ON THE SPACE* (BV)

BY

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1. **Introduction.** For the space (AC) of functions $x(t)$ absolutely continuous on the interval $0 \leqq t \leqq 1$ it is natural, since (AC) then becomes isomorphic with the space (L) of Lebesgue integrable functions, to employ the metric

$$(x, y) = | x(0) - y(0) | + T_0^1(x - y), \tag{1}$$

where $T_0^1(x-y)$ stands for the total variation of $x(t)-y(t)$ on $0 \leqq t \leqq 1$. Until recently the few writers† who have considered the space (BV) of functions of bounded variation have simply carried over to this larger space the metric of (AC). Although with this metric (BV) is a Banach space, it is not separable; perhaps it is partly on that account that (BV) was so little studied.

In I Adams introduced for (BV) the new metric

$$(x, y) = \int_0^1 | x(t) - y(t) | \, dt + | T_0^1(x) - T_0^1(y) |, \tag{2}$$

giving reasons for this choice. These reasons will not be repeated here, but by way of partial motivation one might make two observations: (i) an arbitrary function in (AC) can be approximated (arbitrarily closely) in the metric (1) by an inscribed polygonal function, but no function in the class $(BV)-(AC)$ can be so approximated;‡ (ii) the metric (2) permits such approximation to an arbitrary function in (BV). Hereinafter, unless otherwise specified, (BV) shall always be understood to be metrized with (2); although not then a Banach space, it is boundedly compact and contains a countable dense set of polygonal functions (see I).

Presumably the most important subsets of (BV) are (CBV), the set of continuous functions of bounded variation, and (AC). With the metric (1), (AC) is a closed linear manifold in (CBV) which in turn is a like manifold in (BV), so it may be inferred at once that (AC) is of first category in (CBV) and of second category in itself, while (CBV) is of first category in (BV) and of second category in itself. When the metric (2) is employed, however, the

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† For references see Adams, *The space of functions of bounded variation and certain general spaces*, these Transactions, vol. 40 (1936), pp. 421–438; later referred to as I.

‡ For an indication of this fact see Adams and Lewy, *On convergence in length*, Duke Mathematical Journal, vol. 1 (1935), pp. 19–26; especially p. 23. This paper will be referred to as AL.

questions of category appear to be less trivial, and one of the main objects of this paper is to answer them.

As a preliminary we set forth in §2 a lemma on semi-continuity of the functional $T(x)$. In §1 of I we compared several kinds of convergence and indicated by example that certain kinds do not imply certain others;* in each case the example was one in which the limit function has an external saltus. That only such an example will serve this purpose will appear among the consequences of the lemma in §2, and of an analogous lemma in §5. In §3 we consider the questions of category just mentioned. With (DBV) designating the set of discontinuous functions of bounded variation and (CS) the set of continuous singular functions,† the critical issues in §3 are met by showing that (DBV) is the sum of sets each closed and non-dense in (BV) and that $(CBV)-(CS)$ is the sum of sets each closed and non-dense in (CBV). In §4 we determine a base for (BV). Finally, in §5, we examine similar questions which arise when $T(x)$ is replaced by $L(x)$, the Peano length of $x(t)$.

2. **Semi-continuity of $T(x)$, with applications.** A short time ago Adams and Clarkson‡ observed the rather obvious but none the less fundamental semi-continuity relation: $\lim_{n\to\infty} x_n(t)=x(t)$ *everywhere on*§ $(0, 1)$ *implies*|| $\liminf_{n\to\infty} T(x_n) \geqq T(x)$. Although it was not expressly so stated, this result holds for $T(x)=\infty$ as well as for $T(x)<\infty$. Our first object here is to obtain a similar result when $x_n(t)$ is assumed to converge to $x(t)$ only on a dense set in $0\leqq t\leqq 1$.

To avoid possible misunderstanding let us first set up the

DEFINITION. An arbitrary function $x(t)$ will be said to have *no external saltus* if and only if at each point t_1, $0\leqq t_1\leqq 1$, $x(t_1)$ satisfies the condition $\liminf_{t\to t_1} x(t) \leqq x(t_1) \leqq \limsup_{t\to t_1} x(t)$.

The class of functions having no external saltus will be designated by (N); its intersection with (BV), by (BVN). Clearly $x(t)\varepsilon(BVN)$ implies continuity of $x(t)$ at $t=0$ and $t=1$.

LEMMA 1. *The relation* $x_n(t)\to x(t)$ *on a dense set in* $(0, 1)$, *with* $x(t)\varepsilon(BVN)$, *implies* $\liminf_{n\to\infty} T(x_n) \geqq T(x)$.

The reader should have little difficulty in establishing this lemma.

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* See p. 423 of I.

† See, for example, Saks, *Théorie de l'Intégrale*, Warsaw, 1933, pp. 11 ff.

‡ Adams and Clarkson, *On convergence in variation*, Bulletin of the American Mathematical Society, vol. 40 (1934), pp. 413–417; to be referred to as AC.

§ The notation (a, b) for an interval will always mean the *closed* interval $a\leqq t\leqq b$.

|| In general we employ the symbol $T_a^b(x)$ for the total variation of $x(t)$ on $a\leqq t\leqq b$; in particular the limits a, b will be omitted when it seems that no confusion can arise therefrom.

Although we do not intend to make use of it here, we may assert for the case $T(x)=\infty$

LEMMA 1'. *The relation* $x_n(t)\to x(t)$ *on a dense set in* $(0, 1)$, *when* $T(x)=\infty$ *but* $x(t)\varepsilon(N)$ *and the set E of its discontinuities of the second kind is either vacuous or possesses an isolated point,* implies* $\lim_{n\to\infty}T(x_n)=\infty$.

Proof. For Case 1, E vacuous, a proof can be constructed essentially in the same manner as for Lemma 1. In Case 2, E not vacuous, let t_1 be an isolated point of E. If for example $x(t_1-0)$ fails to exist, there must be an interval (t_0, t_1) which contains no discontinuity of the second kind other than t_1; and $T_{t_0}^{t_1}(x)=\infty$. If $T_{t_0}^{t_2}(x)=\infty$ for any t_2, $t_0<t_2<t_1$, the conclusion follows from Case 1; otherwise, for arbitrary M there exists t_2 such that $T_{t_0}^{t_2}(x)>M$ and the conclusion is a consequence of Lemma 1.

That Lemma 1 fails to hold when $x(t)$ is permitted to have an external saltus at even a single point is clear from the simplest examples. That Lemma 1' ceases to be valid when E has no isolated point is shown by $x_n(t)=0$ for all t, all n; $x(t)$ the characteristic function of the rationals.

The two following lemmas are immediate consequences.

LEMMA 2. *The relation* $\int_0^1|x_n(t)-x(t)|\,dt\to 0$, *with* $x(t)\varepsilon(BVN)$, *implies* $\liminf_{n\to\infty}T(x_n)\geqq T(x)$.

Proof. If the conclusion were false there would exist $\epsilon>0$ and a subsequence $x_p(t)$ such that $T(x_p)<T(x)-\epsilon$ for all p. This subsequence, however, must contain a subsequence $x_q(t)$ convergent almost everywhere to $x(t)$; hence by Lemma 1 we have $\liminf_{q\to\infty}T(x_q)\geqq T(x)$, a contradiction.

LEMMA 2'. *The relation* $\int_0^1|x_n(t)-x(t)|\,dt\to 0$, *when* $T(x)=\infty$ *but* $x(t)$ *is qualified as in Lemma* 1', *implies* $\lim_{n\to\infty}T(x_n)=\infty$.

DEFINITION. We shall employ $x_n(t)-dv\to x(t)$ as an abbreviation for the two conditions: $x_n(t)\to x(t)$ on a dense set in $(0, 1)$, and $T(x_n)\to T(x)$.

THEOREM 1. *The relation* $x_n(t)-dv\to x(t)$ *on* $(0, 1)$ $[(x_n, x)\to 0]$, *with* $x(t)\varepsilon(BVN)$, *implies that relation† for every subinterval whose end-points are points of continuity of* $x(t)$.

Proof. We need only show that $T_0^a(x_n)\to T_0^a(x)$ whenever a is a point of continuity of $x(t)$. By Lemma 1 [Lemma 2] we have $\liminf T_0^a(x_n)\geqq T_0^a(x)$, $\liminf T_a^1(x_n)\geqq T_a^1(x)$; and $\limsup T_0^a(x_n)>T_0^a(x)$ would imply

* From the proof it is evident that this lemma can be strengthened a little by considering right- and left-hand limits.

† We repeat that (x, y) always refers to the metric (2) unless otherwise specified.

$$\lim T_0^1(x_n) \geqq \limsup T_0^a(x_n) + \liminf T_a^1(x_n) > T_0^a(x) + T_a^1(x) = T_0^1(x),$$

in contradiction to hypothesis.

That this theorem breaks down for subintervals having an end-point at a discontinuity of $x(t)$ is apparent from simple examples.

In I we remarked that "if $(x_n, x)\rightarrow 0$ and $x(t)$ is discontinuous at even a single point, $x_n(t)$ may fail to converge to $x(t)$ at every point." This can happen, however, only when $x(t)$ has an external saltus, as we see from

THEOREM 2. *The relation* $x_n(t)-dv\rightarrow x(t)$ *on* $(0, 1)$ $[(x_n, x)\rightarrow 0]$, *with* $x_0(t)\varepsilon(BVN)$, *implies* $x_n(t)\rightarrow x(t)$ *at every point of continuity of* $x(t)$.

Proof. Let a be any such point of continuity. No loss of generality results from assuming $a>0$; for if $a=0$, we can deal with the interval $(0, 1)$ as we now deal with $(0, a)$. Denial of the conclusion implies the existence of $\epsilon>0$ and a subsequence $x_p(t)$ such that $|x_p(a)-x(a)|>\epsilon$ for all p. But there exists $\delta>0$ such that

$$|x(t) - x(a)| \leqq T_t^a(x) = T_0^a(x) - T_0^t(x) < \epsilon/2 \text{ for } a - \delta < t < a;$$

and $a-\delta<t<a$ contains a point t_1 where $x(t)$ is continuous. Hence, in view of Theorem 1, we have $T_{t_1}^a(x_p)\rightarrow T_{t_1}^a(x)<\epsilon/2$. But the interval $t_1<t<a$ contains a point t_2 for which $x_p(t_2)\rightarrow x(t_2)$, and we have for p sufficiently large

$$T_{t_1}^a(x_p) \geqq T_{t_2}^a(x_p) \geqq |x_p(t_2) - x_p(a)| > \epsilon/2.$$

From this contradiction the theorem is to be inferred.

COROLLARY 1. *For* $x(t)\varepsilon(BVN)$, *the relations* $x_n(t)-dv\rightarrow x(t)$ *and* $(x_n, x)\rightarrow 0$ *are equivalent.*

That the first condition implies the second is seen by aid of Theorem 2, the uniform boundedness of $x_n(t)$, and Lebesgue's convergence theorem. The reverse implication is an immediate consequence of Theorem 2.

DEFINITION. For $x(t)\varepsilon(BV)$ we define an *associated function* $\bar{x}(t)$ thus: $\bar{x}(t)=x(t+0)$ for $0\leqq t<1$, $\bar{x}(1)=x(1-0)$.

DEFINITION. For $x(t)\varepsilon(BV)$ we define the *sum of its external saltuses* as $S(x)=T(x)-T(\bar{x})$.

If at an interior point t_1 of $(0, 1)$ the function $x(t)$ has an external saltus, the magnitude of the "external part" of the saltus (or briefly, the external saltus) is commonly defined* as

$$\min \left[|x(t_1) - x(t_1 - 0)|, \ |x(t_1) - x(t_1 + 0)| \right].$$

* See Hobson, *The Theory of Functions of a Real Variable*, 3d edition, vol. 1, Cambridge, 1927, p. 334.

Using two results given by Hobson* one may show without difficulty that

$$S(x) = 2\sum(\text{external saltuses at interior points})$$
$$+ \mid x(0) - x(0+0) \mid + \mid x(1) - x(1-0) \mid .$$

Theorem 3. *The relation* $(x_n, x)\to 0$, *with* $x(t)\varepsilon(BVN)$, *implies* $(\bar{x}_n, x)\to 0$ *and* $S(x_n)\to 0$.

Proof. Since $\bar{x}_n(t) = x_n(t)$ almost everywhere, we have $\int_0^1 | \bar{x}_n(t) - x(t) | dt \to 0$; whence, by Lemma 2, lim inf $T(\bar{x}_n) \geqq T(x)$. But from $T(x_n) = T(\bar{x}_n) + S(x_n) \to T(x)$, $S(x_n) \geqq 0$, we infer $T(\bar{x}_n) \to T(x)$ and $S(x_n) \to 0$.

That $(x_n, x)\to 0$ with $x(t)\varepsilon(BV)-(BVN)$ in general fails to imply $S(x_n)\to S(x)$ is evident from simple examples.

Theorem 4. *The relation* $(x_n, x)\to 0$, *with* $S(x_n) \geqq k > 0$ *for all* n, *implies* $S(x) \geqq k$.

Proof. As in the proof of Theorem 3 we have lim inf $T(\bar{x}_n) \geqq T(\bar{x})$. But the relations

$$T(x) = T(\bar{x}) + S(x) = \lim T(x_n) \geqq \liminf T(\bar{x}_n) + \liminf S(x_n)$$
$$\geqq \liminf T(\bar{x}_n) + k$$

then imply $k \leqq S(x) + T(\bar{x}) - \liminf T(\bar{x}_n) \leqq S(x)$.

3. **Questions of category.** Let us recall from I that when $(x, y) = 0$ we say that x is *metrically equal* to y and write $x \cong y$. An element of the metric space (BV) is therefore a class of functions any two of which are metrically equal; for $x\varepsilon(CBV)$ the class consists of only one function, but for $x\varepsilon(DBV)$ it contains an infinite number of functions. Since category is essentially a metric property, it seems desirable now to consider (BV) merely as a *metric* space rather than as a *vector metric* space, the elements of which are single functions (see I).

From the definition of $\bar{x}(t)$ given above in §2 it is easily seen that if two functions $x(t)$, $y(t)$ satisfy the condition $(x, y) = 0$, then $\bar{x}(t) \equiv \bar{y}(t)$ on $(0, 1)$ and $S(x) = S(y)$. Hence it is proper to set up the following

Definition. If $x\varepsilon(BVN)$ and $x(t)$ is an arbitrary function of the class constituting x, $\bar{x}(t)$ will be called the *representative of the element* x. If $x\varepsilon(DBV)-(BVN)$ and $x(t)$ is an arbitrary function of the class constituting x, any function $x(t) = \bar{x}(t)$ for $0 \leqq t \leqq 1$, $t \neq t_1$; $x(t_1) = \bar{x}(t_1) + k$, where t_1 is any point of $(0, 1)$ and k is so chosen that $|x(t_1) - x(t_1 - 0)| + |x(t_1) - x(t_1 + 0)|$ exceeds the saltus of $\bar{x}(t)$ at t_1 by $S(x)$, will be called a *representative of* x.

Lemma 3. *If* $(x_n, x)\to 0$ *and for each* n *some representative* $x_n(t)$ *of* x_n *has a*

* Hobson, loc. cit., p. 335.

saltus $\geqq k>0$ *at one or more points of* $(0, 1)$, *then some representative* $x(t)$ *of* x *will have at one or more points a saltus* $\geqq k$.

Proof. Let t_n be a point where $x_n(t)$ has a saltus $\geqq k$; let l be a limit point of the sequence t_n; and let the subsequence $t_p \to l$. In case $S(x)>0$ let $x(t)$ be the representative of x obtained by taking t_1 in the foregoing definition as l. Then, δ being positive but arbitrarily small, set $a_\delta = \underset{t}{E}(l-\delta \leqq t \leqq l+\delta)\cdot \underset{t}{E}(0 \leqq t \leqq 1)$ and designate by b_δ the closure of the complement of a_δ with respect to $(0, 1)$. Since b_δ consists either of one closed subinterval of $(0, 1)$, or of two disjoint intervals of this nature, on each of which $x(t)$ has no external saltus, we have $\liminf_{p\to\infty} T_{b_\delta}(x_p) \geqq T_{b_\delta}(x)$ by Lemma 1. Hence the relation

$$\limsup T_{a_\delta}(x_p) + \liminf T_{b_\delta}(x_p) \leqq \lim \left[T_{a_\delta}(x_p) + T_{b_\delta}(x_p)\right] = \lim T_0^1(x_p)$$
$$= T_0^1(x) = T_{a_\delta}(x) + T_{b_\delta}(x)$$

implies

$$\limsup T_{a_\delta}(x_p) \leqq T_{a_\delta}(x) + T_{b_\delta}(x) - \liminf T_{b_\delta}(x_p) \leqq T_{a_\delta}(x).$$

But since $t_p \to l$ we have $\limsup T_{a_\delta}(x_p) \geqq k$; whence $T_{a_\delta}(x) \geqq k$ for all $\delta>0$ and $\lim_{\delta\to 0} T_{a_\delta}(x) \geqq k$. And this last limit is precisely the saltus of $x(t)$ at l.

THEOREM 5. (DBV) *is the sum of a countable number of sets each closed and non-dense in* (BV).

Proof. Let E_m be the set of points $x \varepsilon (DBV)$ each of which has a representative $x(t)$ possessing at some point a saltus $\geqq 1/m$. Clearly $(DBV) = \sum_{m=1}^{\infty} E_m$. By Lemma 3 each E_m is closed in (BV). That each E_m is non-dense in (BV) now follows from the fact (see I, §1) that (CBV) is dense in (BV).

Since (BV), being complete (see I, §1), is of second category in itself, (CBV) is of second category in (BV), and we have in the terminology of Denjoy

COROLLARY 2. (CBV) *is a residual set in* (BV).

COROLLARY 3. (CBV) *is of second category in itself.*

COROLLARY 4. (DBV) *is of first category in itself.*

Otherwise, for the decomposition $(DBV) = \sum_{m=1}^{\infty} E_m$ employed in proving Theorem 5, the closure in (DBV) of at least one set E_m would contain a sphere $K(x_0, r) \subset (DBV)$. Since (DBV) is dense in (BV), the closure in (BV) of this set E_m would then contain the corresponding sphere $K_1(x_0, r) \subset (BV)$, thus yielding a contradiction.

We next consider the set (CS) of continuous singular functions; i.e., of functions $x(t)\varepsilon(CBV)$ for which the derivative $x'(t)$ vanishes almost everywhere* on the interval $(0, 1)$.

THEOREM 6. *$(CBV)-(CS)$ is the sum of a countable number of sets each closed and non-dense in (CBV).*

Proof. For brevity we call the sum of a finite number of disjoint closed intervals $a_i \leqq t \leqq b_i$ $(i=1, 2, \cdots, p)$ an elementary figure, designate such a set generically by R and its measure by $|R|$, and let $T_R(x)$ stand for $\sum_{i=1}^{p} T_{a_i}^{b_i}(x)$.

For each integer $m>0$ we define the subset A_m of (AC) as follows: $\alpha(t)\varepsilon A_m$ if and only if we have

$$T_0^1(\alpha) \geqq 1/m, \qquad \text{and} \qquad T_R(\alpha) \leqq \tfrac{1}{2} T_0^1(\alpha)$$

for every elementary figure $R \subset (0, 1)$ with $|R| \leqq 1/m$. Then the subset E_m of $(CBV)-(CS)$ we define thus: $x(t)\varepsilon E_m$ if and only if

$$\alpha(t)\varepsilon A_m, \qquad \text{where} \qquad \alpha(t) = \int_0^t x'(s)ds.$$

Denoting by $\overline{E}_m$ $(m=1, 2, 3, \cdots)$ the closure of E_m in (CBV), we propose to show that

$$(CBV) - (CS) = \sum_{m=1}^{\infty} \overline{E}_m.$$

It is clear that each point of $(CBV)-(CS)$ belongs to E_m for some m. From the fact (see I) that the polygonal functions are dense in (CBV), it is easily apparent that (CS) is likewise. Therefore if for some $m>0$, $\overline{E}_m$ were not in $(CBV)-(CS)$, there would exist a point $y\varepsilon(CS)$ and a sequence $x_n\varepsilon E_m$ such that $(x_n, y)\to 0$. Let $\alpha_n(t) = \int_0^t x_n'(s)ds$ and $\beta_n(t) = x(t) - \alpha_n(t)$. The relation $y\varepsilon(CS)$ implies the existence of an elementary figure R such that

$$T_R(y) \geqq T_0^1(y) - \frac{1}{4m}, \qquad |R| \leqq \frac{1}{m}.$$

In view of the corollary to Theorem 2 of AC and of the fact that $T(x_n) = T(\alpha_n) + T(\beta_n)$ on every elementary figure, we should then have

$$\begin{aligned} 2T_0^1(y) - \frac{1}{2m} \leqq 2T_R(y) &= \lim 2T_R(x_n) = \lim\, [2T_R(\alpha_n) + 2T_R(\beta_n)] \\ &\leqq \liminf\, [T_0^1(\alpha_n) + 2T_0^1(\beta_n)] \end{aligned}$$

* See, for example, Saks, loc. cit. If one chooses to exclude the functions $x(t)=$const. from the set (CS), the proof of Theorem 6 can easily be modified to show that $(CBV)-(CS)$ is still the sum of non-dense closed sets in (CBV).

$$\leqq \limsup \left[2T_0^1(\alpha_n) + 2T_0^1(\beta_n)\right] - \liminf T_0^1(\alpha_n)$$

$$= \limsup 2T_0^1(x_n) - \liminf T_0^1(\alpha_n) \leqq 2T_0^1(y) - \frac{1}{m},$$

a contradiction which implies the decomposition above stated. That $\overline{E}_m$ $(m=1, 2, 3, \cdots)$ is non-dense in (CBV) follows from the fact that (CS) is dense in (CBV).

From Theorem 6 and Corollary 3 follows

COROLLARY 5. *(CS) is a residual set in (CBV).*

COROLLARY 6. *(CS) is of second category in itself.*

An immediate consequence of Theorem 6 is

COROLLARY 7. *(AC) is of first category in (CBV).*

Since (AC) is dense in (CBV), reasoning of the sort used in proving Corollary 4 may be employed to prove

COROLLARY 8. *(AC) is of first category in itself.*

A complete classification of functions of bounded variation according to external saltus, continuity, absolute continuity, and singularity leads to a decomposition of (BV) into the following seven disjoint sets each of which is easily seen to be dense in (BV): $(DBV)\cdot(S)\cdot(N)$, $(DBV)\cdot(BV-S)\cdot(N)$, $(DBV)\cdot(S)\cdot(BV-BVN)$, $(DBV)\cdot(BV-S)\cdot(BV-BVN)$, (AC), (CS), and $(CBV-AC-CS)$. Combining these sets in all possible ways one obtains 126 distinct proper subsets of (BV) concerning each of which several questions of category and Borel character can be raised. Many of these combinations appear decidedly artificial;* nevertheless it may be worthwhile to point out that all questions of category concerning them can now be answered. For in proving Corollary 4 we have essentially established the lemma: $Q \subset R$ and Q dense in R imply that the category of Q in R is the same as the category of Q in itself. By repeated application of this lemma we infer from the above results that (CS) is a residual set in (BV). Hence *any one of the* 126 *subsets which contains [does not contain] (CS) is of second [first] category in (BV), in itself, and in any other of the subsets in question of which it is a part.*

That (AC) is not a G_δ in (CBV) may readily be inferred; but whether it is an F_σ remains among the numerous open questions concerning Borel character. We may remark, however, that not only (CBV) but also (CS) is a G_δ in

* Curiously enough, our first application of the results of this paper will involve one of these "artificial" combinations.

(BV) and is therefore homeomorphic to a complete metric space, according to a theorem of Alexandroff.*

4. **Bases.** A sequence of elements x_i of a vector metric space S is called a base for S if and only if to an arbitrary element $x \varepsilon S$ there corresponds a unique sequence of constants c_i such that $(\sum_{i=1}^{n} c_i x_i, x) \to 0$ with $1/n$. For the space (C) of continuous functions Schauder† has constructed a base consisting of a sequence of polygonal functions. His argument makes clear that *this sequence also constitutes a base for both* (AC) *and* (CBV) with our metric; that it is *not* a base for either (BV) or (BVN), however, may readily be seen.

In contrast with the point of view in §3, we must now regard (BV) and (BVN) as *vector metric* spaces, whose elements are single functions. As a first step toward the determination of a base for (BV) we shall prove that Haar's system of orthogonal functions‡ serves as a base for (BVN). We recall that each Haar function $x_i(t)$ is continuous at the end points of $(0, 1)$ and that otherwise $x_i(t) = [x_i(t-0) + x_i(t+0)]/2$, whence $x_i \varepsilon (BVN)$ and any linear combination $\sum_{i=1}^{n} c_i x_i \varepsilon (BVN)$.

Now let x be an arbitrary element of (BVN). The function $x(t)$ is one of a set which constitutes an element of the space (L) of Lebesgue integrable functions. Since the Haar system provides a base for (L), there exists a unique sequence of constants c_i satisfying the condition, $\int_0^1 |\sum_{i=1}^{n} c_i x_i - x| \, dt \to 0$ with $1/n$; and these c_i are the coefficients in the Fourier expansion of $x(t)$ in the Haar functions.§ But the partial sum $s_n(t)$ of this expansion is a step-function which, on each of its intervals of constancy, is the mean value of $x(t)$ on that interval.‖ From the following lemma the fact that $T(s_n) \to T(x)$, whence $(\sum_{i=1}^{n} c_i x_i, x) \to 0$, is to be inferred.

LEMMA 4. *If $x(t) \varepsilon (BVN)$ and α_{pq} is any set of numbers satisfying the conditions $m_{pq} \leqq \alpha_{pq} \leqq M_{pq}$ $(p = 1, 2, \cdots, 2^q; q = 0, 1, 2, \cdots)$, where*

$$\left.\begin{aligned} m_{pq} &= \inf\ x(t) \\ M_{pq} &= \sup\ x(t) \end{aligned}\right\} \quad \textit{for } (p-1)/2^q \leqq t \leqq p/2^q;$$

then $\sum_{p=1}^{2^q} |\alpha_{p+1,q} - \alpha_{pq}| \to T_0^1(x)$ as $q \to \infty$.

This is a consequence of a theorem of Hobson, loc. cit., p. 332.

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* See, for example, Kuratowski, *Topologie* I, Warsaw, 1933, p. 200.

† Schauder, *Zur Theorie stetiger Abbildungen in Funktionalräumen*, Mathematische Zeitschrift, vol. 26 (1927), pp. 47–65; especially pp. 48–49.

‡ Haar, *Zur Theorie der orthogonalen Funktionensysteme*, Mathematische Annalen, vol. 69 (1910), pp. 331–371; especially pp. 361 ff.

§ See Schauder, loc. cit., pp. 50–51.

‖ See Haar, loc. cit., p. 367.

If to Haar's system of orthogonal functions we add the function $x_0(t)=0$ *for* $0\leqq t<1$, $=1$ *for* $t=1$, *we obtain a base for* (BV). For an arbitrary function $x(t)\varepsilon(BV)$ is metrically equal to $\bar{x}(t)+s(t)$, where $s(t)=0$ for $0\leqq t<1$, $=S(x)$ for $t=1$. From the above reasoning it is clear that constants c_i $(i=1, 2, 3, \cdots)$ can be uniquely determined so that $(\sum_{i=1}^{n} c_i x_i,\ \bar{x})\rightarrow 0$ with $1/n$, and from the relation $T(\sum_{i=0}^{n} c_i x_i)=T(\sum_{i=1}^{n} c_i x_i)+T(c_0 x_0)=T(\sum_{i=1}^{n} c_i x_i)+c_0$ it follows that with and only with $c_0=S(x)$ shall we have $(\sum_{i=0}^{n} c_i x_i,\ x)\rightarrow 0$. The uniqueness of the entire sequence c_i $(i=0, 1, 2, \cdots)$ may now be deduced without difficulty.

The question as to whether (BV), or perhaps more likely (BVN), has a base consisting wholly of continuous functions remains open.

5. **Concerning the functional $L(x)$.** We now consider the question of analogues of the above results when $T(x)$ is replaced by $L(x)$, the Peano length of $x(t)$, and the metric (2) by

$$(x,\ y)=\int_0^1 |\ x(t)-y(t)\ |\ dt+|\ L_0^1(x)-L_0^1(y)\ |. \tag{3}$$

The reader should have no considerable difficulty in establishing for $L(x)$ the analogue of Lemma 1, which provides a basis for drawing conclusions parallel to those described in §§2, 3 prior to Theorem 6. Although the employment of the metric (3) leaves unaltered the category and Borel properties set forth in Theorem 5 and its corollaries, it does effect a change on the subsets of (CBV). Employing as in the proof of Theorem 6 the notation

$$x_n=\alpha_n+\beta_n,\quad x=\alpha+\beta;\quad \alpha_n,\ \alpha\varepsilon(AC);\quad \beta_n,\ \beta\varepsilon(CS),$$

we have

Lemma 5. *The relation* $(x_n,\ x)\rightarrow 0$, *with* $T(\beta_n)\geqq k>0$ *for all* n, *implies* $T(\beta)\geqq k$.

Proof. According to I, §1 we have in the notation of "convergence in length" $x_n-l\rightarrow x$, $\alpha-l\rightarrow\alpha$; from a theorem of Morse* one may then infer $(x_n-\alpha)-l\rightarrow(x-\alpha)$. By Theorem 1 of AL it follows that $(x_n-\alpha)-v\rightarrow\beta$, in the notation of "convergence in variation." Hence we have $T(x_n-\alpha)=T(\alpha_n-\alpha)+T(\beta_n)\rightarrow T(\beta)$, and from $T(\alpha_n-\alpha)\geqq 0$ the conclusion follows.

Theorem 7. $(CBV)-(AC)$ *is the sum of a countable number of sets each closed and non-dense in* (CBV).

Proof. Allowing E_m to denote the set of points $x\varepsilon(CBV)-(AC)$ each of

* Morse, *Convergence in variation and related topics*, these Transactions, vol. 41 (1937), pp. 48–83, Theorem 5.4.

which satisfies the condition $T(\beta)\geqq 1/m$, we clearly have $(CBV)-(AC) =\sum_{m=1}^{\infty}E_m$. By Lemma 5 each E_m is closed; since with the metric (3) the set (AC) is dense* in (CBV), each E_m is also non-dense.

Corollary 9. *(AC) is a residual set in (CBV), and is of second category in itself.*

Corollary 10. *(CS) is of first category in (CBV), and is of second category in itself.*

Proof. In fact (CS) is a closed non-dense set† in (CBV). For, by Theorem 1 of AL the relation $x_n - l \to x$ implies $x_n - v \to x$; therefore, if $x_n \varepsilon (CS)$ for all n, we have for all t

$$\overset{t}{L_0}(x_n) = \overset{t}{L_0}(0) + \overset{t}{T_0}(x_n) = t + \overset{t}{T_0}(x_n) \to \overset{t}{L_0}(x) = t + \overset{t}{T_0}(x),$$

whence $\{1+[x'(t)]^2\}^{1/2}=1+|x'(t)|$ almost everywhere and $x'(t)=0$ almost everywhere; so that if $x\varepsilon(CBV)$, it also is in (CS).

It is easily seen by example, however, that (CS) has limit points in (DBV). Let $(\overline{CS})$ be the closure of (CS) in (BV). Since $(\overline{CS})$ is a closed set in a complete space (see I, §1), it is of second category in itself. We consider then $(\overline{CS})-(CS)\subset(DBV)$. Let $(DBV)=\sum_{m=1}^{\infty}E_m$ be the decomposition used in showing (DBV) to be of first category in (BV), i.e., the analogue for $L(x)$ of Theorem 5, and let $G_m=E_m\cdot[(\overline{CS})-(CS)]$. Since for each m the set E_m has no limit point in (CBV), G_m has no limit point in (CS). But (CS) is dense in $(\overline{CS})$; therefore $\overline{G}_m$, the closure of G_m in $(\overline{CS})$, contains no sphere of $(\overline{CS})$. Thus $(\overline{CS})-(CS)$ is of first category in $(\overline{CS})$, and (CS) is a residual set in $(\overline{CS})$. (CS) is therefore of second category in itself.

By aid of Theorem 5.2 of Morse (loc. cit.) it can readily be proved that (CS) is dense in (S), so that $(\overline{CS})=(\overline{S})$; reasoning similar to that of the preceding paragraph then shows that (S) is of second category in itself. That (S) is of first category in (BV) is easily seen. It may be remarked that (AC) is a residual set in (BV); that, of the 126 sets mentioned at the close of §3, any combination including (AC) [not including (AC) and not contained in (S)] is of second [first] category in (BV), in itself, and in any other combination of which it is a part; and that most, if not all, of the questions concerning the category of (S) and its subsets can be answered by the aid of results now in hand. We observe also that not only (CBV) but also each of the sets (AC), (CS), and $(CBV)-(CS)$ is a G_δ in (BV) and therefore homeomorphic to a complete metric space.

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* See I, §1.

† By aid of Theorem 8 below one may show similarly that (SN) is a closed non-dense set in (BVN).

With the metric (3), Schauder's base for (C) is again a base for (AC) and for (CBV); and by aid of Theorem 5.2 of Morse (loc. cit.) it follows that the base obtained above in §4 for (BV) with the metric (2) is also a base for (S) with the metric (3). But Haar's system of orthogonal functions is not a base for (BVN), and the question of bases for (BVN) and for (BV) we leave open.

One may use the notation $x_n(t) - dl \rightarrow x(t)$ on $(0, 1)$ to stand for the pair of conditions $x_n(t) \rightarrow x(t)$ on a dense set in $(0, 1)$ and $L(x_n) \rightarrow L(x)$. It is natural to inquire whether $x_n - dl \rightarrow x$ implies $x_n - dv \rightarrow x$. This question may be answered with the aid of the following lemma, whose proof will be left to the reader.

LEMMA 6. *For $x(t) \varepsilon (BVN)$ it is possible to approximate $x(t)$ in length by an inscribed broken line whose "corners" are at points of continuity of $x(t)$.*

THEOREM 8. *The relation $x_n(t) - dl \rightarrow x(t)$ on $(0, 1)$, with $x(t) \varepsilon (BVN)$, implies $x_n(t) - dv \rightarrow x(t)$.*

Proof. By Lemma 1 we have $\liminf T(x_n) \geqq T(x)$. For given $\epsilon > 0$ there exists a broken line B of the sort specified in Lemma 6 such that $L(B) > L(x) - \epsilon$. By aid of the analogue for $L(x)$ of Theorem 1 we may now pursue the argument used in establishing Theorem 1 of AL to complete the proof here.

In Theorem 8 *the qualification $x(t) \varepsilon (BVN)$ cannot be dispensed with*, as the following example shows: $x(t) = 0$ for $0 \leqq t < \frac{1}{2}$, $= 1$ for $t = \frac{1}{2}$, $= \frac{1}{2}$ for $\frac{1}{2} < t \leqq 1$; $x_n(t) = \frac{1}{2}$ for $\frac{1}{2} < t \leqq 1$ and all n; on $(0, \frac{1}{2})$, $x_n(t)$ a sequence of polygonal functions all having the same length $\frac{3}{2}$ and the same total variation $2^{1/2}$, converging uniformly to $x(t) = 0$ and with $x_n(0) = x_n(\frac{1}{2}) = 0$ for all n. We observe that an example of the same type shows that when $x(t)$ has an external saltus at even one point, the condition $x_n(t) \rightarrow x(t)$ on a dense set in $(0, 1)$, plus semicontinuity of $T(x)$ at x, *does not imply* semi-continuity of $L(x)$ at x.

We conclude with the following

COROLLARY 11. *The condition $(x_n, x) \rightarrow 0$ in the metric (3), with $x \varepsilon (BVN)$, implies that condition in the metric (2).*

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