\leqq 1. \tag{2.34}$$

Again

$$a!\geqq 1 \quad\text{and}\quad \frac{1}{a!}\leqq 1. \tag{2.35}$$

Next we notice that

$$\prod{}'|a_i-a_j|^{-s_j-l_j}\leqq\prod{}'|a_i-a_j|^{-S}. \tag{2.36}$$

When T is odd, $T_1=2m+1$, the least $\prod'|a_i-a_j|$ is $m!\,m!$, but when T is even, $T_2=2m$, the least $\prod'|a_i-a_j|$ is $m!(m-1)!$. But*

$$\log x!=(x+\tfrac{1}{2})\log x-x+\log(2\pi)^{1/2}+\frac{\theta}{12x} \qquad (0<\theta<1), \tag{2.37}$$

or

$$\begin{aligned}\log x! &= \log\frac{(x+1)!}{x+1}\\ &=(x+\tfrac{1}{2})\log(x+1)-x-1+\log(2\pi)^{1/2}+\frac{\theta}{12(x+1)}.\end{aligned} \tag{2.38}$$

For the odd case, we obtain from the second expression (2.38):

$$\begin{aligned}2\log m! &= (2m+1)\log(2m+2)-(2m+1)\log 2-2m-2\\ &\quad+2\log(2\pi)^{1/2}+\frac{\theta}{6\,(m+1)}>(2m+1)\log(2m+1)\\ &\quad-4m-2\geqq T_1\log T_1-2T_1.\end{aligned} \tag{2.39}$$

For the even case, using both expressions (2.37) and (2.38):

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* See, for example, Whittaker and Watson, *Modern Analysis*, 4th edition, Cambridge, 1927, p. 251 ff., §12.33.

$$
(2.40)\quad \begin{aligned} \log m! + \log (m-1)! &= 2m \log 2m - 2m \log 2 - 2m + 2 \log (2\pi)^{1/2} + \frac{\theta}{6m} \\ &> 2m \log 2m - 4m \geqq T_2 \log T_2 - 2T_2. \end{aligned}
$$

This shows that in all cases

$$
(2.41)\quad \begin{aligned} &\prod{}' \mid a_i - a_j \mid > \exp [T \log T - 2T] \qquad \text{and} \\ &\prod{}' \mid a_i - a_j \mid^{-S} < \exp [- ST \log T + 2ST]. \end{aligned}
$$

Again, since after the first term the numerators are all less than $2S$, and the denominators are greater than or equal to 2, we have

$$
(2.42)\quad \left| \frac{(s_j + l_j - 1)!}{l_j!(s_j - 1)!} \right| = \left| \frac{s_j}{1} \cdot \frac{s_j + 1}{2} \cdots \frac{s_j + l_j - 1}{l_j} \right| < S^{l_j}.
$$

In consequence of this we have

$$
(2.43)\quad \left| \prod{}' \frac{(s_j + l_j - 1)!}{l_j!(s_j - 1)!} \right| < S^{\Sigma l_j} \leqq S^{S-1}.
$$

To estimate the number of terms, we note that i has T values, while a, b and the $(T-1)$ l_j have at most S values. This gives TS^{T+1} as an upper limit to the number of terms. Consequently, by combining all our appraisals, we find that

$$
(2.44)\quad \left| \frac{P(x)}{\phi(x)} \right| < [\max \mid P^a(a_i) \mid] \exp [- ST \log T + 2ST] TS^{S+T}.
$$

Since $S > T$, and both are integers, $S \geqq T+1$, so that

$$
(2.45)\quad TS^{S+T} < S^{S+T+1} \leqq S^{2S}.
$$

Recalling that M is a bound for $|P^s(t)|$, or $|P^a(a_i)|$, we have

$$
(2.46)\quad \begin{aligned} \mid P(x) \mid < M \exp [&- ST \log T + 2ST + 2S \log S] \\ &\cdot \mid x(x-1) \cdots (x - T + 1) \mid^S, \end{aligned}
$$

which is the inequality of the lemma.

Our final lemma is Darboux's mean value theorem* for functions of a complex variable.

Lemma V. *For any function of a complex variable analytic on the straight line segment joining* z_1 *and* z_2, *we have*

$$
(2.47)\quad \left| \frac{f(z_2) - f(z_1)}{z_2 - z_1} \right| \leqq \max \mid f'(z) \mid ,
$$

the maximum meaning maximum on the segment.

* G. Darboux, Liouville, (3), vol. 2 (1876), p. 291.

The lemma follows at once from the fact that

$$f(z_2) - f(z_1) = \int_{z_1}^{z_2} f'(z)dz, \tag{2.48}$$

so that

$$|f(z_2) - f(z_1)| \leqq \max |f'(z)| \cdot |z_2 - z_1|. \tag{2.49}$$

3. **The general theorem.** Our principal theorem may be formulated as follows:

THEOREM I. *Let α_i, β_i, and η_i, irrational (in particular $\neq 0$ or ∞), be three sequences of algebraic numbers in a fixed field, K_ν, with uniformly bounded conjugates. Let c_i be a sequence of integers, becoming infinite, such that $c_i\alpha_i$, $c_i\beta_i$, and $c_i\eta_i$ are algebraic integers. Then, if the three sequences approach limits A, B, both distinct from zero and unity, and H in such a way that*

$$|A - \alpha_i|, \quad |B - \beta_i|, \quad |H - \eta_i| < c_i^{-(\log c_i)^k}, \tag{3.1}$$

and

$$H = \frac{\log A}{\log B}, \tag{3.2}$$

it is impossible to have $k > 6$.

We shall prove the theorem by assuming it false and deducing a contradiction. For simplicity of writing, we shall generally omit the subscript i on the terms of the sequences. Consider then a particular set α, β, η, and c. Writing, as usual, $[x]$ to denote the greatest integer contained in x, put

$$q = [(\log c)^\kappa], \quad S = [q^{5/3}], \quad T = [\tfrac{1}{2}q^{1/3}], \qquad \left(\frac{1}{\kappa} = \frac{1}{3} - 2\epsilon, \ \epsilon > 0\right). \tag{3.3}$$

Since these values are such that $q^2 > 2ST$, we may apply Lemma I to find a function

$$g(x) = \sum_{k=0}^{q-1}\sum_{l=0}^{q-1} C_{kl}\alpha^{kx}\beta^{lx} \tag{3.4}$$

with coefficients C_{kl} algebraic integers in K_ν, not all zero, and such that

$$g_s(t) = (\log \beta)^s \sum_{k=0}^{q-1}\sum_{l=0}^{q-1} C_{kl}(k\eta + l)^s\alpha^{kt}\beta^{lt} = 0 \tag{3.5}$$

for

$$s = 0, 1, 2, \cdots, S-1;\ t = 0, 1, 2, \cdots, T-1. \tag{3.6}$$

Furthermore, these coefficients will satisfy the inequality

$$\log \|C_{kl}\| < S[\log(\gamma c) + \log q] + 2qT\log(\gamma c)$$

$$< q^{5/3}[\log\gamma + (q+1)^{1/3-\epsilon} + \log q] \tag{3.7}$$

$$+ 2q\cdot\tfrac{1}{2}q^{1/3}\cdot[\log\gamma + (q+1)^{1/3-\epsilon}]$$

$$< q^{2-\epsilon} + o(q^{2-\epsilon}). \tag{3.8}$$

Here, as later, we write $o(\ )$ to mean terms of lower order in q than those written explicitly in the parentheses.

Next, using these same coefficients C_{kl}, form the function

$$f(x) = \sum_{k=0}^{q-1}\sum_{l=0}^{q-1} C_{kl}A^{kx}B^{lx}, \tag{3.9}$$

and construct

$$F(x) = f(x) - P(x), \tag{3.10}$$

with

$$F^s(t) = 0, \tag{3.11}$$

by subtracting off a polynomial with

$$P^s(t) = f^s(t). \tag{3.12}$$

We can show that this polynomial is small by Lemma IV, provided the $f^s(t)$ are small. These will be small if they approximate the $g_s(t)$, since the latter are zero. To guarantee this, we must at this point introduce an assumption as to the degree of approximation of α, β, and η to A, B, and H. Accordingly we put

$$\alpha - A = \Delta A,\quad \beta - B = \Delta B,\quad \eta - H = \Delta H, \tag{3.13}$$

and assume that

$$|\Delta A|,\ |\Delta B|,\ \text{and } |\Delta H| \text{ are all } < e^{-q^{7/3}}. \tag{3.14}$$

As a first step in estimating the size of $f^s(t)$, we consider the crude estimate obtained from Lemma II, which shows that

$$\log|f^s(t)| < \log\|C_{kl}\| + \gamma qt + (s+2)\log q + \gamma s \tag{3.15}$$

$$< q^{2-\epsilon} + o(q^{2-\epsilon}). \tag{3.16}$$

Since we wish to compare the polynomial

$$f^s(t) = \sum\sum C_{kl}(kH+l)^s(\log B)^s A^{kt}B^{lt}, \tag{3.17}$$

with

$$(3.18)\qquad G_s(t) = (\log B)^s g_s(t)(\log \beta)^{-s} = \sum \sum C_{kl}(k\eta + l)^s(\log B)^s \alpha^{kt}\beta^{lt},$$

let us consider a general polynomial in three complex variables x, y, and z with $x_2 - x_1 = \Delta x$, $y_2 - y_1 = \Delta y$, $z_2 - z_1 = \Delta z$,

$$(3.19)\qquad P(x, y, z) = \sum C_{pqr} x^p y^q z^r.$$

If C, P, Q, R, X, Y, and Z are upper bounds for C_{pqr}, p, q, r, x, y, and z respectively, and X, Y, and Z exceed unity, a crude estimate for the size of the polynomial, of the type just written for $f^s(t)$ is

$$(3.20)\qquad U = PQRCX^P Y^Q Z^R.$$

But, on applying Lemma V to an individual term three times we find that

$$(3.21)\qquad \begin{aligned} &| x_2{}^p y_2{}^q z_2{}^r - x_1{}^p y_1{}^q z_1{}^r | \\ &\quad < p\Delta X X^{p-1} Y^q Z^r + q\Delta Y X^p Y^{q-1} Z^r + r\Delta Z X^p Y^q Z^{r-1} \\ &\quad < X^P Y^Q Z^R (P\Delta X + Q\Delta Y + R\Delta Z). \end{aligned}$$

Consequently, we have

$$(3.22)\qquad | P(x_2, y_2, z_2) - P(x_1, y_1, z_1) | < U(P\Delta X + Q\Delta Y + R\Delta Z).$$

By reasoning of the same kind, we find that

$$(3.23)\qquad \begin{aligned} &| G_s(t) - f^s(t) | < U(qT\Delta A + qT\Delta B + S\Delta H) \\ &\qquad < \exp\,[q^{2-\epsilon} + o(q^{2-\epsilon})](q^{4/3} + q^{5/3})e^{-q^{7/3}}, \end{aligned}$$

by making use of the value of U, our crude estimate for $f^s(t)$, (3.16), the values of S and T and our restriction on ΔA, ΔB, and ΔH, (3.14).

But, since $g_s(t) = 0$, we also have $G_s(t) = 0$, so that the inequality just proved amounts to

$$(3.24)\qquad \log | f^s(t) | < - q^{7/3} + o(q^{7/3}),$$

which is the refined estimate for $f^s(t)$ we were seeking.

For the product which occurs in Lemma IV, since $T < q^{1/3}$, $S < q^{5/3}$, if we add the restriction $|x| \leqq q$, we have

$$(3.25)\qquad | x(x-1)\cdots(x-T+1) |^S < (2q)^{ST} < (2q)^{q^2} \leqq \exp q^2 \log 2q.$$

We may now apply Lemma IV, which, in view of the inequalities (3.24) and (3.25) gives

$$(3.26)\qquad \begin{aligned} \log | P(x) | &< - q^{7/3} + o(q^{7/3}) + 2q^2 + 4q^2 \log^2 q + q^2 \log 2q \\ &< - q^{7/3} + o(q^{7/3}). \end{aligned}$$

While the lemma, as written, required $|x| \geqq T$, as soon as the product is replaced by a quantity independent of x, this restriction is no longer essential, from the maximum properties of polynomials. Thus the relation just found holds for all $|x| \leqq q$.

Let us next consider $F(x)$ for values of $|x| \leqq q$. For these values we find from Lemma II, (2.19) with $s=0$,

$$\begin{aligned} \log |f(x)| &< \log \|C_{kl}\| + \gamma_1 q |x| + 2 \log q \\ &< \gamma q^2 + o(q^2), \end{aligned} \tag{3.27}$$

in view of (3.8). It follows at once from (3.10), (3.26), and (3.27) that

$$\log |F(x)| < \gamma q^2 + o(q^2). \tag{3.28}$$

If we write

$$\phi(x) = [x(x-1)\cdots(x-T+1)]^S, \tag{3.29}$$

in consequence of the way in which we constructed $F(x)$, all the zeros of $\phi(x)$ are zeros of $F(x)$ to a multiplicity at least as high. Consequently the function

$$E(x) = \frac{F(x)}{\phi(x)} \tag{3.30}$$

is an entire function. We compare its value for a fixed x_1, and x_2 varying on a circle such that

$$|x_1| = q^{1-2\epsilon}, \qquad |x_2| = q \qquad (\epsilon > 0). \tag{3.31}$$

From the relation

$$|E(x_1)| \leqq \max |E(x_2)|, \tag{3.32}$$

we deduce

$$\begin{aligned} |F(x_1)| &\leqq \max |F(x_2)\phi(x_1)/\phi(x_2)| \\ &< \exp[\gamma q^2 + o(q^2)] \frac{(2q^{1-2\epsilon})^{ST}}{(q/2)^{ST}} \\ &< \exp[-\epsilon q^2 \log q + o(q^2 \log q)]. \end{aligned} \tag{3.33}$$

Since the estimate for $P(x)$ in (3.26) is smaller than this, from (3.10) follows

$$\log |f(x_1)| < -\epsilon q^2 \log q + o(q^2 \log q). \tag{3.34}$$

We proceed from this to the derivatives by the Cauchy integral formula

$$f^s(t_1) = \frac{s!}{2\pi i} \int \frac{f(x)dx}{(x-t_1)^{s+1}}, \tag{3.35}$$

where the path of integration may be taken as a circle about the origin of radius $|x_1| = q^{1-2\epsilon}$. We find from this, (3.3), and (3.34) that

$$(3.36)\qquad |f^s(t_1)| < s^s |f(x_1)| < (q^{5/3})^{q^{5/3}} \exp\left[-\epsilon q^2 \log q + o(q^2 \log q)\right],$$

if $|t_1| < |x_1| - 1$. This will be true, for q large enough, if we require $|t_1| < q^{2/3}$. We are particularly interested in the values

$$(3.37)\qquad t_1 = 0, 1, 2, \cdots, T_1 - 1;\ T_1 = [q^{2/3-\epsilon}].$$

For these values, we have

$$(3.38)\qquad \log |f^s(t_1)| < -\epsilon q^2 \log q + o(q^2 \log q).$$

To go from this to $g_s(t_1)$ we must again use Lemma V. We begin by noting that the new value of T_1 has no effect on the crude estimate, (3.16), so that we still have for the value of U

$$(3.39)\qquad \log |f^s(t_1)| < \log U \leqq 2q^2 + o(q^2).$$

Consequently, we now have in place of (3.23)

$$(3.40)\qquad \begin{aligned} |G_s(t_1) - f^s(t_1)| &< U(qT_1\Delta A + qT_1\Delta B + S\Delta H) \\ &< \exp\left[2q^2 + o(q^2)\right](2q^{5/3} + q^{5/3})e^{-q^{7/3}}. \end{aligned}$$

Thus the analogue of (3.24) is now

$$(3.41)\qquad \log |G_s(t_1) - f^s(t_1)| < -q^{7/3} + o(q^{7/3}).$$

From this, and from (3.38), we find

$$(3.42)\qquad \log |G_s(t_1)| < -\epsilon q^2 \log q + o(q^2 \log q).$$

Since

$$(3.43)\qquad \log |g_s(t_1)| = \log |G_s(t_1)| + s \log |\log \beta| - s \log |\log B|,$$

we also have

$$(3.44)\qquad \log |g_s(t_1)| < -\epsilon q^2 \log q + o(q^2 \log q).$$

But, by Lemma III, unless $g_s(t_1) = 0$, it satisfies the inequality

$$(3.45)\qquad \begin{aligned} \log |g_s(t_1)| > &-\gamma \log \|C_{kl}\| - \gamma(s+2) \log q - \gamma q t_1 \\ &- \gamma s \log c - \gamma q t_1 \log c, \end{aligned}$$

so that, in view of (3.8), (3.3), and (3.37) we have

$$(3.46)\qquad \begin{aligned} \log |g_s(t_1)| &> -2\gamma q^2 - \gamma q^{5/3} \log q - \gamma q^{5/3} - \gamma q^2 - \gamma q^2 - o(q^2) \\ &> -\gamma q^2 - o(q^2). \end{aligned}$$

Since for sufficiently large q this is in contradiction with (3.44), by Lemma III, it proves that

$$g_s(t_1) = 0, \text{ for the } t_1 \text{ of } (3.37). \tag{3.47}$$

We may now repeat our procedure, making use of the additional zeros whose existence has just been established. That is, we construct

$$F_1(x) = f(x) - P_1(x), \tag{3.48}$$

with

$$F_1^s(t_1) = 0, \tag{3.49}$$

by subtracting off a polynomial with

$$P_1^s(t_1) = f^s(t_1). \tag{3.50}$$

As before, we wish to estimate the size of this polynomial by Lemma IV. We begin by using the vanishing of $g_s(t_1)$, and hence $G_s(t_1)$, in conjunction with Lemma V to get an estimate for $f^s(t_1)$. The application of Lemma V has already been made in (3.40), so that we have merely to set $G_s(t_1)=0$ in (3.41) to obtain

$$\log |f^s(t_1)| < -q^{7/3} + o(q^{7/3}). \tag{3.51}$$

We must next consider the product, and find as in (3.25) that

$$\begin{aligned} |x(x-1)\cdots(x-T_1+1)|^S &< (2q)^{ST_1} < (2q)^{7/3-\epsilon} \\ &< \exp[q^{7/3-\epsilon}\log 2q]. \end{aligned} \tag{3.52}$$

We are now in a position to apply Lemma IV, and find from it, using (3.51) and (3.52),

$$\begin{aligned} \log |P_1(x)| &< -q^{7/3} + o(q^{7/3}) + 2q^{7/3-\epsilon} \\ &\quad + 4q^2\log q + q^{7/3-\epsilon}\log 2q \\ &< -q^{7/3} + o(q^{7/3}). \end{aligned} \tag{3.53}$$

From this, (3.27), and (3.48), we conclude that for $|x| \leqq q$,

$$\log |F_1(x)| < \gamma q^2 + o(q^2). \tag{3.54}$$

If we write

$$\phi_1(x) = [x(x-1)\cdots(x-T_1+1)]^S. \tag{3.55}$$

we may now form the entire function

$$E_1(x) = \frac{F_1(x)}{\phi_1(x)}. \tag{3.56}$$

We compare its values for a fixed x_1, and x_2 varying on a circle where

$$|x_1| = q^{1-\epsilon}, \qquad |x_2| = q, \qquad (\epsilon > 0). \tag{3.57}$$

From

$$|E_1(x)| \leqq \max |E_1(x_2)|, \tag{3.58}$$

we deduce

$$\begin{aligned} |F_1(x_1)| &\leqq \max |F_1(x_2)\phi(x_1)/\phi_1(x_2)| \\ &\leqq \exp[\gamma q^2 + o(q^2)] \frac{(2q^{1-\epsilon})^{ST_1}}{(q/2)^{ST_1}} \\ &< \exp[-\epsilon q^{7/3-\epsilon} \log q + o(q^{7/3-\epsilon}\log q)]. \end{aligned} \tag{3.59}$$

As this exceeds the estimate for $P_1(x)$ in (3.53), it follows from (3.48) that

$$\log |f(x_1)| < -\epsilon q^{7/3-\epsilon}\log q + o(q^{7/3-\epsilon}\log q). \tag{3.60}$$

We use this to estimate the derivatives at the origin. We have from Cauchy's integral formula

$$f^s(0) = \frac{s!}{2\pi i}\int \frac{f(x)dx}{x^{s+1}}, \tag{3.61}$$

where the path of integration may be taken as a circle of radius $|x_1| = q^{1-\epsilon}$ with center at the origin. We find from this

$$|f^s(0)| < s^s |f(x_1)|. \tag{3.62}$$

If we take

$$s_1 = 0, 1, 2, \cdots, S_1 - 1, \qquad S_1 = [q^2 \log q], \tag{3.63}$$

we have

$$\log(s_1^{s_1}) < S_1 \log S_1 < q^2 \log q(2\log q + \log\log q). \tag{3.64}$$

Combining (3.60), (3.62), and (3.64) we have

$$\log |f^{s_1}(0)| < -\epsilon q^{7/3-\epsilon}\log q + o(q^{7/3-\epsilon}\log q). \tag{3.65}$$

We next apply Lemma V in the usual way to go from this to $g_{s_1}(0)$. We first get a value to serve as U from Lemma II, namely:

$$\log |g_{s_1}(0)| < q^2(\log q)^2 + o(q^2\{\log q\}^2). \tag{3.66}$$

We then use this, (3.43), (3.14), and the reasoning used for (3.23) to derive

$$\log |G_{s_1}(0) - f^{s_1}(0)| < -q^{7/3} + o(q^{7/3}). \tag{3.67}$$

From this, (3.43) and (3.65), we find that

$$\log |g_{s_1}(0)| < -\epsilon q^{7/3-\epsilon}\log q + o(q^{7/3-\epsilon}\log q). \tag{3.68}$$

But, by Lemma III, unless $g_{s_1}(0)=0$, it satisfies the inequality

$$\log |g_{s_1}(0)| > -\gamma q^{7/3-2\epsilon}\log q + o(q^{7/3-2\epsilon}\log q). \tag{3.69}$$

Since this is in contradiction with (3.68), when q is sufficiently large, it follows from Lemma III that

$$g_{s_1}(0) = 0. \tag{3.70}$$

But from the expression for S_1 in (3.63), we see that for q sufficiently large, S_1 exceeds q^2. For such a value of q, we may regard the equations

$$g_{s_1}(0) = 0 \tag{3.71}$$

as a system of linear equations in the q^2 variables,

$$u_{kl} = \alpha^{k0}\beta^{l0} = 1. \tag{3.72}$$

Eventually, $\alpha\neq 0$, $\beta\neq 0$, since otherwise $H=\infty$ excluded, or $H=0$ excluded by the conditions on A and B. Some of the coefficients C_{kl} may be zero. In this case we omit the corresponding terms and variables u_{kl}. The coefficients C_{kl} are known to be not all zero. If there are Q_1 distinct from zero, the values

$$s = 0, 1, 2, \cdots, Q_1 - 1 \tag{3.73}$$

gives a system of Q_1 equations of the form

$$\sum\sum C_{kl}v_{kl}^{s}u_{kl} = 0, \tag{3.74}$$

where

$$v_{kl} = (k\eta + l). \tag{3.75}$$

Since the u_{kl} are not all zero, the determinant of their coefficients must vanish. On factoring out the C_{kl}, none of which vanish, there is left a Vandermonde determinant, which reduces to

$$\prod_{i,j} (v_{k_il_i} - v_{k_jl_j}) = 0. \tag{3.76}$$

Thus, for some pair of subscripts (k_il_i), (k_jl_j) distinct from one another, we must have a vanishing factor,

$$k_i\eta + l_i - (k_j\eta + l_j) = 0. \tag{3.77}$$

This is impossible, since the coefficients are integers, and we have expressly ruled out the possibility $\eta=0$, ∞, or a rational number.

This contradiction has been derived from the assumptions that

$$(3.78) \qquad q = [(\log c)^k], \quad \frac{1}{k} = \frac{1}{3} - 2\epsilon, \text{ and } |\Delta A|, \ |\Delta B|, \ |\Delta H| < e^{-q^{7/3}},$$

for indefinitely large q. The assumption of the theorem was that

$$(3.79) \qquad |\Delta A|, \ |\Delta B|, \ |\Delta H| < c^{-(\log c)^k}, \qquad k > 6.$$

If we put

$$(3.80) \qquad k = 6 + \eta,$$

and define

$$(3.81) \qquad q = [(\log c)^\kappa], \quad \frac{1}{\kappa} = \frac{1}{3} - 2\epsilon, \text{ so that } \log c \geqq q^{1/\kappa},$$

it follows that

$$(3.82) \qquad \log \left\{c^{-(\log c)^k}\right\} = -(\log c)^{7+\eta} \leqq -q^{(7+\eta)(1/3-2\epsilon)} < -q^{7/3}$$

provided that ϵ is sufficiently small. That is, when (3.79) hold for a particular k, a suitable ϵ and q can be found such that the proof as given leads to a contradiction. This proves the theorem.

COROLLARY. *If any two of the three numbers A, B, and H satisfy the hypothesis of Theorem* I, *i.e., the statement as far as* (3.1), *and* $k>6$, *then the third number given by*

$$(3.83) \qquad H = \frac{\log A}{\log B}, \qquad A = B^H$$

is necessarily transcendental.

In particular, we note that one or both of the two numbers we started with may be actually algebraic.

4. **The first special theorem.** If some of our quantities are actually algebraic, instead of merely approximable by algebraic numbers, a slightly weaker condition suffices. Thus we have

THEOREM II. *Let α and β be fixed algebraic numbers both distinct from zero and unity. Let η_i, irrational* (*in particular not* 0 *or* ∞), *be a sequence of algebraic numbers in a fixed field K_ν with uniformly bounded conjugates. Let c_i be a sequence of integers, becoming infinite, such that c_i, η_i are algebraic integers. Then, if the sequence approaches a limit H in such a way that*

$$(4.1) \qquad |H - \eta_i| < c_i^{-(\log c_i)^k}$$

and

$$H = \frac{\log \alpha}{\log \beta}, \tag{4.2}$$

it is impossible to have $k > 4$.

We shall prove this theorem by the same method as that used for Theorem I. To facilitate reference to the earlier argument, we shall use similar numbers for corresponding equations. Thus these do not run consecutively. The omitted numbers correspond to such equations in §3 as we refer to, without repeating explicitly. Let then η and c be particular values, and put

$$q = [(\log c)^k], \quad S = [q^{3/2}], \quad T = [\tfrac{1}{2}q^{1/2}] \quad \left(\frac{1}{k} = \frac{1}{2} - 4\epsilon, \epsilon > 0\right). \tag{4.3}$$

Apply Lemma I to find a function $g(x)$, (3.4), with coefficients C_{kl} algebraic integers in K_ν, not all zero, and such that $g_s(t) = 0$, (3.5) and (3.6). Furthermore, the coefficients will now satisfy the inequality

$$\log \|C_{kl}\| < S[\log (\gamma c) + \log q] + 2qT \log \gamma. \tag{4.7}$$

To justify the omission of the term $2qT \log c$, we turn back to the proof of Lemma I, and note that this term entered because of the term $c^{2q(T-1)}$ in (2.6), necessitated by the multiplier which made $\alpha^{k(T-1)}$ and $\beta^{k(T-1)}$ algebraic integers. As the α and β are here fixed, so is the multiplier for them, and we may replace $c^{2q(T-1)}$ by $\gamma^{2q(T-1)}$, which may be incorporated with a term already present in (2.6). From (4.7) and (4.3) we conclude that

$$\log \|C_{kl}\| < q^2 + o(q^2). \tag{4.8}$$

We next work with $f(x)$, which is here

$$f(x) = g(x) = \sum_{k=0}^{q-1} \sum_{l=0}^{q-1} C_{kl} \alpha^{kx} \beta^{lx}. \tag{4.9}$$

However, since the derivatives $f^s(x)$ are different from $g_s(x)$, we must again construct $F(x)$ by using (3.10), (3.11), and (3.12).

With regard to the degree of approximation, we assume that for the ΔH defined in (3.13) we have

$$|\Delta H| < e^{-q^{5/2}}. \tag{4.14}$$

The application of Lemma II to obtain a crude estimate proceeds as before. From (3.15), (4.3), and (4.8) we have

$$\log |f^s(t)| < \log U \leqq q^2 + o(q^2). \tag{4.16}$$

For the comparison of

$$f^s(t) = \sum\sum C_{kl}(kH + l)^s(\log \beta)^s \alpha^{kt}\beta^{lt}, \tag{4.17}$$

with

$$G_s(t) = g_s(t) = \sum\sum C_{kl}(k\eta + l)^s \alpha^{kt}\beta^{lt}, \tag{4.18}$$

we use (3.22) to deduce

$$|\, g_s(t) - f^s(t) \,| < US\Delta H < \exp\,[q^2 + o(q^2)]q^{3/2}e^{-q^{5/2}}, \tag{4.23}$$

from which we obtain the refined estimate

$$\log\,|\, f^s(t)\,| < -\,q^{5/2} + o(q^{5/2}). \tag{4.24}$$

Since the values in (4.3) satisfy $ST < q^2$, the inequality (3.25) holds here, and we may combine it with (4.24) and Lemma IV to deduce

$$\log\,|\, P(x)\,| < -\,q^{5/2} + o(q^{5/2}) \text{ for } |\,x\,| \leqq q. \tag{4.26}$$

To proceed to $F(x)$, we first use (4.8) and Lemma II to deduce (3.27). Then this, (3.10), and (4.26) may be combined to give (3.28). From this, as before, (3.33) and, in view of (3.10) and (4.26), (3.34) follow. We then focus our attention on the values

$$t_1 = 0, 1, 2, \cdots, T_1 - 1;\ T_1 = [q^{1-3\epsilon}]. \tag{4.37}$$

For these values, using (3.35), we readily deduce (3.38). Then, with only slight modifications of the previous reasoning, we conclude that (3.44) holds.

We then apply Lemma III, which here becomes

If not 0, $g_s(t_1)$ *satisfies the inequality*

$$\log\,|\, g_s(t_1)\,| > -\,\gamma \log \|C_{kl}\| - \gamma(s+2)\log q - \gamma q t_1 - \gamma s \log c. \tag{4.45}$$

We justify the omission of the term $-\gamma q t_1 \log c$ by examining the proof of Lemma III. This term arises from c^{2qt} which first appears in (2.22). Here, since α and β are fixed, we may take a fixed multiplier for them, and so use γ^{2qt}, which in the final result replaces $-\gamma q t_1 \log c$ by $-\gamma q t_1$, which in turn may be incorporated with the term of this type already present.

From (4.45), in conjunction with (4.8), (4.3), and (4.37) we may conclude that (3.46) holds. As before, the contradiction of this and (3.44) proves that

$$g_s(t_1) = 0, \text{ for the } t_1 \text{ of } (4.37). \tag{4.47}$$

We now come to the second application of the process. We construct $F_1(x)$, defined by (3.48), to satisfy (3.49) and (3.50). In place of (3.51), we now have by similar reasoning

$$\log\,|\, f^s(t_1)\,| < -\,q^{5/2} + o(q^{5/2}). \tag{4.51}$$

For the product our new values give

$$(4.52)\qquad |\, x(x-1)\cdots(x-T_1+1)\,|^S < (2q)^{ST_1} < \exp\,[q^{5/2-3\epsilon}\log 2q].$$

These last two inequalities enable us to apply Lemma IV to obtain

$$(4.53)\qquad \log\,|\, P_1(x)\,| < -\,q^{5/2}+o(q^{5/2}).$$

From this, (3.27), and (3.48), we see that (3.54) again holds. The argument based on the zeros leads from this to the inequality

$$(4.59)\qquad \log\,|\, F_1(x_1)\,| < -\,\epsilon q^{5/2-3\epsilon}\log q + o(q^{5/2-3\epsilon}\log q),$$

and hence to

$$(4.60)\qquad \log\,|\, f(x_1)\,| < -\,\epsilon q^{5/2-3\epsilon}\log q + o(q^{5/2-3\epsilon}\log q).$$

Then using (3.61) and the argument which follows with the value of S_1 again given by (3.63), we find that

$$(4.65)\qquad \log\,|\, f^{s_1}(0)\,| < -\,\epsilon q^{5/2-3\epsilon}\log q + o(q^{5/2-3\epsilon}\log q).$$

Since (3.66) may be again used, and

$$(4.67)\qquad \log\,|\, g_{s_1}(0) - f^{s_1}(0)\,| < -\,q^{5/2}+o(q^{5/2}),$$

we may derive

$$(4.68)\qquad \log\,|\, g_{s_1}(0)\,| < -\,\epsilon q^{5/2-3\epsilon}\log q + o(q^{5/2-3\epsilon}\log q).$$

But the modified inequality of Lemma III given in (4.45) shows that

$$(4.69)\qquad \log\,|\, g_{s_1}(0)\,| > -\,\gamma q^{5/2-4\epsilon}\log q + o(q^{5/2-4\epsilon}),$$

unless $g_{s_1}(0)=0$. Since the last two inequalities are contradictory, we must have the second possibility, (3.70). From here on the earlier argument applies without change, and shows that the assumptions

$$(4.78)\qquad q = [(\log c)^k],\quad \frac{1}{k} = \frac{1}{2} - 4\epsilon,\quad \text{and}\quad |\,\Delta H\,| < e^{-q^{5/2}},$$

for indefinitely large q, lead to a contradiction.

The assumption of Theorem II was that

$$(4.79)\qquad |\,\Delta H\,| < c^{-(\log c)^k},\qquad k > 4.$$

If we put

$$(4.80)\qquad k = 4 + \eta,$$

and define

$$(4.81) \qquad q = [(\log c)^k], \; \frac{1}{k} = \frac{1}{2} - 4\epsilon, \quad \text{so that } \log c \geqq q^{1/k},$$

it follows that

$$(4.82) \qquad \log \left\{ c^{-(\log c)^k} \right\} = -(\log c)^{5+\eta} \leqq - q^{(5+\eta)(1/2-4\epsilon)} < - q^{5/2},$$

provided that ϵ is sufficiently small. That is, when (4.79) holds for a particular k, a suitable ϵ and q can be found such that proof leads to a contradiction. This proves Theorem II.

COROLLARY. *If β and H satisfy the hypothesis of Theorem* II, *i.e., the statement as far as* (4.1) *and $k>4$, then the number given by*

$$(4.83) \qquad \alpha = \beta^H$$

is necessarily transcendental.

5. **The second special theorem.** We next consider the case where the exponent is actually an algebraic number. The theorem here is

THEOREM III. *Let η be a fixed irrational algebraic number* (*in particular distinct from* 0 *and* ∞). *Let α_i and β_i be two sequences of algebraic numbers in a fixed field K_ν, with uniformly bounded conjugates. Let c_i be a sequence of integers, becoming infinite, such that $c_i\alpha_i$, $c_i\beta_i$ are algebraic integers. Then, if these sequences approach limits A and B, each distinct from zero and unity, in such a way that*

$$(5.1) \qquad |A - \alpha_i|, \; |B - \beta_i| < c_i^{-(\log c_i)^k}$$

and

$$(5.2) \qquad \eta = \frac{\log A}{\log B},$$

it is impossible to have $k>1$.

We again use the form of exposition of the preceding section. Let, then α, β, and c be particular values, and put

$$(5.3) \quad q = [(\log c)^k], \quad S = [q^{2-\epsilon}], \quad T = [\tfrac{1}{2}q^\epsilon], \quad \left(\frac{1}{k} = 1 - 2\epsilon, \epsilon > 0\right).$$

Then apply Lemma I to find a function $g(x)$ of (3.4), for which (3.5) and (3.6) hold. The inequality satisfied by the coefficients in this case may be written

$$(5.7) \qquad \log \|C_{kl}\| < S[\log \gamma + \log q] + 2qT \log (\gamma c).$$

Here we have omitted the term $S \log c$, since this comes from the term c^S in (2.6). As η is now fixed, we may take a fixed multiplier for it and so replace the term c^S by γ^S, which may be combined with the term of this type already present in (2.6).

It follows from (5.7) and (5.3) that

$$\log \|C_{kl}\| < q^2 + o(q^2). \tag{5.8}$$

We now form the function $f(x)$ of (3.9), and the $F(x)$ of (3.10), (3.11), and (3.12). This time we take as our assumption on the degree of approximation the following restriction on the size of the ΔA and ΔB defined by (3.13)

$$|\Delta A|, \quad |\Delta B| < e^{-q^{2+2\epsilon}}. \tag{5.14}$$

The crude estimate is obtained as before by applying Lemma II. Thus from (3.15), (5.3), and (5.8) we deduce

$$\log |f^s(t)| < \log U \leqq q^2 + o(q^2). \tag{5.16}$$

For the comparison of

$$f^s(t) = \sum \sum C_{kl}(k\eta + l)^s (\log B)^s A^{kt} B^{lt}, \tag{5.17}$$

with the $G_s(t)$ of (3.18), we use (3.22) to deduce

$$|G_s(t) - f^s(t)| < U(qT\Delta A + qT\Delta B) < e^{q^2 + o(q^2)} q^\epsilon e^{-q^{2+2\epsilon}}. \tag{5.23}$$

This leads to the refined estimate

$$\log |f^s(t)| < - q^{2+2\epsilon} + o(q^{2+2\epsilon}). \tag{5.24}$$

Since the values in (5.3) satisfy $ST < q^2$, the inequality (3.25) holds here, and may be combined with (5.24) and Lemma IV to yield

$$\log |P(x)| < - q^{2+2\epsilon} + o(q^{2+2\epsilon}), \quad \text{for } |x| \leqq q. \tag{5.26}$$

We proceed to $F(x)$, by first using (5.8) and Lemma II to deduce (3.27), and then combining this with (3.10) and (5.26) to give (3.28). From this, as before (3.33) and, in view of (3.10) and (5.26), (3.34) follow.

We then consider the particular values

$$t_1 = 0, 1, 2, \cdots, T_1 - 1; \; T_1 = [q^{2\epsilon}]. \tag{5.37}$$

For these values we may deduce (3.38) by using (3.35). Then, making a few obvious modifications in the earlier argument, we show that (3.44) holds.

Then we apply Lemma III. The inequality here becomes

$$\log |g_s(t)| > - \gamma \log \|C_{kl}\| - \gamma(s + 2) \log q - \gamma q t_1 - \gamma s - q t_1 \log c. \tag{5.45}$$

The only modification is the replacement of $-\gamma s \log c$ by the term $-\gamma s$.

This is permissible, since the multiplier for η is now fixed, and so the factor c^s of (2.22) may be replaced by γ^s. From (5.45), in conjunction with (5.8), (5.3), and (5.37) we may conclude that (3.46) holds. As before, the contradiction of this and (3.44) proves that

$$g_s(t_1) = 0, \text{ for the } t_1 \text{ of } (5.37). \tag{5.47}$$

We are now at that stage of the proof where our process is repeated. We construct $F_1(x)$, defined by (3.48), to satisfy (3.49) and (3.50). In place of (3.51), we now have, similarly,

$$\log |f^s(t_1)| < -q^{2+2\epsilon} + o(q^{2+2\epsilon}). \tag{5.51}$$

For the product of our new values gives

$$|x(x-1)\cdots(x-T_1+1)|^S < (2q)^{ST_1} < \exp[q^{2+\epsilon}\log 2q]. \tag{5.52}$$

These last two inequalities enable us to apply Lemma IV, and so obtain

$$\log |P_1(x)| < -q^{2+2\epsilon} + o(q^{2+2\epsilon}). \tag{5.53}$$

From this, (3.27), and (3.48), we see that (3.54) again holds. The argument on entire functions leads from this to

$$\log |F_1(x_1)| < -\epsilon q^{2+\epsilon}\log q + o(q^{2+\epsilon}\log q), \tag{5.59}$$

and hence to

$$\log |f(x_1)| < -q^{2+\epsilon}\log q + o(q^{2+\epsilon}\log q). \tag{5.60}$$

Then using (3.61) and the argument which follows with the value of S_1 again given by (3.63), we find that

$$\log |f^{s_1}(0)| < -\epsilon q^{2+\epsilon}\log q + o(q^{2+\epsilon}\log q). \tag{5.65}$$

Since we may again use (3.66), and

$$\log |G_{s_1}(0) - f^{s_1}(0)| < -q^{2+2\epsilon} + o(q^{2+2\epsilon}), \tag{5.67}$$

we may, using (3.43), derive the inequality

$$\log |g_{s_1}(0)| < -\epsilon q^{2+\epsilon}\log q + o(q^{2+\epsilon}\log q). \tag{5.68}$$

But the modified inequality of Lemma III given in (5.45) shows that

$$\log |g_{s_1}(0)| > -q^2(\log q)^2 + o(q^2\{\log q\}^2), \tag{5.69}$$

unless $g_{s_1}(0) = 0$. Since the last two inequalities contradict one another, we must have the second alternative, (3.70). From here on the earlier argument applies with no change, and shows that the assumptions

$$q = [(\log c)^k], \quad |\Delta A| \quad \text{and} \quad |\Delta B| < e^{-q^{2+2\epsilon}}, \tag{5.78}$$

for indefinitely large q, lead to a contradiction.

The assumption of Theorem III was that

$$|\Delta A|, \quad |\Delta B| < c^{-(\log c)^k}, \quad k > 1. \tag{5.79}$$

If we put

$$k = 1 + \eta, \tag{5.80}$$

and define

$$q = [(\log c)^k], \quad \frac{1}{k} = 1 - 2\epsilon, \quad \text{so that } \log c \geqq q^{1/k}, \tag{5.81}$$

it follows that

$$\log (c^{-(\log c)^k}) = -(\log c)^{2+\eta} \leqq -q^{(2+\eta)(1-2\epsilon)} < -q^{2+2\epsilon} \tag{5.82}$$

provided that ϵ is sufficiently small. That is, when (5.79) holds for a particular k, a suitable ϵ and q can be found such that the proof leads to a contradiction. This proves Theorem III.

COROLLARY. *If η and B satisfy the hypothesis of Theorem* III, *i.e., the statement as far as* (5.2) *and* $k>1$, *then the number given by*

$$A = B^\eta \tag{5.83}$$

is necessarily transcendental.

6. **Generalized Liouville numbers.** If the numbers approximable by algebraic numbers used in Theorems I, II, and III were algebraic, our theorems would merely be complicated restatements of that due to Gelfond and Schneider. That this is never the case, follows from the theorems of this section.

THEOREM IV. *Let α_i be a sequence of algebraic numbers all belonging to a fixed field K_ν, and with their conjugates uniformly bounded. Let an infinite number of them be distinct from one another. Let c_i and λ_i be two sequences of integers becoming infinite, such that the quantities $c_i\alpha_i$ are algebraic integers, and the numbers α_i approach a limit A in such a way that*

$$|\alpha_i - A| < \frac{1}{c_i^{\lambda_i}}. \tag{6.1}$$

Then A is a transcendental number.

For, suppose A were an algebraic number, and that

$$P(x) \equiv p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0 = 0 \tag{6.2}$$

were the irreducible equation it satisfied. Let V be the bound for $\|\alpha_i\|$ and $|A|$, and put $\sum_{i=0}^{n} p_i = s$. Write α for a particular α_i. Then, by Lemma V, we have

$$\left| \frac{P(A) - P(\alpha)}{A - \alpha} \right| \leqq P'(\xi) < nsV, \tag{6.3}$$

so that, since $P(A) = 0$, we have

$$| P(\alpha) | < | \alpha - A | \, nsV^{n-1}. \tag{6.4}$$

We next notice that

$$\|c^n P(\alpha)\| < c^n s V^n, \tag{6.5}$$

and since the norm of an algebraic integer not zero is numerically at least unity,

$$1 \leqq | c^n P(\alpha) | \, (\|c^n P(\alpha)\|)^{\nu-1} \leqq c^{n\nu} s^{(\nu-1)} V^{n(\nu-1)} \, | P(\alpha) |. \tag{6.6}$$

The possibility that $P(\alpha) = 0$ offers no difficulty, since we have assumed that there are an infinite number of distinct α_i. Consequently, we may omit those equal to A or to one of its conjugates, and still have an infinite sequence left to use in the proof.

Now from (6.4) and (6.6) we have

$$| \alpha - A | > \frac{1}{c^{n\nu} n s^{\nu} V^{n\nu-1}} > \frac{1}{\gamma_1 c^{\gamma_2}}. \tag{6.7}$$

In this relation α and A are two distinct non-conjugate algebraic numbers, and the γ depend only on A, the algebraic field K_ν containing α and the bounds for $\|\alpha\|$.

From this last result, Theorem IV follows at once, since (6.7) contradicts (6.1), if

$$c_i > \gamma_1 \quad \text{and} \quad \lambda_i > \gamma_2 + 1. \tag{6.8}$$

Like their subclass, the ordinary Liouville numbers, the generalized Liouville numbers do not constitute a large fraction of the transcendental. In fact we shall establish

THEOREM V. *The generalized Liouville numbers correspond to a set of points of zero measure in the complex plane, and those on the real axis approximable by real algebraic numbers have zero measure on the line.*

First, consider a fixed field K_ν, and the algebraic numbers in it, α, such that

$$\|\alpha\| < V. \tag{6.9}$$

If c is the smallest integer for which $c\alpha$ is an algebraic integer, and we write, as in Lemma I,

$$c\alpha = \sum_{j=1}^{\nu} B_j \rho_j, \tag{6.10}$$

it follows from $\|c\alpha\| < \|cV\|$ that

$$| B_i | < \gamma c V. \tag{6.11}$$

Consequently, in the field K_ν, subject to the bound V, the number of α is at most

$$(2\gamma_1 c V + 1)^\nu < \gamma_2 c^\nu, \quad \text{for a particular } c. \tag{6.12}$$

Next select a sequence of positive numbers ϵ_m, decreasing to zero, and a sequence of positive integers μ_m increasing to infinity, as m increases. Then for each value of α_i, whose multiplier is c_i we draw a circle about its representative point of radius ϵ_m/c_i^m. Call S_m the set of points inside any of these circles, and define the set T by

$$T = S_1 \cdot S_2 \cdot \cdots \cdot S_m \cdot \cdots . \tag{6.13}$$

For any m, the generalized transcendental number A, with bound V, satisfying (6.1) will be in S_m. For, since c_i and λ_i are becoming infinite, for a sufficiently large i, we shall have

$$c_i^{\lambda_i - m} > \frac{1}{\epsilon_m}, \quad \text{and hence } \frac{1}{c_i^{\lambda_i}} < \frac{\epsilon_m}{c_i^m}. \tag{6.14}$$

Thus T includes all such numbers A, related to K and V.

But the set T has a measure not exceeding that of S_m, and this last, by (6.12) and the definition of S_m, is at most

$$\sum_{c_i=1}^{\infty} \gamma_2 c_i^\nu \pi \left(\frac{\epsilon_m}{c_i^m}\right)^2 = \gamma_2 \pi \epsilon_m^2 \sum c_i^{\nu-2m}. \tag{6.15}$$

When $2m$ exceeds $\nu+2$, the series converges to a sum <2, so that the measure of S_m approaches zero with ϵ_m. Thus the measure of T is zero. Let V be a positive integer. Then there are only an enumerable number of choices for K_ν and V. Hence each generalized Liouville number is included in one of an enumerable number of sets, each of zero measure. This proves the first part of Theorem V.

The statement about the one-dimensional measure of the points on the real axis approximable by real algebraic numbers is proved similarly by enclosing them in intervals. In this case the series corresponding to that in (6.15) is $\sum c^{\nu-m}$, which converges when m exceeds $\nu+2$.

In view of our general theorems, the measure of the B^A is of interest. We have

THEOREM VI. *The set of numbers B^A, where A and B are generalized Liouville numbers with sequences α_i, β_i having a common c_i is of measure zero in the complex plane. The subset with α_i, β_i, $\beta_i^{\alpha_i}$ real has zero measure on the real axis.*

We begin by finding the relation between the degree of the approximation of A and B to that of the power. We have

$$B^A - \beta^\alpha = B^A - \beta^A + \beta^A - \beta^\alpha, \quad \text{and} \tag{6.16}$$

$$| B^A - \beta^A | \leqq \max | Az^{A-1} | \cdot | \Delta B |, \tag{6.17}$$

by Lemma V, where $\Delta B = \beta - B$.

If we restrict $|A|$, $|B|$ and $|\alpha_i|$, $|\beta_i|$ as well as their reciprocals, and $|\log B|$, $|\log \beta_i|$, to be all less than M, necessarily >1, we have from this

$$| B^A - \beta^A | < M^{M+2} | \Delta B |. \tag{6.18}$$

Again, by Lemma V,

$$| \beta^A - \beta^\alpha | \leqq \max | \beta^z \log \beta | \cdot | \Delta A |, \tag{6.19}$$

so that

$$| \beta^A - \beta^\alpha | < M^{M+1} | \Delta A |. \tag{6.20}$$

Next, assume that sequences c_i and λ_i exist, such that simultaneously

$$| \alpha_i - A | < \frac{1}{c_i'^{\lambda_i}} \quad \text{and} \quad | \beta_i - B | < \frac{1}{c_i'^{\lambda_i}}, \tag{6.21}$$

where the c_i' are multipliers for both α_i and β_i. Then, from (6.16), (6.18), (6.20), and (6.21) we deduce that

$$| B^A - \beta^\alpha | < M_1/c_i'^{\lambda_i} \leqq \frac{\epsilon_m}{c_i^m} \frac{M_1}{\epsilon_m c_i^{\lambda_i - m}}, \tag{6.22}$$

where M_1 is a bound depending only on M. For m, M, and hence M_1 fixed, when i becomes infinite, the second factor approaches zero, so that eventually,

$$| B^A - \beta^\alpha | < \epsilon_m/c_i'^m < \epsilon_m/c_i^m, \tag{6.23}$$

where c_i is the least multiplier for both α_i and β_i, and therefore less than or equal to c_i'.

This situation enables us to use the reasoning applied to prove the last theorem. We surround each point β^α with least common multiplier c_i for α_i

and β_i by a circle of radius ϵ_m/c_i^m, and call S_m the set of points inside any of these circles. The set T is then defined by

$$T = S_1 \cdot S_2 \cdot \cdots \cdot S_m \cdot \cdots . \tag{6.24}$$

The inequality (6.23) shows that the point B^A is in each of the sets S_m, and therefore in the set T. To estimate the measure of S_m we observe that c_i is at least as great as the multipliers for α_i or β_i individually, and hence by applying (6.12) to the number of α_i and β_i for a given c_i, we find that the number of points $\beta_i^{\alpha_i}$ for a given c_i is at most $\gamma c^{2\nu}$. Thus the measure of S_m is at most

$$\sum_{c_i=1}^{\infty} \gamma c_i^{2\nu} \pi \left(\frac{\epsilon_m}{c_i^m}\right)^2 \leqq \gamma\pi\epsilon_m^2 \sum c_i^{2\nu-2m}. \tag{6.25}$$

Since the series converges for ν greater than $m+1$, to a sum less than 2, the measure of S_m approaches zero with ϵ_m, and the measure of T is zero.

This shows that when the restriction involving M holds, and if M is taken integral, there are only an enumerable number of choices for M and K_ν, so that the set discussed in the theorem has been shown to be the sum of an enumerable number of sets, each of measure zero.

For the one-dimensional case, we use intervals in place of the circles.

It is worth noting that there are no theorems like those of this section on transcendentality and measure for numbers merely known to be approximable by sequences of numbers $\beta_i^{\alpha_i}$, without α_i and β_i separately approaching limits. The first fact is shown by the example

$$\alpha_i = -\frac{2^{1/2}}{c_i}, \qquad \beta_i = \frac{p_i}{c_i}, \qquad c_i = i,$$

where p_i is an integer so chosen that

$$\left| 2^{i/2^{1/2}} - \frac{p_i}{i} \right| < \frac{1}{i}, \text{ so that } \beta_i = 2^{i/2^{1/2}}\left(1 + \frac{\theta}{i2^{i/2^{1/2}}}\right) \text{ with } |\theta| < 1.$$

Under these conditions we have

$$\left| \tfrac{1}{2} - \beta_i^{\alpha_i} \right| < \frac{1}{c^{\lambda_i}}, \quad \text{where} \quad \lambda_i = \left[\frac{i \log 2}{2^{1/2} \log i}\right]$$

and so becomes infinite with i. Similar sequences may be constructed with any number in place of $\frac{1}{2}$, so that they do not lead to sets of zero measure.

7. **Conclusions and examples.** A simple example of an ordinary Liouville number is obtained by using a series

$$\sum_{n=1}^{\infty} \frac{p_n}{N_n}, \tag{7.1}$$

where the p_n are uniformly bounded integers, and the N_n are integral values increasing with n sufficiently fast. In particular, by taking the p_n less than 10, and the $N_n = 10^{n!}$, Liouville found one whose decimal expansion could readily be written down. In place of this, we might use

$$N_n = 10^{(10^n \log n)} \quad \text{or} \quad a^{(b^E)} \tag{7.2}$$

where a and b are fixed integers greater than unity, and E is a function of n increasing faster than n, i.e., such that $E/n \to \infty$.

If we take as the p_n algebraic integers from a fixed field K_ν, with uniformly bounded conjugates, and again use (7.1) and (7.2), we obtain simple examples of the generalized transcendental numbers of Theorem IV.

To obtain simple examples of the restricted generalized Liouville numbers used in Theorems I, II, and III, we use (7.1) with the p_n algebraic integers from a fixed field K_ν, with uniformly bounded conjugates, and put

$$N_n = 10^{(10^{(10^n)})} \quad \text{or} \quad a^{(b^{(K^n)})}, \tag{7.3}$$

where in the second form, a, b, and K are fixed integers, a and b exceed unity, and K exceeds $k+1$, for the k of the condition (3.1).

Suppose, to be specific, we put x_i equal to the square root of the digit in the ith place of the decimal part of π, and y_i equal to the square root of the digit in the ith place of the decimal part of e. Then write

$$X = \sum_{n=1}^{\infty} x_n 2^{-(2^{(8^n)})}, \qquad Y = \sum_{n=1}^{\infty} y_n 2^{-(2^{(8^n)})}. \tag{7.4}$$

Then, by the corollary to Theorem I, we may assert the transcendentality of the following numbers

$$2^X,\ X^2,\ X^Y,\ \frac{\log X}{\log Y},\ \frac{\log X}{\log 2}. \tag{7.5}$$

The first of these also illustrates the corollary to Theorem II, and would continue to do so if we replaced the 8 in (7.4) by 6. The second of the numbers is an illustration of the corollary to Theorem III, and in this case we could replace the 8 in (7.4) by 3.

We note that if we take the x_n all equal to the 2, and the y_n all equal to 3 in (7.4), the transcendentality of the first three numbers in (7.5) would follow from the theorems of G. Ricci.

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