ON A CLASSIFICATION OF INTEGRAL FUNCTIONS BY MEANS OF CERTAIN INVARIANT POINT PROPERTIES. A SUPPLEMENT*

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In a recent paper† Carmichael, Martin, and Bird noted that the existence or non-existence of their α-sequences had not been established. The purpose of this note is to establish the impossibility of α-sequences. In other words, Theorem I of the paper cited above may be replaced by the following theorem:

In order that $\{t_n\}$ *shall be an I-sequence it is necessary that the following condition shall be satisfied*:

$$\lim_{n=\infty} (t_n)^{1/n} = \infty .$$

It is sufficient for the proof of the theorem to show that the inferior limit of $(t_n)^{1/n}$ cannot be zero. For this purpose we assume that the inferior limit of $(t_n)^{1/n}$ is zero and show that a contradiction arises.

The positive integers can be arranged in ascending order in two infinite sequences $\{m_i\}$ and $\{n_i\}$ such that

$$t_{m_i} > 1, \quad i = 1, 2, 3, \cdots ; \qquad t_{n_i} \leqq 1, \quad i = 1, 2, 3, \cdots .$$

Let β, γ be a pair of integers such that

$$\gamma \geqq \beta \geqq 2.$$

We wish to show that an infinite subsequence $\{\nu_i\}$ of the sequence $\{n_i\}$ exists such that for an infinite subsequence $\{\mu_i\}$ of the sequence $\{m_i\}$ we have

$$\beta\mu_i \leqq \nu_i \leqq \gamma^2\mu_i, \qquad i = 1, 2, 3, \cdots.$$

We take ν_1 to be the least member of the sequence $\{n_i\}$ such that

$$\beta m_1 \leqq \nu_1.$$

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† Carmichael, Martin, and Bird, *On a classification of integral functions by means of certain invariant point properties*, these Transactions, vol. 40 (1936), pp. 462–473.

Let us write ν_1 in the form $\nu_1 = x\gamma^2 m_1$. If x does not exceed 1 we take μ_1 to be equal to m_1. If x exceeds 1 we define μ_1 as the greatest integer which does not exceed $x\gamma m_1$. In either case μ_1 is a member of the sequence $\{m_i\}$ such that we have

$$\beta\mu_1 \leqq \nu_1 \leqq \gamma^2\mu_1.$$

We take m_j' to be the least member of the sequence $\{m_i\}$ such that

$$\beta m_j' > \nu_{j-1}, \qquad j = 2, 3, 4, \cdots.$$

We define ν_j and μ_j with respect to m_j' in the same manner that we defined ν_1 and μ_1 with respect to m_1. In this way we define the monotonically increasing subsequences $\{\nu_i\}$ and $\{\mu_i\}$ of the sequences $\{n_i\}$ and $\{m_i\}$, respectively, which are such that

$$\beta\mu_i \leqq \nu_i \leqq \gamma^2\mu_i, \qquad i = 1, 2, 3, \cdots.$$

We proceed to show that the function defined by the series

$$E_1(x) = \sum_{i=0}^{\infty} x^{n_i},$$

which converges for $|x| < 1$, disputes the relation (1) of the paper cited above. We observe

$$\limsup_{n=\infty} |t_n E_1^{(n)}(0)/n!|^{1/n} \leqq 1.$$

Furthermore, we have

$$E_1^{(\mu)}(a)/\mu! \geqq a^{\nu-\mu} C_{\nu,\mu}, \qquad 0 < a < 1,$$

where ν and μ are corresponding members of the sequences $\{\nu_i\}$ and $\{\mu_i\}$, respectively.

It is easy to prove the inequality

$$C_{\nu,\mu} \geqq \frac{1}{\nu+1}\left(\frac{\nu}{\nu-\mu}\right)^{\nu}\left(\frac{\nu-\mu}{\mu}\right)^{\mu}.$$

This leads us to the inequality

$$E_1^{(\mu)}(a)/\mu! \geqq \frac{1}{\nu+1}\left(\frac{a\nu}{\nu-\mu}\right)^{\nu}\left(\frac{\nu-\mu}{a\mu}\right)^{\mu}.$$

Let us take a to be equal to $1-\gamma^{-2}$. Then we have the inequalities

$$t_\mu E_1^{(\mu)}(a)/\mu! \geqq (\gamma^2\mu + 1)^{-1}(\beta - 1)^{\mu} a^{-\mu} \geqq (\gamma^2\mu + 1)^{-1} a^{-\mu}.$$

Hence we have

$$\liminf_{i=\infty} \left| t_{\mu_i} E^{(\mu_i)}(a)/\mu_i! \right|^{1/\mu_i} \geqq a^{-1} > 1$$

and, consequently,

$$\limsup_{n=\infty} \left| t_n E^{(n)}(a)/n! \right|^{1/n} > \limsup_{n=\infty} \left| t_n E^{(n)}(0)/n! \right|^{1/n}.$$

This contradicts relation (1) of the paper cited above and leads us to conclude that the limit of $(t_n)^{1/n}$ must be infinite.

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